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Similarity Measure between Exponential Trapezoidal Fuzzy Numbers with Metric Distance Method

Salim Rezvani[1] and Majid Mousavi[2]

[1]Big Data Institute College of Computer Science and Software Engineering
Guangdong Key Laboratory of Intelligent Information Processing
Shenzhen, University, Shenzhen 518060, Guangdong, China
[2]Department of Chemistry, Azad University of Shahroud, Shahroud, Iran
E-mail: salim_rezvani@szu.edu.cn

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ABSTRACT

In this paper, we define the metric distance between two exponential trapezoidal fuzzy numbers and we use this metric for similarity measure. The basic properties of the above-mentioned similarity measure are proved in detail. Finally, we rank two generalized fuzzy numbers using distance measure and the similarity measure between them.

Keywords: Distance metric, Exponential fuzzy numbers, Similarity measure.

1. Introduction

The task of measuring similarity occurs in many disciplines. Because the objects to be compared are often represented by sets, this task is frequently performed by means of measures that compare sets. Such measures are usually called similarity measures. While similarity is an essential concept in human reasoning and plays a fundamental role in theories of knowledge, there is no unique and general purpose definition of similarity [14, 25-27] proposed properties of Similarity, separability. Indeed, several studies [13, 15] have shown that similarity measures do not necessarily have to be transitive, implying a contradiction with the most usual approach of comparison, based on geometrical assumptions in the feature space. In many applications, fuzzy sets are more suitable than crisp sets for representing the objects concerned. Consequently, there is a need for fuzzy similarity measure, i.e., measures that compare fuzzy sets [4]. The similarity measure between fuzzy sets has making, pattern recognition, machine learning, and market prediction, etc [11, 17]. The similarity measure between two fuzzy numbers is related to their commonality, in theories of the recognition, identification, and categorization of objects, where a common assumption is that the greater the commonality between a pair objects, more similar they are. Similarity and distance measure between two fuzzy numbers are closely related concepts. So it is possible to express similarity measure and distance measure between fuzzy numbers by a functional relationship. This is one of the oldest and most influential theoretical assumptions that similarity measure is inversely related to the distance measure. Thus the distance between fuzzy sets and as well as fuzzy numbers have gained more attention from researchers [5, 6, 8]. In most of researches, the authors pointed out to construct the distance measure between normal fuzzy numbers, for instance [1, 2, 19] and also in ranking fuzzy numbers [18-23].

Now in this paper, we define the metric distance between two exponential trapezoidal fuzzy numbers and we use this metric for similarity measure. The basic properties of the above-mentioned similarity measure are proved in detail. Finally, we rank two generalized fuzzy numbers using distance measure and similarity measure between them.

2. Preliminaries

Definition 1. A fuzzy number is a fuzzy subset in the universe of discourse *X* that is both convex and normal, and satisfying the following conditions:

(i) $\mu_{\tilde{A}}(X)$ is interval continue.

(ii) $\mu_{\tilde{A}}(X)$ is a normalized fuzzy set and $\mu_{\tilde{A}}(m)=1$, where m is a real number.

The fuzzy number $\tilde{A}$ is a fuzzy set and its membership function is $\mu_{\tilde{A}}(X):U \to [0,1]$.

Definition 2. Generally, a generalized fuzzy number *A* is described as any fuzzy subset of the real line *R*, whose membership function $\mu_{\tilde{A}}$ satisfies the following conditions,

(i) $\mu_{\tilde{A}}$ is a continuous mapping from R to the closed interval $[0,1]$.

(ii) $\mu_{\tilde{A}}(X)=0,\ -\infty < x \le a,.$

(iii) $\mu_{\tilde{A}}(X)=L(x),$ is strictly increasing on $[a,b]$,

(iv) $\mu_{\tilde{A}}(X)=w,\ b \le x \le c,$

(v) $\mu_{\tilde{A}}(X)=R(x),$ is strictly decreasing on $[c,d]$,

(vi) $\mu_{\tilde{A}}(X)=0,\ d \le x < \infty.$

where $0 < w \le 1$ and *a, b, c* and *d* are real numbers. We call this type of generalized fuzzy number a trapezoidal fuzzy number, and it is denoted by $A=(a,b,c,d;w)_{LR}$.

When *w = 1*, this type of generalized fuzzy number is called normal fuzzy number and is represented by $A=(a,b,c,d)_{LR}$.

