Total Pages—12 **UG/II/MATH/H/V/17 (New)**

2017

MATHEMATICS

[Honours]

PAPER – V

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

[NEW SYLLABUS]

GROUP – A

(*Real Analysis*-II)

1. Answer any *two* questions : 15 × 2

(*a*) (*i*) Let $[a, b]$ be a closed and bounded interval and a function $f : [a, b] \rightarrow R$ be bounded on $[a, b]$. Prove that for any $\in (> 0)$ there corresponds $\delta\ (> 0)$ such that

$$U(P,f) < \int_a^b f + \in$$

$$L(Q,f) > \int_{\underline{a}}^b f - \in$$

for all partitions P and Q of $[a, b]$ having norm $\|P\| < \delta$ and $\|Q\| < \delta$.

If a function satisfies the above two conditions then it is Riemann integrable or not – Justify.

(*ii*) If $\alpha(x)$ is continuous in $[a, b]$ and if

$$\int_a^b \alpha(x)\, h(x)\, dx = 0$$

for every function $h(x)$, continuous in $[a, b]$ such that $h(a) = 0 = h(b)$, show that $\alpha(x) = 0$ for all $x \in [a, b]$.

(*iii*) When $-1 < x \leq 1$, show that

$$\lim_{m \to \infty} \int_0^x \frac{t^m}{1+t}\, dt = 0.$$

(3 + 3 + 2) + 4 + 3

(*b*) (*i*) Prove that a bounded function f is integrable on $[a, b]$, if the set of its points of discontinuity has only a finite number of limit points.

Hence or otherwise show that the function $[x]$, where $[x]$ denotes the greatest integer not greater than x is integrable on $[0, 3]$.

(*ii*) State Second Mean Value theorem of integral calculus. A function $f(x)$ is defined on $[0, 1]$ by

$$f(x) = \frac{1}{2^n}, \frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n} (n = 0, 1, 2, ..)$$
$$= 0, \quad x = 0.$$

Prove that $f(x)$ is Riemann integrable on $[0, 1]$ and

$$\int_0^1 f(x)\,dx = \frac{2}{3}.$$

(5 + 3) (2 + 5)

(*c*) (*i*) State and prove Taylor's theorem for two variables with Lagrange's form of remainder. Hence show that for any homogeneous function $f(x, y)$ of degree n, having continuous partial derivatives upto any order

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1) f(x, y).$$

(*ii*) Show that for $\alpha > -1$,

$$\int_0^\infty e^{-x^2} x^\alpha \, dx$$

Converges to

$$\frac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right),$$

Hence deduce that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

(5 + 5) + (3 + 2)

2. Answer any *two* questions : 8 × 2

(*a*) (*i*) Let $f : [a, B]$ be an integrable function for all $B > a$ and $f(x) > 0 \; \forall \; x \geq a$. Prove that the integral

$$\int_a^{\infty} f(x)\, dx$$

is Convergent if and only if there exists a positive real number K such that

$$\int_a^{B} f(x)\, dx < K$$

for all $B > a$. Is this theorem applicable for the convergence of

$$\int_a^{\infty} f(x)\, dx$$

where $f(x)$ is not always positive for all $x \geq a$ or not – *justify*.

(*ii*) Show that the integral

$$\int_a^{\infty} \frac{\sin mx}{x^n}\, dx (m > 0, a > 0)$$

is convergent if $n > 0$ and is absolutely convergent if $n > 1$. (3 + 1) + 4

(*b*) (*i*) Let $f : A \to R$ where $A \subset R^2$ is a region. If $(a, b) \in A$ and if $f(x, y)$ is differentiable at (a, b) then prove that $f(x, y)$ is continuous at (a, b) and both partial derivative f_x and f_y exist at (a, b).

(*ii*) Prove that

$$\iint_R \sqrt{|x^2 - 2y|}\, dx\, dy = 2\pi$$

where $R = [-2, 2 ; 0, 2]$. 4 + 4

(*c*) (*i*) Prove that the necessary condition for the extremum of the function $f(x, y, z)$

where x, y, z satisfies $g(x, y, z) = 0$ (i.e. a constraint) and

$$\frac{\partial g}{\partial z} \neq 0, \text{ are } \frac{\partial(f, g)}{\partial(x, z)} = 0 \text{ and } \frac{\partial(f, g)}{\partial(y, z)} = 0.$$

