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Solving Fuzzy Sequential Quadratic Programming Algorithm for Fuzzy Non-Linear Programming

A. Nagoor Gani[1] and R. Abdul Saleem[2]

[1]PG & Research Department of Mathematics
Jamal Mohamed College (Autonomous), Tiruchirappalli-20, Tamilnadu.
[2]Department of Mathematics, The Quaide Milleth College for Men
Medavakkam, Chennai -600 100, Tamilnadu.
E-mail:[1]ganijmc@yahoo.co.in; [2]rabdulrajak@gmail.com

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ABSTRACT
In this paper, a brief review of on one of the most powerful methods for solving smooth constrained fuzzy non-linear optimization problems via sequence of quadratic sub problems, so called fuzzy sequential quadratic programming problem and some numerical illustrations are given to show the efficiency of algorithms.

Keywords: Triangular Fuzzy Number, Fuzzy Non-linear programming problems, Fuzzy quadratic programming method, Fuzzy Sequential quadratic programming methods, Lagrange Multiplier Methods.

1. Introduction

For nonlinear programming problems, Sequential quadratic programming (SQP) algorithm is viewed at there as one of the most efficient methods. Since the 1970s, many researchers in China and abroad have done many researches into this type of algorithm, and got some appealing result [3, 4]. Through their work, SQP process has engaged an imperative position in solving controlled nonlinear optimization problems. However, the SQP algorithm obtainable so far has a grim limitation, (i.e.) this kind of method require that their quadratic programming sub problems have bounded solution at each iteration [5, 9]. Considering that the constraints of the sub problems are linear approximation of one of the unique problems, feasible sets of such sub problems may be unfilled. So, the exceeding claim is hard to be satisfied. How to conquer this complexity is a hot issue in the study of SQP method [3, 6].

Fuzzy set theory has been practically applied to many disciplines such as control theory and operational research, mathematical modelling and industrial applications [1]. The concept of fuzzy optimization in wide-ranging was first proposed by Tanaka et al, 1974 [13]. (Zimmerman, 1978) [12], Proposed the first formatting of fuzzy linear programming. A most advantageous solution of nonlinear fuzzy programming problems introduced by (Kumar and Kaur, 2010; Kheirfam, 2011) [2,8,10,11].

In this paper, the basic concepts of fuzzy sets and fuzzy non-linear programming problems are given in Section 2 and 3, and Section 4deals with the concepts of fuzzy sequential quadratic programming methodology. In section 5 proposed algorithms is given to solve FSQPP. Finally, in Section 6, the efficiency of the proposed method is explained by means of an example.

2. Preliminary definitions

Definition 2.1. A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu_A(x)) : x \in A, \mu_A(x) \in [0,1]\}$ [12]. In the pair $(x, \mu_A(x))$,the first element x belongs to the classical set A and the second element $\mu_A(x)$belongs to the interval [0, 1] called membership function [7].

Definition 2.2. A fuzzy set $\tilde{A}$ on *R* must possess at least the following three properties to qualify as a fuzzy number [7]:

(i) $\tilde{A}$ must be a normal fuzzy set;

(ii) $\alpha_{\tilde{A}}$must be a closed interval for every α ∈ [0, 1]; and

(iii) the support of $\tilde{A}$ must be bounded.

Definition 2.3. It is a fuzzy number represented with three points as follows: $\tilde{A} = (\alpha_1, \alpha_2, \alpha_3)$ [7].

This representation is interpreted as membership functions and holds the following conditions:

(i) α_1 to α_2 is an increasing function;

(ii) α_2 to α_3 is a decreasing function; and

(iii) $\alpha_1 \le \alpha_2 \le \alpha_3$.

$$\mu_A(x) = \begin{cases} 0 & for\ \ x < \alpha_1 \\ \dfrac{x - \alpha_1}{\alpha_2 - \alpha_1} & for\ \ \alpha_1 \le x \le \alpha_2 \\ \dfrac{\alpha_3 - x}{\alpha_3 - \alpha_2} & for\ \ \alpha_2 \le x \le \alpha_3 \\ 0 & for\ \ x > \alpha_3 \end{cases}$$

Definition 2.4. Let F(R) be a set of fuzzy numbers that defined on set of real numbers. A ranking function is a function Ra: F(R) -> R which maps each fuzzy number in to the real line.