However, these fuzzy numbers always have a fix range as *[c,d]*. Here, we define theirs general forms as follows:

$$f_A(x)=\begin{cases} we^{-[(b-x)/(b-a)]} & a \le x \le b \\ 1 & b \le x \le c \\ we^{-[(x-c)/(d-c)]} & c \le x \le d \end{cases} \qquad (1)$$

where $0 < w \le 1$, *a, b* are real numbers, and *c, d* are positive real numbers. We denote this type of generalized exponential fuzzy number as $A=(a,b,c,d;w)_E$. Especially, when w = 1, we denote it as $A=(a,b,c,d)_E$.

Let the generalized exponential fuzzy number $A=(a,b,c,d)_E$, where $0 < w \le 1$ and *c, d* are positive real numbers, *a, b* are real numbers as in formula *(1)*. Now, let two monotonic functions be

$$L(x) = we^{-[(b-x)/(b-a)]}, \quad R(x) = we^{-[(x-c)/(d-c)]}. \tag{2}$$

For convenience, the fuzzy number denoted by $[a,b,c,d;1]$, and the membership Function $f_{\tilde{A}}$ of the fuzzy number $\tilde{A} = [a,b,c,d;1]$ can be expressed as

$$f_A(x) = \begin{cases} f_{\tilde{A}}^L(x) & a \le x \le b \\ 1 & b = c = 1 \\ f_{\tilde{A}}^R(x) & a \le x \le b \end{cases} \tag{3}$$

where $f_{\tilde{A}}^L : [a,b] \to [0,1]$ and $f_{\tilde{A}}^R : [c,d] \to [0,1]$. Since $f_{\tilde{A}}^L : [a,b] \to [0,1]$ is continuous and strictly decreasing, the inverse function of $f_{\tilde{A}}^R$ also exists.

The inverse functions of $f_{\tilde{A}}^L$ and $f_{\tilde{A}}^R$ are denoted by $g_{\tilde{A}}^L$ and $g_{\tilde{A}}^R$, respectively. Since $f_{\tilde{A}}^L : [a,b] \to [0,1]$ is continuous and strictly increasing, $g_{\tilde{A}}^L : [0,1] \to [a,b]$ is also continuous and strictly increasing. Similarly, if $f_{\tilde{A}}^R : [c,d] \to [0,1]$ is continuous and strictly decreasing, then $g_{\tilde{A}}^R : [0,1] \to [c,d]$ is continuous and strictly decreasing, $g_{\tilde{A}}^L$ and $g_{\tilde{A}}^R$ are continuous on a closed interval *[0, 1]* and they are integrable on *[0, 1]*. That is, both $\int_0^1 g_{\tilde{A}}^L(y)dy$ and $\int_0^1 g_{\tilde{A}}^R(y)dy$ are exist. Then we can use $f_{\tilde{A}}^L : [a,b] \to [0,1]$ and $g_{\tilde{A}}^L : [0,1] \to [a,b]$ to derive the inverse functions of $f_{\tilde{A}}^L$ and $f_{\tilde{A}}^R$, which Are $g_{\tilde{A}}^L : [0,1] \to [a,b]$ and $g_{\tilde{A}}^R : [0,1] \to [c,d]$, respectively.

The inverse functions of $f_{\tilde{A}}^L$ and $f_{\tilde{A}}^R$ are

$$g_{\tilde{A}}^L = b + (b-a)\ln\frac{y}{w}, \quad g_{\tilde{A}}^R = c - (d-c)\ln\frac{y}{w}.$$

3. Fuzzy numbers by metric distance method

Definition 3. A metric space $\langle X, \rho \rangle$ in which X is a nonempty and ρ is a real number function, both X and ρ are on the $X \times X$, where x, y, and $z \in X$:

(i) $\rho(x,y) \ge 0$,

(ii) $\rho(x,y) = 0$ if and only if $x = y$,

(iii) $\rho(x,y) = \rho(y,x)$,

(iv) $\rho(x,y) \le \rho(x,z)$,

then function ρ is called a metric.

If there are two metric spaces $\langle X, \rho \rangle$ and $\langle Y, \omega \rangle$ form a new metric space called Cartesian product $X \times Y$, its point set is a set of $X \times Y = \{\langle x, y \rangle : x \in X, y \in Y\}$, and its metric τ is defined as follows:

$$\tau(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = [\rho(x_1, x_2)^2 + \omega(y_1, y_2)^2]^{1/2} \quad (4)$$

This metric can be adapted to fuzzy numbers, and it can be redefined as then

$$f_A(x) = \begin{cases} f_{\tilde{A}}^L(x) & for \quad x \le m_1 \\ f_{\tilde{A}}^R(x) & for \quad x \ge m_1 \end{cases} \quad (5)$$

$$f_B(x) = \begin{cases} f_B^L(x) & for \quad x \le m_2 \\ f_B^R(x) & for \quad x \ge m_2 \end{cases} \quad (6)$$

$$d(A,B) = [\int_0^1 (g_A^L - g_B^L)^2 dy + \int_0^1 (g_A^R - g_B^R)^2 dy]^{1/2} \quad (7)$$

is the distance of $A(x)$ and $B(x)$, and the inverse functions of $f_{\tilde{A}}^L$ and $f_{\tilde{A}}^R$ are $g_{\tilde{A}}^L$ and $g_{\tilde{A}}^R$.

