NEW

2017

Part-II 3-Tier

MATHEMATICS

(General)

PAPER—II

Full Marks : 90

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group—A

(*Differential Calculus*)

[Marks : 45]

1. Answer any *one* question : 1×15

(a) (i) Show that if $x_n = \dfrac{3n+1}{n+2}$ then the sequence $\{x_n\}$ is strictly ıncreasing. Is the sequence convergent ? Justiiy your answer. Also find its limits. 3+2+1

(Turn Over)

(ii) Give example to show that the sum and product of two irrational numbers may be rational or irrational. 2

(iii) Use Cauchy's criterion to prove that the sequence $\{x_n\}$, where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, is divergent. 5

(iv) Evaluate $\lim_{n\to\infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n}$. 2

(b) (i) If $f(x) = 2|x| + |x-2|$, find $f'(1)$. 3

(ii) State D' Alembert ratio test for positive term series. Examine the convergence of the series

$$2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \cdots \quad (x > 0).$$ 2+5

(iii) If $\lim_{x\to 0} \frac{\sin 2x + 9\sin x}{x^3}$ be finite, find the value of a and the limit. 2+2

(iv) The function f is defined as

$$\begin{aligned} f(x) &= -2\sin x, \quad -\pi \le x \le -\pi/2 \\ &= a.\sin x + b, \quad -\pi/2 < x < \pi/2 \\ &= \cos x, \quad \pi/2 \le 6 \le \pi \end{aligned}$$

If the function f(x) is continuous in the interval $-\pi \le x \le \pi$, find the values of a and b. 3

2. Answer any *one* question : 1×8

(a) (i) State Cauchy's Mean Value Theorem. Deduce Lagrange's Mean Value Theorem from Cauchy's Mean Value Theorem. 2+2

(ii) In the Mean Value Theorem, $f(h) = f(0) + h\, f'(\theta h)$, $0 < \theta < 1$. Show that the limiting value of θ as $h \to 0$ is $\frac{1}{2}$, when $f(x) = \cos x$. 4

(b) (i) Expand the polynomial $f(x) = x^3 - 2x^2 + 3x + 5$ in a series of positive integral powers of $(x - 2)$. 4

(ii) Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$. 4

3. Answer any *four* questions : 4×4

(a) Investigate for what values of x, the function $f(x) = x^3 - 9x^2 + 24x - 12$ has maximum or minimum. 4

(b) Show that the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ at the point (a, b), whatever be the value of n. 4

(c) If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$ then show that

$$\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}.$$ 4

(d) Find the asymptotes of the cubic
$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0.$ 4

(e) Find the envelope of the family of parabolas
$\alpha y^2 = 2x + 12\alpha^3$, where α is a parameter. 4

(f) Verify the Schwarz's Theorem for the following function at (0, 0).

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & x^2 + y^2 = 0 \end{cases}$$ 4

4. Answer any *three* questions : 2×3

(a) If $z = e^{xy^2}$, $x = \cos t$, $y = \sin t$ find $\dfrac{\partial z}{\partial t}$ at $t = \pi/2$. 2

(b) Show that the length of the normal chord at any point on the catenary $y = c \cosh (x/c)$ is y^2/c. 2

(c) State the necessary condition for convergence of an infinite series Σu_n ($u_n > 0$) and justify it with an example. 2

(d) Give an example of a function which is discontinuous everywhere. Justify your answer. 1+1

(e) Find the n^{th} derivative of $\sin(ax + b)$. 2

Group—B

(*Integral Calculus*)

[Marks : 30]

5. Answer any *one* question : 1×16

A. (a) Evaluate any *two* : 2×4

(i) $\int \frac{x^3 dx}{(x-a)(x-b)(x-c)}$ 4

(ii) $\int \sin^{-1}\sqrt{\frac{x}{x+a}}\,dx$ 4

(iii) $\int \frac{x^2+1}{x\sqrt{1+x^4}}\,dx$ 4

(b) Obtain the reduction formula for $\int \tan^n x\,dx$, n being positive integer greater than 1. Hence find the value of

$\int_0^{\pi/4} \tan^6 x\,dx$. 4+4

B. (a) Answer any *two* questions : 2×4

(i) Evaluate

$$\lim_{n \to \infty} \left[\frac{1^2}{n^3 + 1^3} + \frac{2^2}{n^3 + 2^3} + \cdots + \frac{n^2}{n^3 + n^3} \right].$$ 4

(ii) Evaluate $\int_0^{\pi/2} \frac{dx}{5 + 3\cos x}$. 4

(iii) Show that $\int_0^2 |1 - x| \, dx = 1$. 4

(b) (i) Apply Beta and Gamma functions show that

$$\int_0^1 x^{3/2}(1-x)^{3/2} = \frac{3\pi}{128}$$ by putting $x = \sin^2\theta$. 4

(ii) Show that $\int_0^\infty e^{-x^4} x^2 dx \times \int_0^\infty e^{-y^4} dy = \frac{\pi}{8\sqrt{2}}$. 4

6. Answer any *one* question : 1×9

(a) (i) Find the area in the 1st quadrant included between the parabola $y^2 = bx$ and the circle $x^2 + y^2 = 2bx$. 4

(ii) Fine the volume of the solid generated by revolving one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about its base. 5

(b) (i) Find the surface of the sphere generated by the circle $x^2 + y^2 = a^2$ about the x-axis. 4

(ii) Show that length of the arc of the parabola $y^2 = 16x$ measured from the vertex to an extrimity of the latus rectum is $4\left\{\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right\}$. 5

7. Answer any *one* question : 1×5

(i) Prove that $\iiint \left(x^2 + y^2 + z^2\right) dxdydz = \frac{4\pi}{5}$ when the integration is taken throughout the region $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1\}$. 5

(ii) Evaluate

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_a^b \left(y^2 + z^2\right) dzdydx .$$ 4

Group—C

(*Differential Equation*)

[Marks : 15]

8. Answer any *two* questions : 2×6

(a) (i) Show that a constant a can be found so that $(x + y)^a$ is an integrating factor of $(4x^2 + 2xy + 6y)dx + (2x^2 + 9y + 3x)dy = 0$ and hence solve the equation. 3

(ii) Solve : $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$. 3

(b) (i) Show that if $\frac{d^2\theta}{dt^2} = g\theta = 0$ and if $\theta = \alpha$ and $\frac{d\theta}{dt} = 0$

when t = 0 then $\theta = \alpha \cos\sqrt{\frac{g}{l}}\, t$, g, l being positive.

3

(ii) Solve : $3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 5\cos x$. 3

(c) (i) Solve :

$$\frac{d^3y}{dx^3} - \frac{4}{x}\frac{d^2y}{dx^2} + \frac{5}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 1.$$ 3

(ii) Find the eigen values and eigen functions for the differential equation $\frac{d^2y}{dx^2} + \lambda y = 0$ $(\lambda > 0)$ satisfying the boundary conditions $y(0) = 0$ and $y(\pi/2) = 0$.

3

9. Answer any *one* question : 1×3

(i) Find the equation to the cone whose slope at any point is equal to y + 2x and which passes through the origin.

3

(b) Find the integrating factor of $xdy - ydx = \cos\left(\frac{1}{x}\right) dx$.

3
