Journal of Physical Sciences, Vol. 23, 2018, 241-248
ISSN: 2350-0352 (print), www.vidyasagar.ac.in/publication/journal
Published on 24 December 2018

Application of Method of Separation of Variables to Find Wave Motion in Waveguides with Variable Boundaries and Medium

John Daniel

M.R.Complex, Kattumavadi Road, Aranthangi, Pudukottai (Dt.)
Tamil Nadu-614616, India
E-Mail: johndaniel.india@gmail.com

Received 10 May 2018; accepted 30 November 2018

ABSTRACT

Perturbation methods were used to study the sound wave propagation in several acoustic wave guides with weak surface undulations. These methods are highly complicated and applicable only in a very narrow domain of the wave guide. Moreover, such solutions create singularities in the solution. Singularities in solution of physical problems originate due to unrealistic assumptions. Therefore, a much simpler and widely applicable analytical method of solution was developed using separation of variables method and presented recently to such wave guide problems. In this article, application of that solution to find wave propagation in several periodic waveguide structures is enumerated.

Keywords: Method of separation of variables and wave motion in periodic structures.

1. Introduction

Nayfeh [6] had developed the method of multiple scales which is a perturbation method. This method of multiple scales is used to analyze the wave propagation in two-dimensional hard-walled ducts with sinusoidal walls in [7]. For traveling waves, resonance occurs whenever the wall wave number is equal to the difference of the wave numbers of any two duct acoustic modes. The results show that neither of these resonating modes could occur without strongly producing the other.

Nayfeh and Kandil [8] applied the method of multiple scales to analyze the wave propagation in cylindrical hard-walled ducts having weak undulations which need not be periodic. Results are presented for two and three interacting modes. In the case of modes traveling in the same direction in a uniform duct, two interacting, spinning or non spinning modes propagate un attenuated in an undulated duct. Moreover, neither of them can exist without strongly exciting the other. On the other hand, in the case of modes propagating in opposite directions, they may be cut off as a result of the interaction.

Nusayr [9] applied the method of multiple scales is to analyze the wave propagation in a rectangular hard-walled duct whose walls have weak periodic undulations. Interacting modes traveling in the same direction propagate un attenuated. The energy is continuously exchanged between the two modes. The modes that travel in opposite directions are attenuated and, there, may be cut off.

This cutoff will depend upon the geometry of the cross section as well as the phase-angle differences between the undulations of the opposite walls.

Hawwa [3] considers acoustic wave propagation in ducts with rigid walls having square-wave wall corrugations in the context of a perturbation formulation. Using the ratio of wall corrugation amplitude to the mean duct half width, a small parameter is defined and two levels of approximations are obtained. The first-order solution produces an analytical description of the pressure field inside the duct. The second-order solution yields an analytical estimate of the phase speed of waves transmitting through the duct. The effect of wall corrugation density on acoustic impedance and wave speeds is highlighted. The analysis reveals that waves propagating in a duct with square-wave wall corrugation are slower than waves propagating in a duct with sinusoidal wave corrugation having the same corrugation wavelength.

Anand and George [1] have studied the sound wave propagation in a shallow water waveguide with a sinusoidal surface waves by applying the method of multiple scales. V.Sundaravadivel [11] studied the sound wave propagation in a three dimensional Oceanic waveguide by applying perturbation method.

All the above mentioned authors used perturbation methods which are applicable only to wave guides with low frequency and smaller amplitude surface waves. Moreover, perturbation method of solution generates singularity in the solution and complicates the determination of solution at points close to the singularity. Therefore, later on John Daniel [4] developed a simpler analytical solution based on the method of separation of variables. In this article, application of that solution to find wave propagation in several periodic waveguide structures is enumerated. First the theory based on the method of separation of variables to solve the wave propagation problem in a ocean acoustic waveguide with a wavy surface is presented and then the application of the method to solve wave propagation problems in several periodic acoustic, radio and optical waveguides is explained.