Eq. (7) is the generalized ranking fuzzy number method, and our method can be used in negative and positive ranking of fuzzy numbers. For the convenience of ranking n positive fuzzy number A_1, A_2 and A_n, let $B(x) = 0$ in *Eq. (7)*, so *D* can be rewritten as

$$d(A_i, 0) = [\int_0^1 (g_A^L)^2 dy + \int_0^1 (g_A^R)^2 dy]^{1/2} \quad (8)$$

4. A approach for ranking exponential trapezoidal fuzzy numbers by distance method

This section is based on the concept of minimum metric *D* and *Eq. (7)* to produce the parameter of fuzzy mean $\mu(m = \mu)$ and fuzzy standard deviation $\sigma(\alpha = \beta = \sigma)$. A exponential trapezoidal fuzzy number $A = (a,b,c,d)$ and be approximated as a symmetry fuzzy number $S[\mu, \sigma], \mu$ denotes the mean of *A*, σ denotes the standard deviation of *A*, and the membership function of *A* is defined as follows:

$$f_B(x) = \begin{cases} \dfrac{x - (\mu - \sigma)}{\sigma} & if \quad \mu - \sigma \le x \le \mu \\ \dfrac{(\mu + \sigma) + x}{\sigma} & if \quad \mu \le \mu + \sigma \end{cases} \quad (9)$$

The inverse functions $g_{\tilde{A}}^L$ and $g_{\tilde{A}}^R$ of $f_{\tilde{A}}^L$ and $f_{\tilde{A}}^R$ respectively, are shown as follows:
We can get *A* symmetry function by immunizing this metric *D*, namely:

$$g_{\tilde{A}}^L = (\mu - \sigma) + \sigma\mu \quad (10)$$

$$g_{\tilde{A}}^R = (\mu + \sigma) - \sigma\mu \quad (11)$$

$$d(A_i,S[\mu,\sigma])=[\int_0^1(g_A^L-S[\mu,\sigma]^L)^2dy+\int_0^1(g_A^R-S[\mu,\sigma]^R)^2dy]^{1/2} \quad (12)$$

$S[\mu,\sigma]$ is a *LR* type, $S[\mu,\sigma]^L$ is the left membership function and $S[\mu,\sigma]^R$ is the right membership function.

Then, setting $\frac{\partial D}{\partial \mu}$ and $\frac{\partial D}{\partial \sigma}$ in *Eq. (12)*, we obtain

$$\begin{cases} \sigma=\dfrac{\int_0^1\left(g_A^R-g_A^L\right)(1-y)\,dy}{2\int_0^1(1-y)\,dy} \\ \\ \mu=\dfrac{1}{2}\int_0^1\left(g_A^R+g_A^L\right)dy \end{cases} \quad (13)$$

From *Eq. (13)* we can use the exponential trapezoidal fuzzy number $A=(a,b,c,d)_E$ to obtain an approximate parameter of the symmetry fuzzy number:

$$\mu=\frac{1}{2}\int_0^1\left(g_A^R(y)+g_A^L(y)\right)dy=\frac{1}{2}\int_0^1([c-(d-c)\ln\frac{y}{w}]+[b+(b-a)\ln\frac{y}{w}])dy$$

$$=\frac{1}{2}\int_0^1\left((c+b)+(b-a-d+c)\ln\frac{y}{w}\right)dy=\frac{1}{2}\left((c+b)y+(b-a-d+c)\left(y\ln\frac{y}{w}-y\right)\right)\Bigg|_0^1$$

$$=\frac{1}{2}\left[c+b+(b-a-d+c)\left(\ln\frac{1}{w}-1\right)\right]-\frac{1}{2}[0]\Rightarrow\mu=\frac{a+d}{2}+\left(\frac{b-a-d+c}{2}\right)\ln\frac{1}{w}. \quad (14)$$

Then μ and σ are calculated as follows:

$$\begin{cases} \sigma=\dfrac{9(d-a)+3(b-c)}{8}+\left(\dfrac{a-b-d+c}{2}\right)\ln\dfrac{1}{w} \\ \\ \mu=\dfrac{a+d}{2}+\left(\dfrac{a-b-d+c}{2}\right)\ln\dfrac{1}{w} \end{cases}$$