2.5. Operation of triangular fuzzy number

Let $\tilde{A} = (\alpha_1, \alpha_2, \alpha_3)$ and $\tilde{B} = (\beta_1, \beta_2, \beta_3)$ Then

(i) $\tilde{A} \oplus \tilde{B} = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3)$.

(ii) $\tilde{A} - \tilde{B} = (\alpha_1 - \beta_3, \alpha_2 - \beta_2, \alpha_3 - \beta_1)$

(iii) $$\tilde{A} \otimes \tilde{B} = \begin{cases} \alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3 & \alpha_1 \ge 0 \\ \alpha_1\beta_3, \alpha_2\beta_2, \alpha_3\beta_3 & \alpha_1 < 0, \alpha_3 \ge 0 \\ \alpha_1\beta_3, \alpha_2\beta_2, \alpha_3\beta_3 & \alpha_3 < 0 \end{cases}$$

(iv) $-\alpha = (-\alpha_3, -\alpha_2, -\alpha_1)$

Definitions 2.6.

Let $\tilde{A}=(\alpha_1,\alpha_2,\alpha_3)$ and $\tilde{B}=(\beta_1,\beta_2,\beta_3)$ be two triangular fuzzy numbers then

(i) $\tilde{\alpha}\leq\tilde{\beta}$ *iff* $\alpha_1\leq\beta_1, \alpha_2-\alpha_1\leq\beta_2-\beta_1, \alpha_3-\alpha_2\leq\beta_3-\beta_2$

(ii) $\tilde{\alpha}\geq\tilde{\beta}$ *iff* $\alpha_1\geq\beta_1, \alpha_2-\alpha_1\geq\beta_2-\beta_1, \alpha_3-\alpha_2\geq\beta_3-\beta_2$

(iii) $\tilde{\alpha}=\tilde{\beta}$ *iff* $\alpha_1=\beta_1, \alpha_2=\beta_2, \alpha_3=\beta_3$

3. Fuzzy non- linear programming problem

$$Maximize\ (or\ Minimize)\ f(\tilde{x}) = \sum_{j=1}^{n} C_j \tilde{x}^n$$

$$Subject\ to$$

$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j^{\,n} \leq (or\ \geq)\tilde{b}_i,\ i-1,2,3.....m\ .... \qquad (3.1)$$

$$\tilde{x}_j \geq 0,\ j=1,2,3.....n$$

where $\tilde{C}_j(j=1,2,3....n)$, $\tilde{a}_{ij}$ *and* $\tilde{b}_i(i=1,2,3....m)$ are triangular fuzzy numbers and $\tilde{x}_j(j=1,2,3....n)$ are crisp variables. We use the ranking function on the problem (3.1) to get

$$Max\ (or\ Min)\ f(\tilde{x}) = \sum_{j=1}^{n} Ra(C_j\tilde{x}^n)$$

$$Subject\ to$$

$$\sum_{j=1}^{n} Ra(a_{ij}\tilde{x}_j^{\,n}) \leq (or\ \geq) Ra(\tilde{b}_i),\ i-1,2,3.....m.......... \qquad (3.2)$$

$$\tilde{x}_j \geq 0,\ j=1,2,3.....n$$

This is equal to:

$$Max\ (or\ Min)\ f(\tilde{x}) = \sum_{j=1}^{n} Ra(C_j)\tilde{x}^n$$

$$Subject\ to$$

$$\sum_{j=1}^{n} Ra(\tilde{a}_{ij})\tilde{x}_j^{\,n} \leq (or\ \geq) Ra(\tilde{b}_i),\ i-1,2,3.....m\ .... \qquad (3.3)$$