2. Theory

Consider an oceanic wave guide with a wavy surface as shown in the following Figure-1. The surface wave is a single frequency wave propagating in x direction which can be expressed mathematically as

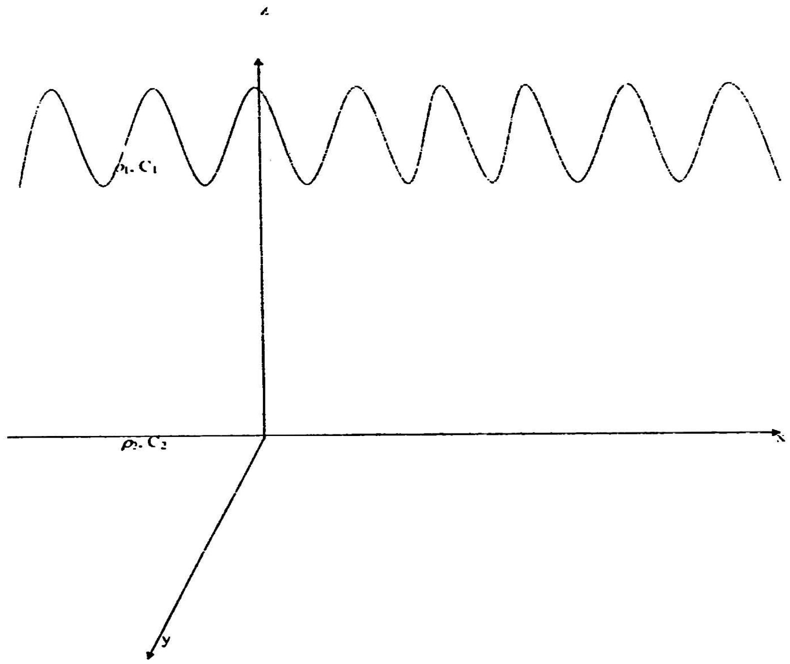
$$Z = h(x, t) = h_0 + a \operatorname{Cos}(\alpha x - \Omega t) \tag{1}$$

Where h_0 is average channel depth, α is wave number, Ω is circular frequency and a is amplitude of the surface wave. Let ρi, Ci ($i = 1,2$) denote respectively the density and velocity of sound in the two media. Medium 2 is assumed to be a semi-infinite one. The interface between medium 1 and medium 2 is assumed to be flat.

Let us assume that a plane sound wave propagates in the wave guide in the direction x. The acoustic pressure P (x, y, z, t) can be determined by solving the wave equation.

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}-\frac{1}{C^2}\ \frac{\partial^2 P}{\partial t^2}=0 \tag{2}$$

with boundary conditions



P (x, y, z, t) = 0 at z = h (x, t) (3a)
P(x , y, + 0, t) = P (x, y, -0, t) (3b)
where Vz is z component of particle velocity P → 0 a z → - ∞ (3c)

where +0 and -0 indicates that the interface z = 0 is approached from the sides z > 0 and z < 0 respectively.
Since there is no variation of pressure in y direction, equation (2) can be written as

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}-\frac{1}{C^2}\;\frac{\partial^2 P}{\partial t^2}=0 \tag{4}$$

Let us assume that Ω << ω where ω is angular frequency of sound wave. Therefore, equation (4) can be written as

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}+k^2 P \approx 0 \tag{5}$$

where

$$k=\frac{\omega}{c_1}=k_1 \;\; for \; 0<z<h \tag{6a}$$

$$=\frac{\omega}{c_2}=k_2 \;\; for \;\; z<0 \tag{6b}$$

Therefore, by separation of variables methods,

$$P(x, z) = \Psi_n(z).P_x(x) \tag{7a}$$

$$P_x(x) = \mathrm{Sin}(\zeta_n.x) \tag{7b}$$

The function $\Psi_n(z)$ must satisfy the equation

$$\frac{\partial^2 \Psi_n}{\partial z^2} + (K^2 - \xi_n^2\,(x,t))\,\Psi_n = 0 \tag{8}$$

with the boundary conditions

$$\psi_n = 0 \text{ at } z = h\,(x, t) \tag{9a}$$

$$\psi_n\,(+0, x, t) = \psi_n\,(-0, x, t) \tag{9b}$$

$$\frac{1}{\rho_1}\,\frac{\partial \psi_n}{\partial z}(+0, x, t) = \frac{1}{\rho_2}\,\frac{\partial \psi_n}{\partial z}(-0, x, t) \tag{9c}$$

$$\Psi_n \rightarrow 0 \text{ as } z \rightarrow -\infty \tag{9d}$$

The solution to the equation (8) is

$$\Psi_n = N_n \sin \chi_n\,(z/h - 1) \text{ for } 0 < z < h$$
$$= Cn\, e^{Dnz} \text{ for } z < 0 \tag{10}$$

where N_n, C_n, χ_n & D_n, are functions of x, t, Ψ_n must satisfy the orthonormality condition.

$$\int_{-\infty}^{h} \rho\,(z)^{-1} \psi_n\,\psi_m\,dz = \delta_{mn} \tag{11}$$

where $\rho(z) = \rho_1$ for $0 < z < h$

$= \rho_2$ for $0 < z < h$

and δ_{mn} is kronecker delta.