The inverse functions $g_{\tilde{A}}^L$ and $g_{\tilde{A}}^R$ of $f_{\tilde{A}}^L$ and $f_{\tilde{A}}^R$ respectively, are shown as follows:

$$g^{L}=(\mu-\sigma)+\sigma\mu$$
$$=[\frac{a+d}{2}+\left(\frac{b-a-d+c}{2}\right)\ln\frac{1}{w}-\frac{9(d-a)+3(b-c)}{8}-\left(\frac{a-b-d+c}{2}\right)\ln\frac{1}{w}]+$$
$$[\frac{9(d-a)+3(b-c)}{8}+\left(\frac{a-b-d+c}{2}\right)\ln\frac{1}{w}]y=\left[\frac{13a-3b+3c-5d}{8}+(b-a)\ln\frac{1}{w}\right]$$
$$+\left[\frac{9(d-a)+3(b-c)}{8}+\left(\frac{a-b-d+c}{2}\right)\ln\frac{1}{w}\right]y. \qquad (16)$$

and

$$g^{R}=(\mu+\sigma)-\sigma\mu$$
$$=[\frac{a+d}{2}+\left(\frac{b-a-d+c}{2}\right)\ln\frac{1}{w}+\frac{9(d-a)+3(b-c)}{8}-\left(\frac{a-b-d+c}{2}\right)\ln\frac{1}{w}]-$$
$$[\frac{9(d-a)+3(b-c)}{8}+\left(\frac{a-b-d+c}{2}\right)\ln\frac{1}{w}]y=\left[\frac{-5a+3b-3c+13d}{8}+(c-d)\ln\frac{1}{w}\right]$$
$$-\left[\frac{9(d-a)+3(b-c)}{8}+\left(\frac{a-b-d+c}{2}\right)\ln\frac{1}{w}\right]y. \qquad (17)$$

So

$$\begin{cases} g_{A}^{L}=\left[\frac{13a-3b+3c-5d}{8}+(b-a)\ln\frac{1}{w}\right]+\left[\frac{9(d-a)+3(b-c)}{8}+\left(\frac{a-b-d+c}{2}\right)\ln\frac{1}{w}\right]y, \\ g_{A}^{L}=\left[\frac{-5a+3b-3c+13d}{8}+(c-d)\ln\frac{1}{w}\right]-\left[\frac{9(d-a)+3(b-c)}{8}+\left(\frac{a-b-d+c}{2}\right)\ln\frac{1}{w}\right]y. \end{cases}$$

5. Distance measure for two exponential trapezoidal fuzzy numbers
Let *A*, *B* and *C* are three exponential trapezoidal fuzzy numbers denoted by $A=(a_1,b_1,c_1,d_1)_E$, $B=(a_2,b_2,c_2,d_2)_E$ and $C=(a_3,b_3,c_3,d_3)_E$.

Theorem 1. For A, B and C, we have
(i) $d(A,B)\geq 0$ and $d(A,A)=0$,
(ii) $d(A,B)=d(B,A)$,
(iii) $d(A,B)\leq d(A,C)+d(C,B)$.
Proof:
(i). is clearly.

(ii). $\Rightarrow$ Left side:

$$d(A,B)=[\int_0^1\left(g_A^L(y)-g_B^L(y)\right)^2dy+\int_0^1\left(g_A^R(y)-g_B^R(y)\right)^2dy]^{1/2}$$
$$=[\int_0^1((g_A^L(y))^2+(g_B^L(y))^2-2g_A^L(y)g_B^L(y))dy+\int_0^1((g_A^R(y))^2+(g_B^R(y))^2$$
$$-2g_A^R(y)g_B^R(y))dy]^{1/2}$$

Right side:

$$d(B,A)=[\int_0^1\left(g_B^L(y)-g_A^L(y)\right)^2dy+\int_0^1\left(g_B^R(y)-g_A^R(y)\right)^2dy]^{1/2}$$
$$=[\int_0^1((g_B^L(y))^2+(g_A^L(y))^2-2g_B^L(y)g_A^L(y))dy+\int_0^1((g_B^R(y))^2+(g_A^R(y))^2$$
$$-2g_B^R(y)g_A^R(y))dy]^{1/2}$$

So left side equal to right side, then $d(A,B)=d(B,A)$.

