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PG/IVS/MTM-401/15

M.Sc. 4th Semester Examination, 2015

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

(*Functional Analysis*)

PAPER — MTM - 401

Full Marks : 50

Time : 2 hours

Answer Q.No.1 and any four from Q.No.2 to Q.No.6

The figures in the right-hand margin indicate marks

1. Answer any *four* questions : 2×4

(*a*) State, with justification, whether the following statement is true or false. Let X and Y be normed linear spaces and $F : X \to Y$ be a bounded linear map. If $\{x_n\}_{n \geq 1}$ is a cauchy sequence in X, then $\{F(x_n)\}_{n \geq 1}$ is a cauchy sequence in Y.

(*b*) Let $X = C^1[0, 1]$ and $Y = C[0, 1]$, with the supremum norms. Let $F : X \to Y$ be defined

(Turn Over)

as $F(x) = x'$, where x' denotes the derivatives of x. Is F continuous ? Justify your answer.

(*c*) "Every normed space is a Hilbert space". Is this statement is true or false ? Justify your answer.

(*d*) Define seperable metric space. Give an example of a metric space which is not separable.

(*e*) State uniform boundedness principle.

(*f*) Let $T : X \to Y$ be a continuous linear operator. Show that the null space $N(T)$ is closed.

2. (*a*) Let X be a linear space over $\mathbb{C}$ and f be a complex-linear functional on X. Then Ref is a real-linear functional on X, regarded as a linear space over $\mathbb{R}$. Show that :

(*i*) Ref determines f as follows :

$$f(x) = \text{Ref}(x) - i\,\text{Ref}(ix),\ x \in X.$$

(*ii*) If $\|\ \|$ is a norm on X, then $\|\text{Ref}\| = \|f\|$.

(*b*) Let $X = C[0, 1]$ with the supremum norm. Consider the sequence

$$x_n(t) = \frac{t^n}{n^2}, \quad t \in [0,1].$$

Check whether the series $\sum_{n=1}^{\infty} x_n$ is summable in X. 4 + 4

3. (*a*) Show that the Function space $C[a, b]$ is a Banach space.

(*b*) If A^* is the adjoint of the operator $A : H \to H$ then show that $\| A^* \| = \| A \|$, where H is a Hilbert space. 4 + 4

4. (*a*) State and prove Riesz representation theorem.

(*b*) If $A \in BL(H)$ is self-adjoint ; then show that

$$\| A \| = \text{Sup} \{ |\langle Ax, x \rangle| : \| x \| \leq 1, x \in H \}$$ 5 + 3

5. (*a*) If $\{ x_1, x_2, \ldots\ldots, x_n \}$ is an orthogonal set of an inner product space then prove that

$$\| x_1 + x_2 + \ldots\ldots + x_n \|^2 = \| x_1 \|^2 + \| x_2 \|^2 + \ldots\ldots + \| x_n \|^2.$$

What is the name of this result ?

(*b*) If $\{e_1, e_2, \ldots\ldots, e_n\}$ is a finite orthonormal set in an inner product space X and x is any element of X, then prove that

$$\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2.$$

When equality holds ? 4 + 4

6. (*a*) State the closed graph theorem. Let H be a Hilbert space and F be a closed subspace of H. Let $P : H \to F$ be the orthogonal projection defined as follows. Any $x \in H$ can be uniquely written as

$$x = y + z, \quad y \in F, \; z \in F^{\perp}.$$

Define $Px = y$. Show that P is continuous.

(*b*) Let $T \in BL(\mathrm{H,K})$ where H, K are Hilbert spaces. Then show that

$$\mathrm{Ker}(T) = \mathrm{Ran}(T^*)^{\perp} = \mathrm{Ker}(T^* T)$$ 2 + 3 + 3

[*Internal Assessment* : 10 Marks]