$$\tilde{x}_j \geq 0,\ j=1,2,3.....n$$

$$Ra(\tilde{C}_j) = c_j^{'}$$

$$Ra(\tilde{a}_{ij}) = a_{ij}^{'}$$

$$Ra(\tilde{b}_i) = b_i^{'}\quad (i-1,2,3.....m\quad and\quad j=1,2,3.....n\ )$$

Then we have

$$Max\ (or\ Min)\ \tilde{f}(\tilde{x}) = \sum_{j=1}^{n} c'_j x^n$$

$$Subject\ to$$

$$\sum_{j=1}^{n} a'_{ij}\tilde{x}_j^n \le (or\ \ge)\, b'_i,\ i-1,2,3.....m\ .... \qquad (3.4)$$

$$\tilde{x}_j \ge 0,\ j = 1,2,3.....n$$

The above relation we conclude that the optimal solutions of (3.1) and (3.4) are equivalent [7].

4. Fuzzy sequential quadratic programming methodology

The Fuzzy Sequential Quadratic Programming methodology to fuzzy non-linear optimization problem of the form

$$Minimum\ f(\tilde{x}) \qquad \tilde{x}\in R^n$$

$$Subject\ to$$

$$h(\tilde{x}) = 0$$

$$g(\tilde{x}) \le 0$$

$$\because\Rightarrow\ f(\tilde{x}): R^n \to R\ ; h(\tilde{x}) = R^n \to R^m; g(\tilde{x}) = R^n \to R^p$$

described the equality and inequality constraints.

Definition 4.1. The set of all points that satisfy the equality and inequality constraints $f = \{\tilde{x}\in R^n / h(\tilde{x}) = 0, g(\tilde{x}) \le 0\}$ is called the feasible set of fuzzy non-linear programming.

Definitions 4.2.

$$L: R^{n\times m\times p} \to R \quad defined\ by\ \ L(\tilde{x},\tilde{\lambda},\tilde{\mu}) = f(\tilde{x}) + \tilde{\lambda}^T h(\tilde{x}) + \tilde{\mu}^T g(\tilde{x})$$

Where $\tilde{\lambda}\in \tilde{R}^m, \tilde{\mu}\in \tilde{R}^p$ are referred to as Lagrangian multipliers.

$$f: R^n \to R, \nabla\tilde{f}(\tilde{x})\ \ is\ a\ gradient\ of\ f\ \ at\ \tilde{x}\in R^n$$

$$\nabla f(\tilde{x}) = (\frac{\partial f(\tilde{x})}{\partial\tilde{x}_1}, \frac{\partial f(\tilde{x})}{\partial\tilde{x}_2}, ...\frac{\partial f(\tilde{x})}{\partial\tilde{x}_n})$$

$$\nabla\ is\ the\ Jacobian\ of\ \tilde{h}\ according$$

$$\nabla h(\tilde{x}) = (\nabla h_1(\tilde{x}), \nabla h_2(\tilde{x}).......\nabla h_m(\tilde{x}))$$

Definition 4.3.
The fuzzy index set

$$I_{ac}(\tilde{x}) = \{i\in\{1,2,3......p\} / g_i(\tilde{x}) = 0\}$$

$$I_{in}(\tilde{x}) = \{i\in\{1,2,3......p\} / I_{ac}(\tilde{x}) = 0\},\ \tilde{x}\in R^n$$

Definition 4.4.

If $\tilde{x} \in R^n$ is a local minimum of the Fuzzy non-linear programming, the condition $q_{\tilde{x}} = \left| I_{ac}(\tilde{x}) \right|$ and assuming by $I_{ac}(\tilde{x}) = \{i_1, i_2........i_{q(\tilde{x})}\}$ will denote $G(\tilde{x}) \in \tilde{R}^{n\times(m+qx)}$ the matrix is given by

$G(\tilde{x}) = (\nabla h_1(\tilde{x}), \nabla h_2(\tilde{x})\ldots\ldots\nabla h_m(\tilde{x}), \nabla g_{i1}(\tilde{x}), \nabla g_{i2}(\tilde{x})\ldots\ldots\nabla g_{iq}(\tilde{x}))$

The functions $f(\tilde{x}), g(\tilde{x})$ *and* $h(\tilde{x})$ are three times continuously differentiable.