(iii).

$$d(A,B)=[\int_0^1\left(g_A^L(y)-g_B^L(y)\right)^2dy+\int_0^1\left(g_A^R(y)-g_B^R(y)\right)^2dy]^{1/2}$$
$$=[\int_0^1([\frac{13a_1-3b_1+3c_1-5d_1}{8}+(b_1-a_1)\ln\frac{1}{w}]+[\frac{9(d_1-a_1)+3(b_1-c_1)}{8}+(\frac{a_1-b_1-d_1+c_1}{2})\ln\frac{1}{w}]y$$
$$-[\frac{13a_2-3b_2+3c_2-5d_2}{8}+(b_2-a_2)\ln\frac{1}{w}]-[\frac{9(d_2-a_2)+3(b_2-c_2)}{8}+(\frac{a_2-b_2-d_2+c_2}{2})\ln\frac{1}{w}]y)^2dy$$
$$+\int_0^1([\frac{-5a_1+3b_1-3c_1+13d_1}{8}+(c_1-d_1)\ln\frac{1}{w}]-[\frac{9(d_1-a_1)+3(b_1-c_1)}{8}+(\frac{a_1-b_1-d_1+c_1}{2})\ln\frac{1}{w}]y$$
$$-[\frac{-5a_2+3b_2-3c_2+13d_2}{8}+(c_2-d_2)\ln\frac{1}{w}]+[\frac{9(d_2-a_2)+3(b_2-c_2)}{8}+(\frac{a_2-b_2-d_2+c_2}{2})\ln\frac{1}{w}]y^2dy]^{1/2}$$
$$=[\int_0^1([\frac{13a_1-3b_1+3c_1-5d_1}{8}+(b_1-a_1)\ln\frac{1}{w}]+[\frac{9(d_1-a_1)+3(b_1-c_1)}{8}+(\frac{a_1-b_1-d_1+c_1}{2})\ln\frac{1}{w}]y$$
$$-[\frac{13a_2-3b_2+3c_2-5d_2}{8}+(b_2-a_2)\ln\frac{1}{w}]-[\frac{9(d_2-a_2)+3(b_2-c_2)}{8}+(\frac{a_2-b_2-d_2+c_2}{2})\ln\frac{1}{w}]y)$$
$$\pm[\frac{13a_3-3b_3+3c_3-5d_3}{8}+(b_3-a_3)\ln\frac{1}{w}]\pm[\frac{9(d_3-a_3)+3(b_3-c_3)}{8}+(\frac{a_3-b_3-d_3+c_3}{2})\ln\frac{1}{w}]y)^2dy$$

$$+\int_0^1([\frac{-5a_1+3b_1-3c_1+13d_1}{8}+(c_1-d_1)\ln\frac{1}{w}]-[\frac{9(d_1-a_1)+3(b_1-c_1)}{8}+(\frac{a_1-b_1-d_1+c_1}{2})\ln\frac{1}{w}]y$$

$$-[\frac{-5a_2+3b_2-3c_2+13d_2}{8}+(c_2-d_2)\ln\frac{1}{w}]+[\frac{9(d_2-a_2)+3(b_2-c_2)}{8}+(\frac{a_2-b_2-d_2+c_2}{2})\ln\frac{1}{w}]y)$$

$$\pm[\frac{-5a_3+3b_3-3c_3+13d_3}{8}+(c_3-d_3)\ln\frac{1}{w}]\pm[\frac{9(d_3-a_3)+3(b_3-c_3)}{8}+(\frac{a_3-b_3-d_3+c_3}{2})\ln\frac{1}{w}]y)^2dy]^{1/2}$$

$$=[\int_0^1([\frac{13a_1-3b_1+3c_1-5d_1}{8}+(b_1-a_1)\ln\frac{1}{w}]+[\frac{9(d_1-a_1)+3(b_1-c_1)}{8}+(\frac{a_1-b_1-d_1+c_1}{2})\ln\frac{1}{w}]y$$

$$-[\frac{13a_3-3b_3+3c_3-5d_3}{8}+(b_3-a_3)\ln\frac{1}{w}]-[\frac{9(d_3-a_3)+3(b_3-c_3)}{8}+(\frac{a_3-b_3-d_3+c_3}{2})\ln\frac{1}{w}]y$$

$$+[\frac{13a_3-3b_3+3c_3-5d_3}{8}+(b_3-a_3)\ln\frac{1}{w}]+[\frac{9(d_3-a_3)+3(b_3-c_3)}{8}+(\frac{a_3-b_3-d_3+c_3}{2})\ln\frac{1}{w}]y$$

$$-[\frac{13a_2-3b_2+3c_2-5d_2}{8}+(b_2-a_2)\ln\frac{1}{w}]-[\frac{9(d_2-a_2)+3(b_2-c_2)}{8}+(\frac{a_2-b_2-d_2+c_2}{2})\ln\frac{1}{w}]y)^2dy$$