By substituting the equating (10) into equation (1) and (8) we get

$$\int_0^h N_n^2\,\rho_1^{-1}\sin^2\chi_n\left(\frac{z}{h} - 1\right)dz + \int_{-\infty}^{h} \rho_2^{-1}\,C_n^2\,e^{2Dnz}\,dz = 1 \tag{12}$$

$$\xi_n^2 = K_1^2 - \left[\frac{\chi_n}{h}\right]^2 \tag{13}$$

$$D_n = (\xi_n^2 - K_2^2)^{1/2} \tag{14}$$

Substitution of equation (10) into equations (9b) & (9c) gives,

$$-N_n \sin\chi_n = C_n \tag{15}$$

$$\frac{1}{\rho_1} N_n \left(\frac{\chi_n}{h}\right) \mathrm{Cos}\chi_n = C_n \frac{D_2}{\rho_2} \tag{16}$$

Equations (13) to (16) can be combined to get

$$\mathrm{Cot}\,\chi_n = -\,(q\,\chi_n)^{-1} (h^2 (K_1^2 - K_2^2) - \chi_n^2)^{1/2}. \tag{17}$$

where $q = \rho_2 / \rho_1$ From equation (12) we get

$$N_n = \sqrt{\frac{2}{h\rho_1^{-1}(1-\frac{\sin 2\chi_n}{2\chi_n}) + \frac{\rho_1^{-1}\sin^2\chi_n}{D_n}}} \quad (18)$$

χ_n can be found by solving the equation (17).

The solution as in the equation (10) can be expanded using Taylor's series as in the following line.

$\Psi_n = N_n \sin \chi_n (z/h - 1) = N_n.\sin\chi_n(z/h_0.(1- (a/h_0). \mathrm{Cos}(\alpha x - \Omega t))^{-1})$ (19)

Since a/h_0 is less than 1, as per Taylor's series, $F(y) = 1/ (1- y) = \Sigma y^m$ where m is an integer varies from 0 to ∞ and $y = (a/h_0). \mathrm{Cos}(\alpha x - \Omega t)$. Therefore, equation (19) could be written as,

$\Psi_n = N_n.\sin\chi_n(z/h_0. \Sigma ((a/h_0). \mathrm{Cos}(\alpha x - \Omega t))^m)$ (20)

Similarly, $\zeta_n = (K_1^2 - (\chi_n/h)^2)^{1/2}$ and $D_n = (\zeta_n^2 - K_2^2)^{1/2} = (K_1^2 - K_2^2 - (\chi_n/h)^2)^{1/2}$ (21) could be expanded using Taylor's series and trigonometric expressions. If the summation series in the equation (20) is expanded using trigonometric equations, it could equated to a Fourier series. Similarly, ζ_n and D_n could be expressed in Fourier series form. F(y), ζ_n and D_n are periodic functions of φ, where $\varphi = (\alpha x - \Omega t)$. Therefore, F(y), ζ_n and D_n could be expressed in Fourier series form and the frequency spectrum of the acoustic signal could be derived.

Thus the complete analytical solution to the problem finding frequency spectrum of sound wave in the waveguide is obtained.

$C_n = 0$, $\Psi_n = 0$ at $z = 0$ (22)

Therefore, $\sin \chi_n = n\pi$, where n is a positive integer (23)

The solution is $P (x, z, t) = \sin \chi_n (z/h - 1). \mathrm{Sin}(\zeta_n.x - \omega t)$ (24)

Therefore, the acoustic signal is phase modulated in the direction of z and x by the surface wave. In addition there will be an amplitude modulation of normal mode acoustic signal by the surface wave.

If the condition $\Omega << \omega$ is not satisfied, then the solution can be obtained by Fourier transforming $P_t(t)$, if $P(x, z, t) = P_x(x).P_z(z).P_t(t)$ is assumed as the solution. Then by inverse Fourier transforming $P_\omega(\omega)$ where $P_\omega(\omega)$ is Fourier transformation of $P_t(t)$. Therefore, $P_t(t)$ will be directly proportional to $\int 1/\omega^2.e^{j\omega t}\, d\omega$, ω varies from 0 to 2π. Acoustic signal is phase modulated in t dimension also by the surface waves.