Definition 4.5.

$$\nabla f(\tilde{x}) + \sum_{i=1}^{m} \tilde{\lambda}_i \nabla h_i(\tilde{x}) = 0$$

$$h(\tilde{x}) = 0$$

$$\tilde{\lambda} \in R^m$$

$$\Rightarrow L(\tilde{x}, \tilde{\lambda}) = f(\tilde{x}) + \sum_{i=1}^{m} \tilde{\lambda}_i \nabla h_i(\tilde{x})$$

The Fuzzy KKT condition can be written as $\begin{pmatrix} \nabla_{\tilde{x}} L(\tilde{x}, \tilde{\lambda}) \\ \nabla_{\tilde{\lambda}} L(\tilde{x}, \tilde{\lambda}) \end{pmatrix} = 0$ with fuzzy Newton method unconstrained optimization. The fuzzy sequential quadratic programming is to model problem at a given point $\tilde{x}^k$ by a fuzzy quadratic sub problem and then use the solution of this problem to construct a more accurate approximation $\tilde{x}^{k+1}$.

If we perform a Taylor series expansions of the system (FEQKKT) about $(\tilde{x}^k, \tilde{\lambda}^k)$.

$$\Rightarrow \begin{pmatrix} \nabla_{\tilde{x}} L(\tilde{x}^k, \tilde{\lambda}^k) \\ \nabla_{\tilde{\lambda}} L(\tilde{x}^k, \tilde{\lambda}^k) \end{pmatrix} + \begin{pmatrix} \nabla^2{}_{\tilde{x}} L(\tilde{x}^k, \tilde{\lambda}^k) & \nabla^2 h(\tilde{x}^k) \\ \nabla h(\tilde{x}^k)^T & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{x}^k \\ \delta\tilde{\lambda}^k \end{pmatrix} = 0$$

where

$$\delta\tilde{x}^k = \tilde{x}^{k+1} - \tilde{x}^k, \delta\tilde{\lambda}^k = \tilde{\lambda}^{k+1} - \tilde{\lambda}^k$$

and

$$\nabla_x^2 L(\tilde{x}, \tilde{\lambda}) = \nabla^2 f(\tilde{x}) + \sum_{i=1}^{m} \lambda_i \nabla^2 h_i(\tilde{x})$$

Is the Hessian Matrix of the Fuzzy Lagrangian function. The Taylor series expansion can be written as

$$\begin{pmatrix} \nabla^2{}_{\tilde{x}} L(\tilde{x}^k, \tilde{\lambda}^k) & \nabla^2 h(\tilde{x}^k) \\ \nabla h(\tilde{x}^k)^T & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{x}^k \\ \delta\tilde{\lambda}^k \end{pmatrix} = \begin{pmatrix} -\nabla f(\tilde{x}^k) - \nabla h(\tilde{x}^k)\tilde{\lambda}^k \\ -h(\tilde{x}^k) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \nabla^2{}_{\tilde{x}} L(\tilde{x}^k, \tilde{\lambda}^k) & \nabla^2 \tilde{h}(\tilde{x}^k) \\ \nabla \tilde{h}(\tilde{x}^k)^T & 0 \end{pmatrix} \begin{pmatrix} \tilde{d} \\ \tilde{\lambda}^{k+1} \end{pmatrix} = \begin{pmatrix} -\nabla \tilde{f}(\tilde{x}^k) \\ -\tilde{h}(\tilde{x}^k) \end{pmatrix}$$

By setting

$$d = \delta\tilde{x}^k, \quad \tilde{\lambda}^{k+1} = \delta\tilde{\lambda} + \tilde{\lambda}^k$$

