M.Sc. 2nd Semester Examination, 2013

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

(*Functional Analysis*)

PAPER—MTM - 205

Full Marks : 50

Time : 2 hours

Answer **Q. No.1** and **2** and any **four** from **Q. No. 3** to **8**

The figures in the right-hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

1. Answer any *two* questions : 2 × 2

(*a*) Define seperable metric space. Give an example of a metric space which is not seperable.

(*b*) Let H be a Hilbert space and fix $Y \in H$. Define $f(x) = \langle x, y \rangle$, $\forall X \in H$. Find $\|f\|$.

(Turn Over)

(*c*) Prove that any function from a discrete metric space into a metric space is continuous.

2. Answer any *one* : 8 × 1

(*a*) If X is a normed space, M is a closed subspace of X, $x_0 \in X \setminus M$ and $d = \text{dist}(x_0, M)$, show that there is an $f \in X^*$ such that $f(x_0) = d$, $f(x) = 0$ for all $x \in M$, and $\|f\| = 1$. 8

(*b*) Using Banach fixed point theorem determine the solution of the system of equations 8

$$x = 0{\cdot}2x - 0{\cdot}5y + 1{\cdot}3$$
$$y = 0{\cdot}4x + 0{\cdot}3y + 0{\cdot}3$$

3. Let V, W and U be normed spaces. Prove that :

(*i*) $B(V, W)$ is a normed space.

(*ii*) If $T \in B(V, W)$ and $S \in B(W, U)$ Prove that $ST \in B(V, U)$ and $\|ST\| \leq \|S\| \, \|T\|$. 3 + 4

4. (*a*) State uniform Boundedness principle.

(*b*) Suppose $\{T_n : n = 1, 2, 3, \ldots\} C\, B(X,Y)$ is a sequence of bounded operators, where X is a Banach space and Y is a normed space, and suppose the sequence $\{T_n x\}_{n=1}^{\infty}$ is a convergent sequence in Y, for each $x \in X$. Show that the equation $T(x) = \lim_{n\to\infty} T_n(x)$ defines a bounded operator $T \in B(X, Y)$. 2 + 5

5. (*a*) Prove : (polarization identity) For $x, y \in V$,

$$4 < x.y >= \sum_{k=0}^{3} i^k \|x + i^k y\|^2.$$

where V is a inner product space.

(*b*) Assume that $\{u_\alpha\}_{\alpha \in I}$ is an orthonormal set in the inner product space X and $x \in X$. Let $E_x = \{u_\alpha : < x, u_\alpha > \neq 0\}$. Then show that E_x is a countable set. 3 + 4

6. (*a*) Define positive operator. Prove that positive operators are self-adjoint operators.

(*b*) If H be a complex Hilbert space and $A \in BL(H)$ and $< Ax, x > = 0$, $\forall x \in H$, then show that $A = 0$. 1 + 2 + 4

7. (*a*) State and Prove Projection theorem.

   (*b*) Let H be a Hilbert space and ECH. Prove that Span $E = E^{\perp\perp}$. 4 + 3

8. (*a*) If $f(x) = f(y)$ for every bounded linear functional f on a normed space X, show that $x = y$.

   (*b*) Is dual of a non-zero normed space non-zero ? 2 + 5

[*Internal Assessment* : 10 Marks]