2015

M.Sc.

3rd Semester Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—MTM-301

Full Marks : 50

Time : 2 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

(Partial Differential Equations and Generalized Functions)

Answer *Q. No. 1* and any *two questions* from the rest.

1. Answer any *two* of the following : 2×4

(a) Show that :

$$x^n D^m \delta(x) = \begin{cases} 0, & m<n \\ \frac{(-1)^n m!}{(m-n)!} D^{m-n}\, \delta(x), & m > n \end{cases}$$

$\delta(x)$ being the Direct-delta function.

(Turn Over)

(b) Define Cauchy problem for a first order Quasi-linear Partial Differential equation.

Give example of Cauchy problem for the first order Quasi-linear PDE which have unique solution and does not have any solution.

(c) Find the general integral of :

$$(y + zx)p - (x + yz)q = x^2 - y^2$$

2. (a) Classify and reduce the following equation to its Canonical form :

$$e^x u_{xx} + e^y u_{yy} = u.$$ 8

(b) Let u be a Continuously differentiable function on the closure of D where D is given by

$$D = \{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\}.$$

Let u be a solution of $a(x, y)u_x + b(x, y)u_y = -u$, where a, b are positive differentiable functions in the entire plane. Then,

(i) Show that if u is positive on the boundary of D, then it is positive at every point in D.

(ii) Suppose that u attains a local minimum (maximum) at a point $(x_0, y_0) \in D$. Evaluate $u(x_0, y_0)$. 2+2

(c) If $F(D,D') \equiv (\alpha_r D + \beta_r D' + \gamma_r)^2$ and $\varphi_r(\xi)$ and $\psi_r(\xi)$ are arbitrary functions of ξ, show that if $\alpha_r \neq 0$,

$$u_r = \left[x\varphi_r(\beta_r x - \alpha_r y) + \psi_r(\beta_r x - \alpha_r y)\right] \exp\left(-\frac{\gamma_r x}{\alpha_r}\right)$$

is a solution of the equation $F(D, D')z = 0$ where $\alpha_r, \beta_r, \gamma_r$ are constants and $D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}$. 4

3. (a) Solve the problem : 8+2

$$u_{tt} - u_{xx} = 0, \; 0 < x < \infty, \; t > 0,$$
$$u(0, t) = \frac{t}{1+t}, \; t \geq 0,$$
$$u(x, 0) = u_t(x, 0) = 0, \; 0 \leq x \leq \infty.$$

Also show that the limit

$$\lim_{x \to \infty} u(cx, x) = \phi(c)$$ exists for all $c > 0$.

What is the limit?

(b) Using the method of separation of variables solve the following problem :

$$u_{tt} - c^2 u_{xx} = 0, \; 0 < x < 2, \; t > 0$$
$$u(0, t) = u(2, t) = 0, \; t \geq 0$$
$$u(x, 0) = \sin^3\left(\frac{\pi x}{2}\right), \; 0 \leq x \leq 2$$
$$u_t(x, 0) = 0, \; 0 \leq x \leq 2$$ 6

4. (a) Define Fourier transform of a generalised function. Use it to obtain Fourier transform of $\delta(x - x_0)$. 2+3

(b) Let $D \subseteq \mathbb{R}^2$ be a domain and u be a harmonic function in D. Then show that u has the mean value property on D. 5

(c) (i) Show that the Robin problem for the Poisson's equation has at most one solution if $\alpha \geq 0$.

(ii) Let u(x, y) be a non-constant harmonic function in the disk $x^2 + y^2 < R^2$.

Define for each $0 < r < R$,

$$\mu(r) = \max_{x^2 + y^2 = r^2} u(x, y)$$

Prove that M(r) is a monotone increasing function in the interval (0, R). 3+3

(Internal Assessment — 10 Marks)