The Fuzzy sequential quadratic programming method for general fuzzy nonlinear problems.

$$Minimum\ f(\tilde{x})$$
$$Subject\ to$$
$$h(\tilde{x}) = 0$$
$$g(\tilde{x}) \le 0$$

The steps are the same as with the fuzzy Quadratic programming sub problems defined as

$$\underset{d}{Minimize}\, \nabla f(\tilde{x}^k)^T \tilde{d} + \frac{1}{2}\tilde{d}^T \nabla^2{}_{\tilde{x}} L(\tilde{x}^k, \tilde{\lambda}^k)\tilde{d}$$
$$Subject\ to$$
$$h(\tilde{x}^k) + \nabla h(\tilde{x}^k)^T \tilde{d} = 0$$
$$g(\tilde{x}^k) + \nabla g(\tilde{x}^k)^T \tilde{d} \ge 0$$

5. Proposed algorithm

Step 1: Determine ($\tilde{x}^0, \tilde{\lambda}^0$)
Step 2: Set $k = 0$
Step 3: Approximate the problem with the linear constrained fuzzy quadratic
Problem at $\tilde{x}^k$, to determine ($\tilde{d}^k, \tilde{\lambda}^{k+1}$)
Step 4: Set $\tilde{x}^{k+1} = \tilde{x}^k + \tilde{d}^k$
Step 5: Set k=k+1.
Step 6: Until the convergences.

6. Numerical example

The Fuzzy non-linear programming problem

$$Z \Rightarrow \underset{d}{Minimize}\ f(\tilde{x}) = [(0,1,2)\tilde{y} - (1,2,3)]^2 - (0,1,2)\tilde{x}^2$$
$$Subject\ to$$
$$h(\tilde{x}) = (2,3,4)\tilde{x}^2 + (0,1,2)\tilde{y}^2 = (0,1,2)$$
$$\tilde{x}, \tilde{y} \ge 0.$$

Solve, using the Fuzzy Sequential Quadratic Programming method.

Solution: Using the Ranking function, the equivalent crisp model of the above example is as follows

$$Z^{'} \Rightarrow \underset{d}{Minimize}\ f(\tilde{x}) = (\tilde{y} - 2)^2 - \tilde{x}^2$$
$$Subject\ to$$
$$h(\tilde{x}) = 4\tilde{x}^2 + \tilde{y}^2 = 1 \quad \tilde{x}, \tilde{y} \ge 0.$$

Starting Point $\tilde{x}^{(0)} = (2,4), \tilde{\lambda} = 0.5$.

$$\nabla f(\tilde{x}) = \begin{pmatrix} -2\tilde{x} \\ 2(\tilde{y}-2) \end{pmatrix}$$

$$\nabla h(\tilde{x}) = \begin{pmatrix} 8\tilde{x} \\ 2\tilde{y} \end{pmatrix}$$

$$\nabla^2 L(\tilde{x},\tilde{\lambda}) = \begin{pmatrix} -2-8\tilde{\lambda} & 0 \\ 0 & 2-2\tilde{\lambda} \end{pmatrix}$$

The Fuzzy quadratic programming sub problems at an iteration k can be written as

$$\underset{d}{Minimize}\, \nabla f(\tilde{x}^k)^T \tilde{d} + \frac{1}{2}\tilde{d}^T \nabla^2{}_{\tilde{x}} L(\tilde{x}^k,\tilde{\lambda}^k)\tilde{d}$$

Subject to

$$h(\tilde{x}^k) + \nabla h(\tilde{x}^k)^T \tilde{d} = 0$$

In algebraic form

$$Z' = \underset{d}{Minimize} \quad -2\tilde{x}^{(k)}\tilde{d}_1 + 2(\tilde{y}^{(k)}-2)\tilde{d}_2 - \frac{(2+8\tilde{\lambda}^{(k)})}{2}\tilde{d}_1^{\,2} + \frac{(2-2\tilde{\lambda}^{(k)})}{2}\tilde{d}_2^{\,2}$$