3. Application

3.1. Application-I

Nayfeh [6 7] developed perturbation method to determine acoustic field inside a two dimensional acoustic waveguide with weakly undulating hard walls. This method is valid only for a waveguide with smaller waveguide undulations and the direct perturbation expansion results in solution which contains singularities at certain frequencies. At these resonant frequencies, method of multiple scales is used to find solutions at frequencies close to singularities. Such solutions are highly inaccurate at frequencies close to the singularities. As per the theory presented in this paper, solution to such wave motion problems is

$P(x, z, t) = \sin((z/h_1-1).\gamma_z). \sin(\gamma_x x - \omega t)$, where $\gamma_z = n\pi./((h_2/h_1) - 1)$,
$\gamma_x = ((\gamma_z)^2 - (\omega/c)^2)^{1/2}$ (25)

and boundaries are assumed to be at $z = h_1(x) = a.\cos(\alpha x)$ and $z = h_2(x) = h_0+b.\cos(\beta x+\theta)$, where a, b, θ, α, β and h_0 are constants (26)

Nayfeh assumed that $h_1(x) = a.\cos(\alpha x)$ and $h_2(x) = a.\cos(\alpha x+\theta)$ (27)

Also, h_1 and h_2 need not be periodic functions of x. Therefore, solution presented here is much more generalized solution.

3.2. Application-II

Nayfeh and Kandil [8] developed perturbation method to find the acoustic field inside a circular cylindrical waveguide with weakly undulating waveguide walls. As per theory presented in this paper the generalized solution to such wave propagation problems is

$P(r, \varphi, z, t) = J_n(r.h).(A_n\cos(n.\varphi) + B_n.\sin(n.\varphi)).\ \sin(\gamma_z.z - \omega t)$ where $J_n(r_0.h) = 0$, $r_0(x) = a + b.\Sigma.\alpha_m.\sin(k_m x)$, $\gamma_z = (k^2 - h^2)^{1/2}$, $k = \omega/c$, m is an integer, α_m, k_m, a and b are constants and $r_0(x)$ need not be periodic. $J_n(r.h)$ is a Bessel's function of first kind of order n.

3.3. Application-III

Nusayr [9] applied perturbation methods developed by A. H. Nayfeh to solve wave propagation problem in a rectangular waveguide with weak hard wall undulations. To this wave propagation problem generalized solution is

$P(x, y, z, t) = \sin((z/h_1-1).\gamma_z).\sin((y/h_3-1).\gamma_y).(\ \sin(\gamma_x x - \omega t)$, where $\gamma_z = n\pi./((h_2/h_1) - 1)$, $\gamma_y = m\pi/((h_4/h_3) -1)$, $\gamma_x = ((\gamma_z)^2 +(\gamma_y)^2- (\omega/c)^2)^{1/2}$ and boundaries are assumed to be at $z = h_1(x) = a.\cos(\alpha x)$ and $z = h_2(x) = h_0+b.\cos(\beta x+\theta)$, $y = h_3(x) = c.\cos(\alpha_1.x)$ and $y = h_4(x) = w_0+d.\cos(\beta_1.x+\theta_1)$, where a, b, c, d, θ, θ_1, α, α_1, β, β_1, h_0 and w_0 are constants Waveguide dimensions h_1, h_2, h_3, and h_4 need not be periodic functions of x. Therefore, solution presented here is much more generalized solution.

3.4. Application-IV

Hawwa [3] has applied perturbation methods developed by Nayfeh, et al to study the wave propagation with a periodic square wall profile. Since the method developed in this article is applicable to waveguide with any type of boundary variations, solutions developed in sections applications-I, II, III are valid to two dimensional, rectangular and circular cylindrical waveguides with any type of boundary variations.

3.5. Application-V[1 4 11]

In the theory presented in the first section, an underwater acoustic waveguide with a travelling sinusoidal surface on the top boundary of the waveguide and a semi infinite medium from the bottom flat boundary were assumed. However, in practice bottom floor of the ocean can't be flat. Therefore, assumption of variable boundary at the oceanic floor will be closer to natural situation. Since the method developed in this article is applicable to waveguides with any type of boundary variations, the generalized solution is applicable to ocean acoustic waveguides with any type of surface and ocean floor variations.

3.6. Application-VI [10 2]

One dimensional Photonic Band Gap (P.B.G.) materials with periodic defects and bound by metal planes on both sides are analyzed for their filtering behavior to develop nano scale electronic filters. Since dielectric medium varies periodically in the direction of propagation of electromagnetic wave, the velocity of electromagnetic signal could be described in the Fourier series form and the method of separation of variables could be applied to determine the solution to the wave propagation problem.

