OLD

2017

Part-II 3-Tier

MATHEMATICS

(General)

PAPER—II

Full Marks : 90

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group—A

(Differential Calculus)

[Marks : 45]

1. Answer any *one* question : 1×15

(a) (i) Prove that $\sqrt{2}+\sqrt{3}$ is an irrational number. 4

(ii) If $x_n = (-1)^n$ and $y_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$. Prove that the sequence $\{y_n\}$ converges although the sequence $\{x_n\}$ is not convergent. 4

(Turn Over)

(iii) *Evaluate* : $\lim\limits_{n\to\infty} \left\{\frac{1^3}{n^4}+\frac{2^3}{n^4}+\cdots+\frac{n^3}{n^4}\right\}$ 3

(iv) Let $f(x) = x\sin\left(\frac{1}{x}\right), \quad x \neq 0$

$= 0, \qquad x = 0$

Show that $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$. 2+2

(b) (i) Show that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. 3

(ii) Examine the convergence or divergence of the series :

$$x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+ \cdots\cdots$$ 4

(iii) Show that $f(x) = |x| + |x-2|$ is continuous but not derivable at $x = 0$ and at $x = 2$. 5

(iv) Show that

$$\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots+\frac{1}{(2n)^2}\right]=0$$ 3

2. Answer any *one* question : 1×8

(a) (i) If $u = \phi(H_n)$ where H_n is a homogeneous function of degree n in x, y then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)} \text{ where } F(u) = H_n.$$ 4

(ii) If $y = \cos(10\cos^{-1}x)$ show that
$$(1 - x^2)\, y_{12} = 21xy_{11}.$$ 4

(b) (i) State Rolle's theorem. Prove by example that the conditions of Rolle's theorem are a set of sufficient conditions but not necessary. 2+2

(ii) Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$. 4

3. Answer any *four* questions : 4×4

(a) Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their point of contact lie on
$$x^2y^2 = x^2 - y^2.$$ 4

(b) Find the envelope of the straight line $y = mx + \frac{a}{m}$, m being variable parameter $(m \neq 0)$.

(c) Show that maximum value of $x + \frac{1}{x}$ is less than its minimum value.

(d) Prove that the asymptotes of a cubic $(x^2 - y^2)y - 2ay^2 + 5x - 7 = 0$ form a triangle of area a^2 sq. unit. 4

(e) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3.$$ 4

(f) Find the radius of curvature and centre of curvature of the curve

$$x^{2/3} + y^{2/3} = a^{2/3}.$$ 4

4. Answer any *three* questions : 3×2

(a) Examine whether Lagrange's Mean Value Theorem can be applied to be function $f(x) = |x|$ in the interval $[-1, 1]$ or not. 2

(b) Show that the function $u(x, y) = x^3 + y^3 + 3x^2y + 3xy^2$ is a homogeneous function. Hence verify Euler's theorem for this function. 1+1

(c) Prove that the function $f(x, y) = |x| + |y|$ is continuous at $(0, 0)$. 2

(d) State Taylor's Theorem with Canchy's form of Remainder. 2

(e) State Schwarz's theorem on commutativity of mixed partial derivative. 2

Group—B

(*Integral Calculus*)

[Marks : 27]

5. Answer any *one* question : 1×15

A. (a) Evaluate any *two* : 2×4

(i) $\int e^x \frac{2-\sin 2x}{1-\cos 2x}dx$ 4

(ii) $\int \frac{\sqrt{x}}{x-1}dx$ 4

(iii) $\int_0^{2\pi} \frac{dx}{5+3\cos x} = \frac{\pi}{2}$ 4

(b) If $I_n = \int_0^{\pi/2} \sin^n x dx$ show that $I_n = \frac{n-1}{n} I_{n-2}$, where n is a positive integer > 1.

Hence deduce $I_n = \frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$,

n is even positive integer. 4+3

B. (a) Answer any *two* questions : 2×4

(i) Evaluate

$$\lim_{n\to\infty}\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2\cdots\left(1+\frac{n^2}{n^2}\right)^n\right\}^{\frac{1}{n}}.$$

(ii) Show that $\int_0^1 \frac{\log(1+x)dx}{1+x^2} = \frac{\pi}{8}\log 2$.

(iii) Express $\int_a^b e^{-x}dx$ as the limit of a sum and hence evaluate it.

(b) (i) Define Beta function. Show that $B(m, n) = 2$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta,$$

$\forall\ m, n > 0$ where $B(m, n)$ is a Beta function. 1+2

(ii) Show that $\int_0^1 \frac{dx}{\left(1-x^6\right)^{1/6}} = \frac{\pi}{3}$. 4

6. Answer any *one* question : 1×8

(a) (i) Find the arc length of the curve $x = t^2$, $y = t^3$ between $(1, 1)$ and $(4, 8)$. 4

(ii) Compute the area of the region enclosed by the graphs of the equations $y = e^x$, $y = e^{-x}$ and $x = 2$. 4

(b) (i) Find the volume of the ring generated by revolving the circle $x^2 + (y - b)^2 = a^2$ about x-axis. 4

(ii) Find the area of one loop of the curve $\rho = a\cos 2\theta$. 4

7. Answer any *one* question : 1×4

(i) Show that $\iint\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy=\frac{\pi ab}{8}$ over the positive quadrant within the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. 4

(ii) Evaluate $\iiint z^2\, dxdydz$ extended over the hemisphere $z \geq 0$ and $x^2 + y^2 + z^2 \leq a^2$. 4

Group—C

(*Differential Equation*)

[Marks : 18]

8. Answer any *two* questions : 2×8

(a) (i) Solve : $\frac{dy}{dx}+\frac{1}{x}\sin 2y = x^3\cos^2 y$. 4

(ii) Solve : $y = p^2x + p,\ p = \frac{dy}{dx}$. 4

(b) (i) Obtain the general and singular solutions of

$$y = px + p - p^2 \text{ where } p = \frac{dy}{dx}.$$ 4

(ii) Solve : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 5\cos x$, given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$. 4

(c) (i) Find the eigen values and eigen functions of $\frac{d^2y}{dx^2} + \lambda y = 0$ $(\lambda > 0)$ where $y(0) = 0$ and $\frac{dy}{dx} = 0$ at $x = \pi$. 4

(ii) Solve the following simultaneous equations :

$$\frac{dx}{dt} + 4x + 3y = t$$

$$\frac{dy}{dt} + 2x + 5y = e^t$$ 4

9. Answer any *one* question : 1×2

(i) One kilogram of some radioactive element becomes 990 g.m. after 24 hours. Find its half life.

(b) Deduce the differential equation of a simple pendulum.