$$+\int_0^1([\frac{-5a_1+3b_1-3c_1+13d_1}{8}+(c_1-d_1)\ln\frac{1}{w}]-[\frac{9(d_1-a_1)+3(b_1-c_1)}{8}+(\frac{a_1-b_1-d_1+c_1}{2})\ln\frac{1}{w}]y$$

$$+\int_0^1([\frac{13a_3-3b_3+3c_3-5d_3}{8}+(b_3-a_3)\ln\frac{1}{w}]+[\frac{9(d_3-a_3)+3(b_3-c_3)}{8}+(\frac{a_3-b_3-d_3+c_3}{2})\ln\frac{1}{w}]y$$

$$-[\frac{13a_2-3b_2+3c_2-5d_2}{8}+(b_2-a_2)\ln\frac{1}{w}]-[\frac{9(d_2-a_2)+3(b_2-c_2)}{8}+(\frac{a_2-b_2-d_2+c_2}{2})\ln\frac{1}{w}]y)^2dy$$

$$+\int_0^1[\frac{-5a_3+3b_3-3c_3+13d_3}{8}+(c_3-d_3)\ln\frac{1}{w}]-[\frac{9(d_3-a_3)+3(b_3-c_3)}{8}+(\frac{a_3-b_3-d_3+c_3}{2})\ln\frac{1}{w}]y$$

$$-[\frac{-5a_2+3b_2-3c_2+13d_2}{8}+(c_2-d_2)\ln\frac{1}{w}]+[\frac{9(d_2-a_2)+3(b_2-c_2)}{8}+(\frac{a_2-b_2-d_2+c_2}{2})\ln\frac{1}{w}]y)^2dy]^{1/2}$$

$$\leq[\int_0^1\left(g_A^L(y)-g_C^L(y)\right)^2dy+\int_0^1\left(g_A^R(y)-g_C^R(y)\right)^2dy+\int_0^1\left(g_C^L(y)-g_B^L(y)\right)^2dy+\int_0^1\left(g_C^R(y)-g_B^R(y)\right)^2dy]^{1/2}$$

$$\leq d(A,C)+d(C,B).\ \square$$

$$-[\frac{-5a_3+3b_3-3c_3+13d_3}{8}+(c_3-d_3)\ln\frac{1}{w}]+[\frac{9(d_3-a_3)+3(b_3-c_3)}{8}+(\frac{a_3-b_3-d_3+c_3}{2})\ln\frac{1}{w}]y$$

$$+[\frac{-5a_3+3b_3-3c_3+13d_3}{8}+(c_3-d_3)\ln\frac{1}{w}]-[\frac{9(d_3-a_3)+3(b_3-c_3)}{8}+(\frac{a_3-b_3-d_3+c_3}{2})\ln\frac{1}{w}]y$$

$$-[\frac{-5a_2+3b_2-3c_2+13d_2}{8}+(c_2-d_2)\ln\frac{1}{w}]+[\frac{9(d_2-a_2)+3(b_2-c_2)}{8}+(\frac{a_2-b_2-d_2+c_2}{2})\ln\frac{1}{w}]y)^2dy]^{1/2}$$

$$\leq[\int_0^1([\frac{13a_1-3b_1+3c_1-5d_1}{8}+(b_1-a_1)\ln\frac{1}{w}]+[\frac{9(d_1-a_1)+3(b_1-c_1)}{8}+(\frac{a_1-b_1-d_1+c_1}{2})\ln\frac{1}{w}]y$$

$$-[\frac{13a_3-3b_3+3c_3-5d_3}{8}+(b_3-a_3)\ln\frac{1}{w}]-[\frac{9(d_3-a_3)+3(b_3-c_3)}{8}+(\frac{a_3-b_3-d_3+c_3}{2})\ln\frac{1}{w}]y)^2dy$$

$$+\int_0^1([\frac{-5a_1+3b_1-3c_1+13d_1}{8}+(c_1-d_1)\ln\frac{1}{w}]-[\frac{9(d_1-a_1)+3(b_1-c_1)}{8}+(\frac{a_1-b_1-d_1+c_1}{2})\ln\frac{1}{w}]y$$
$$-[\frac{-5a_3+3b_3-3c_3+13d_3}{8}+(c_3-d_3)\ln\frac{1}{w}]+[\frac{9(d_3-a_3)+3(b_3-c_3)}{8}+(\frac{a_3-b_3-d_3+c_3}{2})\ln\frac{1}{w}]y^2dy]^{1/2}$$

Example 1. Let $A=(0.1,\ 0.2,\ 0.3,\ 0.4;\ 0.3)$, $B=(0.2,\ 0.4,\ 0.5,\ 0.7;\ 0.5)$ and $C=(0.1,\ 0.3,\ 0.5,\ 0.7;\ 0.7)$ are three trapezoidal fuzzy numbers, so we have

(i) $d(A,B)\geq 0$ and $d(A,A)=0$,

(ii) $d(A,B)=d(B,A)$,

(iii) $d(A,B)\leq d(A,C)+d(C,B)$.

