NEW

2017

Part-I 3-Tier

MATHEMATICS

(General)

PAPER—I

Full Marks : 90

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group—A

(*Classical Algebra*)

[Marks : 25]

1. Answer any *one* question : 15×1

(a) (i) If $\cosh^{-1}(x + iy) + \cosh^{-1}(x - iy) = \cosh^{-1}a$, where $a > 1$ is a constant, then show that (x, y) lies on an ellipse. 5

(Turn Over)

(ii) Find the general values and the principal value of $i^{\log(1+i)}$. 5

(iii) Solve, by Ferrari's method $x^4 + 12x = 5$. 5

(b) (i) Show that the roots of the equation

$$\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=\frac{1}{x},$$ where $a > b > c > 0$ are real.

5

(ii) Solve the equation $2x^3 - 21x^2 + 42x - 16 = 0$ the roots being in G.P. 5

(iii) If $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ then evaluate $(A^3 - A^2 - I)$,

where I is the 3×3 unit matrix. 5

2. Answer any *one* question : 8×1

(a) (i) Show that every skew-symmetric determinent of even order is a perfect square. 4

(ii) If α be a root of the cubic equation $x^3 - 3x + 1 = 0$, then show that the other roots are $(\alpha^2 - 2)$ and $(2 - \alpha - \alpha^2)$. 4

(b) (i) Find the special roots' of the equation $x^6 - 1 = 0$.

4

(ii) Solve by Cramer's rule

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 4y + z &= 7 \\ 3x + 2y + 9z &= 14. \end{aligned}$$ 4

3. Answer any *one* question : 2×1

(a) Show that the equation $x^3 - 3x^2 - 9x + 27 = 0$ has a multiple root. 2

(b) Prove that any square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix. 2

Group B

(Modern Algebra)

[Marks : 20]

4. Answer any *two* questions : 8×2

(a) (i) Show that the set $G = \{1, -1, i, -i\}$ of the fourth roots of unity is a group with multiplicative composition. 4

(ii) If A and B be two sets, then show that $(A \cup B)' = A' \cap B'$ where A' denotes the complement of A. 4

(b) (i) Prove that the set H of all real matrices

$\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1\right\}$ form a commutative group

with respect to matrix multiplication. 4

(ii) Show that the set of all integers form a ring with respect to usual addition and multiplication. 4

(c) (i) Is every subjective mapping $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ an injective mapping ? Justify ($\mathbb{Z}^+$ denotes the set of all positive integers). 3

(ii) Define sub-field.

Check whether the set

$Q(\sqrt{21}) = \{a + b\sqrt{21} \ ; \ a, \ b \in Q\}$ is a sub-field of the field of all rational numbers Q. 5

5. Answer any *one* question : 4×1

(a) If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, the verify that A satisfies its own characteristic equation. 4

(b) Let $f : A \to B$ and $g : B \to C$ be two mappings. If $gof : A \to C$ is bijective, prove that f is injective and g is subjective. 4

Group C

(Analytical Geometry)

[Marks : 30]

6. Answer any *one* question : 15×1

(a) (i) Prove that if one of the straight lines $ax^2 + 2hxy + by^2 = 0$ be perpendicular to one line of the pair of straight lines represented by $a'x^2 + 2h'xy + b'y^2 = 0$ then $(aa' - bb')^2 + 4(ah' + hb')(a'h + h'b) = 0.$ 7

(ii) Reduce the equation $x^2 + 4xy + y^2 - 2x + 2y + 6 = 0$ to its canonical form and determine the type of the conic represented by it. 8

(b) (i) Show that the equation of the tangent to the conic $\frac{l}{r} = 1 + e\cos\theta$, parallel to the tangent at $\theta = \alpha$, is given by

$$l(e^2 + 2e\cos\alpha + 1) = r(e^2 - 1)\{\cos(\theta - \alpha) + e\cos\theta\}.$$

7

(ii) Prove that the lines $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and $\frac{x-3}{4} = \frac{y-4}{4} = \frac{z-5}{6}$ are coplanar. Find also the equation of the plane in which they lie. 4

(iii) If PSP′ and QSQ′ be two perpendicular focal chords of a conic with focus S, then prove that

$\frac{1}{SP.SP'} + \frac{1}{SQ.SQ'}$ is constant. 4

7. Answer any *one* question : 8×1

(a) (i) Find the equation of the sphere which passes througl the points (1, 0, 0), (0, 1, 0), (0, 0, 1) and which touches the plane $2x + 2y - z = 15$. 5

(ii) Find the equation of the plane passing through the points (1, 1, 2) and (2, 4, 3) and perpendicular to the plane $x - 3y + 7z + 5 = 0$. 3

(b) (i) Find the image (projection) of the straight line

$\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-4}{2}$ in the plane $2x - y + z + 3 = 0$.

5

(ii) Find the equation of the right circular cone with vertex at the point (1, –2, –1), semi-vertical angle 60°

and axis $\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z+1}{5}$. 3

8. Answer any *one* question : 4×1

(a) Find the value of C for which the plane $x + y + z = C$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.
4

(b) Find the equation of the cone whose vertex is the origin and base is the circle $x = a$, $y^2 + z^2 = b^2$. 4

9. Answer any *one* question : 3×1

(a) The gradient of one of the straight lines of $ax^2 + 2hxy + by^2 = 0$ is twice that of the other. Show that $8h^2 = 9ab$. 3

(b) Find the points of intersection of the straight line $r\cos\theta = a$ and the circle $r = 2a\cos\theta$. 3

Group D

(*Vector Algebra*)

[Marks : 15]

10. Answer any *one* question : 8×1

(a) (i) Establish the necessary and sufficient condition for three distinct points with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ to be collinear. 6

(ii) Find the unit vector perpendicular to both $\vec{a} = 2i + 3j - k$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$. 2

(b) (i) If $\vec{a} \times \vec{b} = \vec{c}$, $\vec{b} \times \vec{c} = \vec{a}$ and $\vec{c} \times \vec{a} = \vec{b}$, then show that the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular. 4

(ii) If $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ be three vectors from the origin to the points A, B, C respectively, then show that the vector $\left(\vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} + \vec{\alpha}\times\vec{\beta}\right)$ is perpendicular to the plane ABC. 4

11. Answer any *one* question : 4×1

(a) Prove that the medians of a triangle are concurrent and find the point of concurrence. 4

(b) If $\vec{\alpha}\times\vec{\beta} + \vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} = 0$, then show that the vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ are coplaner. 4

12. Answer any *one* question : 3×1

(a) Find the vector equation of the plane through the point (2, 3, −1) and perpendicular to the vector $\left(3\hat{i}-4\hat{j}+7\hat{k}\right)$. 3

(b) Show that the shortest distance between the straight lines $\vec{r} = \vec{a}+t\vec{\alpha}$ and $\vec{r} = \vec{b}+t\vec{\beta}$ where $\vec{a} = 6i+2j+2k$, $\vec{b} = -4i-k$, $\vec{\alpha} = i-2j+2k$ and $\vec{\beta} = 3i-2j-2k$ is 9 units. 3

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