PG/IS/MTM-103/14

M.Sc. 1st Semester Examination, 2014

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

(Ordinary Differential Equations and Special Functions)

PAPER – MTM- 103

Full Marks : 50

Time : 2 hours

Answer **Q.No.1** and any **three** questions from **Q.No.2** to **Q.No.5**

The figures in the right hand margin indicate marks

The Symbols have their usual meanings.

1. Answer any *five* questions : 2 ×5

(*Turn Over*)

(*a*) Let $P_n(z)$ be the Legendre's polynomial of degree n and $P_{m+1}(0) = -\frac{m}{m+1} P_{m-1}(0)$,

$m = 1, 2, 3, \ldots.$

If $P_n(0) = -\frac{5}{16}$, then find the value of

$$\int_{-1}^{1} P_n^2(z)\, dz.$$

(*b*) Show that the Green's function of a given problem is everywhere continuous.

(*c*) Write down the hypergeometric series represented by $F(a, b, c\,; z)$. Prove that

$$F(1, b, b\,; z) = \frac{1}{1-z}.$$

(*d*) What are meant by regular and irregular singularities of the differential equation :

$$a_0(z)\frac{d^2 w}{dz^2} + a_1(z)\frac{dw}{dz} + a_2(z)w = 0.$$

(*e*) What do you mean by fundamental matrix of system of linear homogeneous differential equation ?

(*f*) Write the important features of Sturm-Liouville problem

2. (*a*) Prove that if $f(z)$ is continuous and has continuous derivatives in [-1, 1] then $f(z)$ has unique Legendre series expansion is given by

$$f(z) = \sum_{n=0}^{\infty} C_n P_n(z)$$

where P_n' s are Legendre Polynomials and

$$C_n = \frac{2n+2}{2} \int_{-1}^{1} f(z)\ P_n(z) dz, \quad n = 1, 2, 3 \ldots.$$ 6

(*b*) Show that

$$J_0^2(z) + 2\sum_{n=1}^{\infty} J_n^2(z) = 1$$

and prove that for real z, $|J_0(z)| \le 1$, and $|J_n(z)| < \frac{1}{\sqrt{2}}$, for all $n \ge 1$. 4

3. (*a*) Show that the solution of the differential equation

$$\frac{d^2u}{dx^2} = f(x)$$

subject to the boundary conditions $u(0) = u(a) = 0$ is given by

$$u(x) = \int_0^a G(x,\xi) f(\xi) d\xi,$$

where $G(x, \xi)$ is known as Green's function to be determined by you. 6

(*b*) Show that

(*i*) $z J_n'(z) = z J_{n-1}(z) - n J_n(z)$

(*ii*) $(n+1) P_{n+1}(z) - (2n+1) z P_n(z) + n P_{n-1}(z) = 0.$ 4

4. (*a*) Find the general solution of the ODE $2zw''(z) + (1+z)w'(z) - kw = 0$. (where k is a real constant) in series form for which values of k is there a polynomial solution ? 6

(*b*) Deduce the integral formula for hypergeometric function. 4

5. (*a*) Obtain the first five terms in the expansion of the following function f in terms of Legendre's polynomial

$$f(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ x, & \text{if } 0 < x < 1. \end{cases}$$ 5

(*b*) Find the general solution of the homogeneous equation :

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \vec{x}, \quad \text{where} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$ 5

[*Internal Assessment* — 10 *Marks*]