Solution:

$$g_{\tilde{A}}^{L}=[\frac{1.3-0.6+0.9-2}{8}+(0.1)(1.2)]+[\frac{2.7-0.3}{8}+\frac{-0.2}{2}(1.2)]y=0.07+0.18y,$$
$$g_{\tilde{B}}^{L}=[\frac{2.6-1.2+1.5-3.5}{8}+(0.2)(0.69)]+[\frac{4.5-0.3}{8}+\frac{-0.4}{2}(0.69)]y=0.06+0.03y,$$
$$g_{\tilde{C}}^{L}=[\frac{1.3-0.9+1.5-3.5}{8}+(0.2)(0.36)]+[\frac{5.4-0.6}{8}+\frac{-0.4}{2}(0.36)]y=-0.13+0.53y,$$

and

$$g_{\tilde{A}}^{R}=[\frac{-0.5+0.6-0.9+5.2}{8}+(-0.1)(1.2)]-[\frac{2.7-0.3}{8}+\frac{-0.2}{2}(1.2)]y=0.43-0.18y,$$
$$g_{\tilde{B}}^{R}=[\frac{-1+1.2-1.5+9.1}{8}+(-0.2)(0.69)]-[\frac{4.5-0.3}{8}+\frac{-0.4}{2}(0.69)]y=0.83-0.38y,$$
$$g_{\tilde{C}}^{R}=[\frac{-0.5+0.9-1.5+9.1}{8}+(-0.2)(0.36)]-[\frac{5.4-0.6}{8}+\frac{-0.4}{2}(0.36)]y=0.93-0.53y.$$

(i). is obvious.

(ii).

$$d(A,B)=[\int_0^1(0.07+0.18y-0.06-0.03y)^2\,dy+\int_0^1(0.43-0.18y-0.83+0.38y)^2\,dy]^{1/2}$$
$$=[\int_0^1(0.01+0.15y)^2\,dy+\int_0^1(-0.4+0.2y)^2\,dy]^{1/2}$$
$$=[\int_0^1(0.0001+0.0225y^2+0.003y)dy+\int_0^1(0.16+0.04y^2-0.16y)^2\,dy]^{1/2}$$

$$=[0.0001y+0.0225\frac{y^3}{3}+0.003\frac{y^2}{2}+0.16y+0.04\frac{y^3}{3}-0.16\frac{y^2}{2}]^{1/2}]_0^1$$

$$=[\frac{0.96+0.12-0.48}{6}]^{1/2}=[0.1]^{1/2}\Rightarrow d\left(A,B\right)=[0.1]^{1/2}.$$

Now

$$d\left(B,A\right)=[\int_0^1\left(0.06+0.03y-0.07-0.18y\right)^2dy+\int_0^1\left(0.83-0.38y-0.43+0.18y\right)^2dy]^{1/2}$$

$$=[\int_0^1\left(-0.01-0.15y\right)^2dy+\int_0^1\left(0.4-0.2y\right)^2dy]^{1/2}$$

$$=[\int_0^1\left(0.0001+0.0225y^2+0.003y\right)dy+\int_0^1\left(0.16+0.04y^2-0.16y\right)^2dy]^{1/2}$$

$$=[0.0001y+0.0225\frac{y^3}{3}+0.003\frac{y^2}{2}+0.16y+0.04\frac{y^3}{3}-0.16\frac{y^2}{2}]^{1/2}]_0^1$$

$$=[\frac{0.96+0.12-0.48}{6}]^{1/2}=[0.1]^{1/2}\Rightarrow d\left(B,A\right)=[0.1]^{1/2}.$$

Then $d\left(A,B\right)=d\left(B,A\right)$.

(iii). Of *(ii)* we know $d\left(A,B\right)=[0.1]^{1/2}$.