(*ii*) Show that

$$\int_1^{\infty} e^{-xt} \cdot \frac{\sin t}{t}\, dt, x \geq 0$$

is uniformly convergent. 4 + 4

3. Answer any *one* question : 4 × 1

(*a*) Show that the function

$$f(x, y) = 2x^4 - 3x^2 y + y^2$$

has neither maximum nor a minimum at (0, 0). 4

(*b*) Show that

$$\iiint \frac{dx\, dy\, dz}{x^2 + y^2 + (z-2)^2} = \pi\left(2 - \frac{3}{2}\log 3\right)$$

extended over the sphere $x^2 + y^2 + z^2 \leq 1$. 4

GROUP – B

(*Metric Space*)

4. Answer any *one* question : 8 × 1

(*a*) (*i*) Let M denote the set of all bounded sequences of real numbers. If $x = \{x_n\}$ and $y = \{y_n\}$ are points in M, define

$$d(x,y) = \underset{1 \le n < \infty}{\text{l.u.b}} |x_n - y_n|.$$

Show that (M, d) is a metric space.

(*ii*) Prove that a subset G of the metric space (X, d) is closed if and only if its complement G^c is open. 4 + 4

(*b*) (*i*) Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a metric space (X, d) and $d(x_n, y_n) \to 0$ as $n \to \infty$. Show that if $\{x_n\}$ is a Cauchy sequence, then $\{y_n\}$ is a Cauchy sequence or if $\{x_n\}$ converges to x of X then $\{y_n\}$ converges to x.

(*ii*) State Cantor's intersection theorem and also show that both the conditions in this theorem are necessary. (2 + 2) + (2 + 2)

5. Answer any *one* question : 4 × 1

(*a*) If d is a metric on a set X, then show that d_1 is also a metric on a set X where $d_1(x, y) = \min\{1, d(x, y)\}\ \forall x, y \in X$. 4

(*b*) Let (X, d) be a metric space. Prove that

$$\text{Int}\, A = X - Cl\,(X - A).$$ 4

6. Answer any *one* question : 3 × 1

(*a*) Let (X, d) be any metric space and A is a non-empty subset of X. Then for all $x, y \in X$

$$|d(x, A) - d(y, A)| \leq d(x, y).$$ 3

(*b*) In a discrete space (X, d) show that every subset X is closed. 3

GROUP – C

(*Complex Analysis*)

7. Answer any *one* question : 8 × 1

(*a*) (*i*) Define stereographic projection of the unit sphere in Euclidean space R^3 to the extended complex plane.

(*ii*) Examine the continuity of the function at $z = 0$ where $f(z)$ is given by

$$f(z) = \frac{\bar{z}}{z}, z \neq 0$$
$$= 0, z = 0.$$

4 + 4

(*b*) (*i*) Let $f(z) = u + iv$ be an analytic function in a region G. Show that both u and v satisfy the following equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0.$$

(*ii*) Show that the function $u = x^3 - 3xy^2$ is harmonic and also find the corresponding analytic function. 4 + 4

8. Answer any *one* question : 2 × 1

(*a*) If f is an analytic function prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2. \qquad 2$$

(*b*) Show that $f(z) = \text{Re}(z)$ is not differentiable anywhere in C. 2

GROUP – D

(*Tensor Calculus*)

9. Answer any *one* question : 8 × 1

(*a*) (*i*) Prove that co-variant derivative of Tensor to type (0, 1) is a tensor of type (0, 2).

(*ii*) If a^{ij} is a symmetric tensor, show that

$$a^{jk}[ij,k] = \frac{1}{2} a^{jk} \frac{\partial g_{jk}}{\partial x^i}$$

where g_{ij} are the components of the fundamental tensor. 4 + 4

(*b*) (*i*) If $a_{ij} u^i u^j$ is an invariant for an arbitrary contravariant vector u^i, show that $a_{ij} + a_{ji}$ is a tensor.

(*ii*) Show that the Christoffel symbols of 1st and 2nd kind are symmetric in *i* and *j*. 4 + 4

10. Answer any *one* question : 4 × 1

(*a*) If $a_{ij} (\neq 0)$ are the components of a co-variant tensor of order 2 such that $ba_{ij} + ca_{ji} = 0$

where b and c are non-zero scalars show that either $b = c$ and a_{ij} is skew-symmetric or $b = -c$ and a_{ij} is symmetric. 4

(*b*) Prove that the fundamental tensors behave in covariant differentiation as though they were constants. 4

11. Answer any *one* question : 3 × 1

(*a*) Show that in a Riamannian space

$$\left\{ \begin{matrix} i \\ i\ j \end{matrix} \right\} = \frac{\partial}{\partial x^j}\left(\log\sqrt{g}\right)$$

where $g = |g_{ij}| \neq 0$. 3

(*b*) What is quotient law ? Define inner product of tensors. 3