UG/I/MATH/H/II/18

2018

MATHEMATICS

[Honours]

PAPER – II

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

Notations have their usual meanings

GROUP – A

(*Real Analysis*)

[*Marks* : 35]

1. Answer any *one* question : 15 × 1

(*a*) (*i*) State Peano's axioms for natural numbers. Using these, prove that the set of natural numbers has no upper bound. 5

(*ii*) Prove that every convergent sequence of real numbers is necessarily bounded. Is the converse true ? Justify your answer. 4 + 2

(*iii*) State order properties of real numbers. Prove that $\sqrt{17}$ is an irrational numbers. 1 + 3

(*b*) (*i*) Define a Cauchy sequence of real numbers. Prove that in R, every Cauchy sequence is convergent. Is it true in Q ? Justify your answer. 2 + 4 + 2

(*ii*) Prove that the set of rational numbers Q is enumerable. 4

(*iii*) Let A and B be subsets of R and $A \subset B$, then prove that $A' \subset B'$, where A' is the derived set of A. 3

2. Answer any *two* questions : 8 × 2

(*a*) (*i*) Define subsequence. Define limsup and liminf of a sequence. Prove

$$\lim\left(1+\frac{1}{2n}\right)^n = \sqrt{e}.$$ 4

(*ii*) Show that infinite series

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 x^2 + \cdots$$

converges if $0 < x < 1$ and diverges if $x \geq 1$. 4

(*b*) State Maclaurin's infinite series and obtain the expansion of $(1+x)^m$ where m is any real number other than positive integer and $|x| < 1$. 2 + 6

(*c*) (*i*) Let $f: [a, b] \to [a, b]$ be continuous on $[a, b]$. Then show that for some $\xi \in [a, b]$, $f(\xi) = \xi$. Is ξ unique ? 2 + 2

(*ii*) From Cauchy's Mean Value Theorem, deduce Lagrange's Mean Value Theorem. 4

3. Answer any *one* question : 4 × 1

(*a*) Find

$$\lim_{x \to \infty} \left\{x - \sqrt[n]{(x-a_1)(x-a_2)\ldots\ldots(x-a_n)}\right\}.$$ 4

(*b*) Give example to show that the condition stated in Rolle's Theorem are sufficient only but not necessary for the validity of the result. 4

GROUP – B

(*Several Variables and Applications*)

[*Marks* : 20]

4. Answer any *two* questions : 8 × 2

(*a*) Let

$$f(x,y)=\begin{cases} xy, & \text{when } y>0 \\ -xy^2, & \text{when } y\le 0 \end{cases}$$

which of the four Second-order derivatives exist at origin ? 8

(*b*) (*i*) Show that the family of circles

$$(x-a)^2+y^2=a^2$$

has no envelope. 3

(*ii*) If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$ show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\frac{y^2 - x^2}{uv(u-v)}.$$ 5

(*c*) (*i*) If $v = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then verify that $\sin v = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of x, y of degree $\frac{1}{2}$. Hence prove that

$$x\frac{dv}{dx} + y\frac{dv}{dy} - \frac{1}{2}\cdot\frac{1}{2}\tan v = 0.$$ 4

(*ii*) Find the envelope of the lines whose equation are

$$x\sec^2\theta + y\,\mathrm{cosec}^2\theta = c,\ \theta$$

being parameter and c is a constant. 4

5. Answer any *one* question : 4 × 1

(*a*) Find the pedal equation of the curve
$$r = a + b\cos\theta.$$
with respect to the pole. 4

(*b*) Show that the origin is a node, cusp or on isolated point on the curve $y^2 = ax^2 + bx^3$ according as $a >=$ or < 0. 4

GROUP – C

(*Analytical Geometry for two Dimensions*)

[*Marks* : 20]

6. Answer any *two* questions : 8 × 2

(*a*) If one of the straight lines $ax^2 + 2hxy + by^2 = 0$ coincides with one of the straight lines $a'x^2 + 2h'xy + b'y^2 = 0$ and the remaining two straight lines are at right angles, then prove that
$$h\left(\frac{1}{b}-\frac{1}{a}\right) = h'\left(\frac{1}{b'}-\frac{1}{a'}\right).$$ 8

(*b*) Reduce the equation

$$4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0$$

to its canonical form and show that it represents a parabola. Find the latus rectum and the equation of the axis of the parabola. 8

(*c*) Prove that the two conics

$$\frac{l_1}{r} = 1 - e_1 \cos\theta \text{ and } \frac{l_2}{r} = 1 - e_2 \cos(\theta - \alpha)$$

will touch one another if

$$l_1^2\left(1-e_2^2\right)+l_2^2\left(1-e_1^2\right)=2\,l_1\,l_2\left(1-e_1e_2\cos\alpha\right).$$ 8

7. Answer any *one* question : 4 × 1

(*a*) If by a rotation of co-ordinate axes the expression $ax^2 + 2hxy + by^2$ changes to $a'x'^2 + 2h'x'y' + b'y'^2$, then prove that $a + b = a' + b'$ and $ab - h^2 = a'b' - h'^2$. 4

(*b*) If the sum of the ordinates of two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be b, show that the locus of the pole of the chord which joins them is $b^2x^2 + a^2y^2 = 2a^2by$. 4

GROUP –D

(Differential Equations-I)

[*Marks* : 15]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) Show that the differential equation of a general parabola is

$$\frac{d^2}{dx^2}\left[\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right] = 0.$$ 5

(*ii*) Show that in an exact equation $M(x, y)\,dx + N(x, y)\,dy = 0$, if M and N be homogeneous functions of x and y and of degree $n\,(\neq -1)$, having continuous first order partial derivatives, then the primitive is $Mx + Ny =$ constant. 5

(*iii*) Solve by the method of variation of parameter

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}\sec x.$$ 5

(*b*) (*i*) Solve

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-1=0$$

given that $x+\frac{1}{x}$ is its one integral. 5

(*ii*) Find the eigenvalues and eigenfunctions of $\frac{d^2y}{dx^2}+\lambda y=0,\ (\lambda>0)$ under boundary conditions $y(0)+y'(0)=0$ and $y(1)+y'(1)=0$. 5

(*iii*) Find the orthogonal trajectories of the family of co-axial circles

$$x^2+y^2+2gx+c=0$$

where g is the parameter and c is constant. 5