Subject to

$$8\tilde{x}^{(k)}\tilde{d}_1 + 2\tilde{y}^{(k)}\tilde{d}_2 + 4(\tilde{x}^{(k)})^2 + (\tilde{y}^{(k)})^2 = 1$$

Iteration 0: $x^{(0)} = (2,4), \lambda = 0.5$

$$Z' = \underset{d}{Minimize} \quad -2(2)\tilde{d}_1 + 2(4-2)\tilde{d}_2 - \frac{(2+8(0.5))}{2}\tilde{d}_1^{\,2} + \frac{(2-2(0.5))}{2}\tilde{d}_2^{\,2}$$

Subject to

$$8(2)\tilde{d}_1 + 2(4)\tilde{d}_2 + 4(2)^2 + (4)^2 = 1$$

Solving by KKT Condition

$\tilde{d}_1^{(0)} = -0.804, \tilde{d}_2^{(0)} = -2.267, \tilde{\lambda}^{(1)} = -0.350$

The new solution of $\tilde{x}^{(1)} = 1.196,\ \tilde{y}^{(1)} = 1.732$

The solution is not optimal, so continue until converges.
Concluded after 9 iterations, the given system is converging.
Therefore, the optimal solution is

$\tilde{x} = 0, \tilde{y} = 1, \tilde{\lambda} = -1$

By substituting this problem into the fuzzy form of the problem:

$\tilde{x} = (0,0,0) : \tilde{y} = (1,1,1) : \tilde{\lambda} = -(1,1,1)$

7. Conclusion

In this paper, we obtain the optimum solution of the fuzzy non-linear programming problems using fuzzy sequential quadratic programming method. First, we convert the problem into crisp model and then the crisp form was solved by the methods of fuzzy sequential quadratic programming. Illustrated numerical examples approve the effectiveness of the proposed method.

REFERENCES

1. A.Akrami, M.M.Hosseini and S.Medicarpins, Applications of genetic algorithm for solving of non-linear fuzzy programming problems with linear constraints. SAUSSUREA, 6(1) (2017)118-127.
2. R.E.Bellman and L.A.Zadeh, Decision making in a fuzzy environment,Management Science, 17 (1970) 141-160.
3. J.V.Burke and S.P.Han, A robust sequential quadratic programming method, Math. Prog.,43(2) (1989).
4. M.Fukushirna, A successive quadratic programming algorithm with global and super linear convergence properties, Math. Prog., (1986) 35: 253.
5. H.E.Guoping, G.A.O.Ziyou and L.A.Yanlian**,** New sequential quadratic programming algorithm with consistent sub problems, Science n China (Series A) 4(2)(1997).
6. B.Kheirfam, A method for solving fully fuzzy quadratic programming problems, Acta universities Apulensis, 27 (7) (2011) 69-76.
7. A.Kumar, J.Kaur and P.Singh, Fuzzy optimal solution of fully fuzzy linear programming problems with inequality constraints, International Journal of Applied Mathematics and Computer Sciences, 6 (3) (2010).37-40.
8. A.Nagoor Gani and W.Ritha, A parametric non-linear programming approach to fuzzy decision problem of a queuing system with finite capacity, Journal of Physical Sciences, (11) (2007) 17-28.
9. S.H.Nasseri, Fuzzy nonlinear optimization, J. Nonlinear Sci. Applications 1(4) (2008) 230-235.
10. S.H.Nasseri, Fuzzy nonlinear optimization, J. Nonlinear Sci. Applications 1(4) (2008) 230-235.
11. H.Tanaka, T.Okuda and K.Asai, On fuzzy mathematical programming, The Journal of Cybernetic, 3 (2) (1974) 37-46.
12. M.G.Voskoglou, An application of triangular fuzzy numbers to learning assessment, Journal of Physical Sciences, (20) (2015) 63-79.
13. H.J.Zimmerman, Fuzzy programming and linear programming with several objective functions, Fuzzy Sets and Systems, 1 (1) (1978) 45-55.