UG/I/MATH/H/I/18

2018

MATHEMATICS

[Honours]

PAPER –I

Full Marks : 90

Time : 4 hours

The figures in the right-hand margin indicate marks

GROUP – A

(*Classical Algebra*)

[Marks : 30]

1. Answer any *one* question : 15 × 1

(*a*) (*i*) Show that the equation $x^4 - 14x^2 + 24x + k = 0$ has

(*I*) four real and unequal roots if $-11 < k < -8$.

(*II*) two distinct real roots if $-8 < k < 117$

(*III*) no real root if $k > 117$.

Discuss the cases when $k = 117$, $k = -8$ and $k = -11$. 5

(*ii*) If a, b, c be positive real numbers such that the sum of any two is greater than the third, prove that

$$abc \geq (a+b-c)(b+c-a)(c+a-b)$$ 4

(*iii*) State and prove De Moivre's theorem. 6

(*b*) (*i*) If $(1+x)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots$, then show that

$$a_0 - a_2 + a_4 - \cdots\cdots = 2^{n/2} \cos\frac{n\pi}{4}$$

and $a_1 - a_3 + a_5 - \cdots\cdots = 2^{n/2} \sin\frac{n\pi}{4}$. 5

(*ii*) If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots b_n$ are any real numbers, then prove that

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots\cdots + a_n b_n)^2$$ 6

(*iii*) Find the relation among p, q, r, s so that the product of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ is unity. 4

2. Answer any *one* question : 8 × 1

(*a*) (*i*) If α, β, γ be the roots of the equation $x^3 + qx + r = 0$ $(r \neq 0)$, find the equation whose roots are

$$\frac{\alpha}{\beta}, \frac{\beta}{\alpha}, \frac{\beta}{\gamma}, \frac{\gamma}{\beta}, \frac{\gamma}{\alpha}, \frac{\alpha}{\gamma}.$$ 4

(*ii*) Find the roots of the equation $z^n = (z+1)^n$ and show that the points which represent them in the Argand diagram are collinear. 4

(*b*) (*i*) Let a, b, c be three positive numbers in harmonic progression. Prove that

$$\frac{a+b}{2a-b} + \frac{c+b}{2c-b} > 4.$$ 3

(*ii*) If $1, \alpha, \beta, \gamma, \ldots$ are the roots of the equation $x^n - 1 = 0$, then show that

$$(1-\alpha)(1-\beta)(1-\gamma)\cdots\cdots = n.$$ 3

(*iii*) Find the values of $(1+i)^{1/5}$. 2

3. Answer any *one* question : 4 × 1

(*a*) Solve the equation by Cardan's method

$$x^3+3x^2+6x+4=0.$$ 4

(*b*) Find the general values and the principal value of $i^{(x+iy)}$ where x, y are real. Show that the principal value is purely real or purely imaginary according as x is an even or an odd integer. 4

4. Answer any *one* question : 3 × 1

(*a*) If α be a multiple root of order 3 of the equation $x^4+bx^2+cx+d=0$, then show that

$$\alpha=-\frac{8d}{3c}.$$ 3

(*b*) If $a.\exp(i\theta)+b.\exp(3i\theta)=c$ where a, b, c are all real, prove that either $a+b=\neq c$, or $b(b-a)=c^2$. 3

GROUP – B

(*Abstract Algebra*)

[Marks : 35]

5. Answer any *three* questions : 8 × 3

(*a*) (*i*) Correct or justify the statement : Every relation is a mapping but every mapping is not a relation. 4

(*ii*) Use mathematical induction to establish

$$\sum_{i=1}^{n}(i+1)2^{i} = n\cdot 2^{n+1}.$$ 4

(*b*) (*i*) Let

$$G=\left\{\begin{pmatrix} a & b \\ a & b\end{pmatrix} \middle| \begin{matrix} a,b\in R \\ a+b\neq 0\end{matrix}\right\},$$

then show that

(1) G is a semigroup under matrix multiplication.

(2) G has left identity.