The solution is $E(x, z, t) = \sin(\gamma_z.z).\ \sin(\gamma_x x - \omega t)$, where $\gamma_z = n\pi/h_0$, $\gamma_x = ((n\pi/h_0)^2 - k^2)^{1/2}$, $k = \omega/c(x)$ and $c(x) = 1/(\varepsilon(x).\mu)^{1/2}$ (30)

This solution is applicable to any type of dielectric medium $\varepsilon(x)$ variations. The solution clearly indicates the dependency of filtering characteristics on dielectric medium variations. Therefore, by proper choice of $c(x)$ all types of filters could be constructed.

3.7. Application-VII [10 2 5]

One and two dimensional PBG materials with air-dielectric interface and periodic defects are analyzed for their filtering behavior. The solution for TM wave propagation in one dimensional PBG material bound by air-dielectric interface is

$H(x, z, t) = \sin(\gamma_z.z).\ \sin(\gamma_x x - \omega t)$ or $\cos(\gamma_z.z).\sin(\gamma_x.x - \omega t)$ for $0 < z < h_0$ and $H(x, z, t) = H_0.e^{-\gamma z}$ for $z > h_0$ and $= H_0 e^{\gamma z}$ for $z < 0$ where h_0 is the height of the dielectric material of the 1-D PBG material and γ_x, γ_z, γ and H_0 are constants (31)

The unknown constants could be found by substituting the solution at the boundaries and in the wave equation. In the wave equation for inside the wave equation, $k = \omega/c(x)$ and $c(x) = 1/(\varepsilon(x).\mu)^{1/2}$. For outside the waveguide region (air/vacuum), $k = \omega/c$ where $c = 3x10^8$ m/s. This solution is applicable to any type of dielectric medium $\varepsilon(x)$ variations. The solution clearly indicates the dependency of filtering characteristics on dielectric medium variations. Therefore, by proper choice of $c(x)$ all types of filters could be constructed.

4. Conclusion

A much simpler analytical solution to the problem of finding wave propagation in several acoustic, radio and optical waveguides with periodic structures is explained which is applicable to periodic and non periodic structures of any dimensions and parameter values without any singularity in the solution. The method of solution could be extended easily to 2 and 3 dimensions also. The method could be extended to analyze the ionosphere radio wave propagation and also to analyze the light wave propagation in optical waveguides with surface irregularities.

Acknowledgement. The author is grateful to Prof. G. V. Anand, Indian Institute of Science, Bangalore, India and Prof. R. H. Macphie, University of Waterloo, Canada for introducing ocean acoustic waveguide problems and Waveguide discontinuity scattering problems, Indian Institute of Science, Bangalore, India and University of Waterloo, Canada for a research scholarship/assistantship during 1987-1990.

The decisions of the reviewers help me to further researches to a great extent.

REFERENCES

1. G.V.Anand and M.K.George, Normal mode sound propagation in an ocean with sinusoidal surface waves, JASA, 80 (1) (1986) 238 – 243.

2. Fabrizio Frezza, et al, Numerical Investigation on the Filtering behaviour of 2-D PBG with multiple periodic defects, IEEE transactions on Nano technology, .4(6) (2005) 730-738.
3. M.A.Hawwa, Sound propagation in a duct with wall corrugations having square wall profiles, Hindawii publishing corporation, Mathematical problems in Engineering, (2015)
4. J.Daniel, Frequency Spectrum of normal mode sound propagation in an ocean with a wavy surface, Submitted to Applied Acoustics, Elsevier publishers, Netherlands
5. E.C.Jordan and K.G.Balmain, Electromagnetic Waves and Radiating Systems, Prentice Hall of India, New Delhi, (2007) 273-275.
6. A.H.Nayfeh, Perturbation Methods, Wiley Inter-science, (1973).
7. A.H.Nayfeh, Sound waves in two dimensional ducts with sinusoidal walls, J.A.S.A., 56(3) (1974) 768-770.
8. A.H.Nayfeh, O.A.Kandil,Propagation of waves in cylindrical hard walled ducts with generally weak undulations, AIAA Journal, 16(10) (1978) 1041-1045.
9. A.M.Nusayr, Propagation of waves in rectangular ducts with sinusoidal undulations, J.A.S.A., 67(5) (1980) 1472-1476
10. K.Srivatsava, Omnidirectional Reflection Band in 1-Dimensional Photonic Crystal Structures using Fullerene Films, Progress in Electromagnetic Research, 74 (2007) 181-194.
11. V.Sundaravadivel, Normal mode wave propagation in an ocean with a wavy surface, M.Sc (Engg.) thesis, E.C.E.Dept., I.I.Sc., Bangalore, India (1990)