(3) Each element of G has a right inverse. 4

(*ii*) Prove that every field is an integral domain, but the converse is not true. 4

(*c*) (*i*) Define cosets of a subgroup in a group. Let G be a group and H be a subgroup of G. Prove that any two left cosets of H in G are either identical or disjoint. 1 + 3

(*ii*) Define normal subgroup of a group. If $Z(G)$ be the centre of the group G. Prove that $Z(G)$ is a normal subgroup of G. 1 + 3

(*d*) (*i*) Let G be a group and the mapping $f: G \to G$ be defined by $f(a) = a^{-1}, a \in G$. Show that f is an automorphism if and only if G is abelian. 4

(*ii*) Define subfield of a field. Prove that intersection of two subfield is a subfield. 4

(*e*) (*i*) Prove that order of each subgroup of a finite group G is a divisor of the order of the group G. 4

(*ii*) State well ordering principle for the set of natural numbers. Prove that every integer $n > 1$ has a prime factor. 4

6. Answer any *two* questions : 4 × 2

(*a*) State and prove Lagrange's theorem on finite groups. 4

(*b*) If $a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ are two elements of S_3, then find the solution of the equation $ax = b$ in S_3. 4

(*c*) If in a ring $(B, +, \cdot)$, $a^2 = a$ for all $a \in B$, show that $2a = a + a = 0$ for all a where B is a commutative ring. Show that in a commutative ring R, $2a = 0$ for all $a \in R$ need not hold. 2 + 2

7. Answer any *one* question : 3 × 1

(*a*) Define cyclic group. Prove that cyclic group is abelian. 3

(*b*) Let G be a group and $a \in G$. If $O(a) = 24$, find $O(a^4)$, $O(a^7)$ and $O(a^{10})$. 3

GROUP – C

(*Linear Algebra*)

[Marks : 25]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) If

$$U = L\{(1, 2, 1), (2, 1, 3)\},$$
$$W = L\{(1, 0, 0), (0, 0, 1)\}$$

show that U and W are subspace of $\mathbb{R}^3$. Determine $\dim U$, $\dim W$, $\dim(U \cap W)$. Deduce that $\dim(U + W) = 3$. 5

(*ii*) Show that a linearly independent set of vectors of a finite dimensional vector space V over a field F is either a basis of V or can be extended to a basis of V. 6

(*iii*) If a, b, c are unequal real numbers then find the rank of the matrix

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{pmatrix}.$$ 4

(*b*) (*i*) A real symmetric matrix A is positive definite. Prove that $A = PP'$ for some non-singular matrix P. Show that

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

is positive definite and find the matrix P such that $A = PP'$. 5

(*ii*) Find the eigen values and the corresponding eigen vectors of the following matrix

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}.$$ 5

(*iii*) Define eigen value of a matrix. If λ be an eigen value of an orthogonal matrix, then show that $\dfrac{1}{\lambda}$ is also an eigen value of this matrix. 5

9. Answer any *one* question : 8 × 1

(*a*) (*i*) If α and β be vectors in an inner product space, then show that

$$\| \alpha + \beta \|^2 + \| \alpha - \beta \|^2 = 2 \| \alpha \|^2 + 2 \| \beta \|^2$$ 4

(*ii*) Determine the values of k so that the following system of equations has (1) no solution, (2) more than one solution, (3) a unique solution.

$$x + y - z = 1$$
$$2x + 3y + kz = 3$$
$$x + ky + 3z = 2$$ 4

(*b*) (*i*) Reduce the following quadratic form to normal form and examine whether the quadratic form is positive definite or not.

$$6x^2 + y^2 + 18z^2 - 4yz - 12zx$$ 4

(*ii*) Use Gram-Schmidt process to obtain an orthogonal basis from the basis set $\{(1, 0, 1), (1, 1, 1), (1, 3, 4)\}$ of the

Euclidean space $\mathbf{R}^3$ with standard inner product. 4

10. Answer any *one* question : 2×1

(*a*) Use Cayley-Hamilton theorem to compute A^{-1}, where $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$. 2

(*b*) Find a basis and the dimension of the subspace W of $\mathbf{R}^3$, where

$$W = \{(x, y, z) \in \mathbf{R}^3 : x + y + z = 0\}.$$ 2