$$d\left(A,C\right)=[\int_0^1\left(0.07+0.18y+0.13-0.53y\right)^2dy+\int_0^1\left(0.43-0.18y-0.93+0.53\right)^2dy]^{1/2}$$

$$=[\int_0^1\left(0.2-0.35\right)^2dy+\int_0^1\left(-0.5+0.35\right)^2dy]^{1/2}$$

$$=[\int_0^1\left(0.04+0.12y^2-0.14y\right)dy+\int_0^1\left(0.25+0.12y^2-35y\right)^2dy]^{1/2}$$

$$=[0.04y+0.12\frac{y^3}{3}-0.14\frac{y^2}{2}+0.25y+0.12\frac{y^3}{3}-35\frac{y^2}{2}]^{1/2}]_0^1$$

$$=[0.12]^{1/2}\Rightarrow d\left(A,C\right)=[0.12]^{1/2}.$$

and

$$d\left(C,B\right)=[\int_0^1\left(-0.13+0.53y-0.06-0.03y\right)^2dy+\int_0^1\left(0.93-0.53y-0.83+0.38y\right)^2dy]^{1/2}$$

$$=[\int_0^1\left(-0.19+0.5y\right)^2dy+\int_0^1\left(0.1-0.15y\right)^2dy]^{1/2}$$

$$=[\int_0^1 (0.04+0.25y^2-0.19y)dy+\int_0^1 (0.01+0.0225y^2-0.03y)dy]^{1/2}$$

$$=[0.04y+0.25\frac{y^3}{3}-0.19\frac{y^2}{2}+0.01y+0.0225\frac{y^3}{3}-0.03\frac{y^2}{2}]_0^1$$

$$=[\frac{0.3+0.54-0.66}{6}]^{1/2}=[0.03]^{1/2}\Rightarrow d(C,B)=[0.03]^{1/2}.$$

Then

$$d(A,B)\le d(A,C)+d(C,B)\Rightarrow[0.1]^{1/2}\le[0.12]^{1/2}+[0.03]^{1/2}$$

$$\Rightarrow 0.32\le 0.35+0.17\Rightarrow 0.32\le 0.52\ .$$

Theorem 2. For three exponential trapezoidal fuzzy numbers *A, B* and *C,* we have

(i) $d(A+B,B+C)=d(A,C)$,

(ii) $d(A+B,0)\le d(A,0)+d(B,0)$,

(iii) $d(\lambda A,\lambda B)\le\lambda d(A,B),\ \lambda\in R.$

Proof:

(i).

$$d(A+B,B+C)=[\int_0^1 ([g_A^L(y)+g_B^L(y)]-[g_B^L(y)+g_C^L(y)])^2dy$$

$$+\int_0^1 ([g_A^R(y)+g_B^R(y)]-[g_B^R(y)+g_C^R(y)])^2dy]^{1/2}$$

$$=[\int_0^1 (g_A^L(y)+g_B^L(y)-g_B^L(y)-g_C^L(y))^2dy$$

$$+\int_0^1 (g_A^R(y)+g_B^R(y)-g_B^R(y)-g_C^R(y))^2dy]^{1/2}$$

$$=[\int_0^1 (g_A^L(y)-g_C^L(y))^2dy+\int_0^1 (g_A^R(y)-g_C^R(y))^2dy]^{1/2}=d(A,C).$$

(ii).

$$d(A+B,0)=[\int_0^1 ([g_A^L(y)+g_B^L(y)]-0)^2dy+\int_0^1 ([g_A^R(y)+g_B^R(y)]-0)^2dy]^{1/2}$$

$$\le[\int_0^1 (g_A^L(y)-0)^2dy+\int_0^1 (g_A^R(y)-0)^2dy]^{1/2}+\int_0^1 (g_B^L(y)-0)^2dy+\int_0^1 (g_B^R(y)-0)^2dy]^{1/2}$$

$$=d(A,0)+d(B,0)\Rightarrow d(A+B,0)\le d(A,0)+d(B,0).$$

(iii).

$$d(\lambda A,\lambda B)=[\int_0^1(\lambda g_A^L(y)-\lambda g_B^L(y))^2dy+\int_0^1(\lambda g_A^R(y)-\lambda g_B^R(y))^2dy]^{1/2}$$

$$=[\int_0^1\lambda^2(g_A^L(y)-g_B^L(y))^2dy+\int_0^1\lambda^2(g_A^R(y)-g_B^R(y))^2dy]^{1/2}$$

$$=[\lambda^2\int_0^1(g_A^L(y)-g_B^L(y))^2dy+\int_0^1(g_A^R(y)-g_B^R(y))^2dy]^{1/2}$$

$$=(\lambda^2)^{1/2}[\int_0^1(g_A^L(y)-g_B^L(y))^2dy+\int_0^1(g_A^R(y)-g_B^R(y))^2dy]^{1/2}$$

$$=\lambda d(A,B)\Rightarrow d(\lambda A,\lambda B)=\lambda d(A,B). \qquad \square$$

Example 2. Let $A=(0.1,\ 0.2,\ 0.3,\ 0.4;\ 0.3)$, $B=(0.2,\ 0.4,\ 0.5,\ 0.7;\ 0.5)$ and $C=(0.1,\ 0.3,\ 0.5,\ 0.7;\ 0.7)$ are three trapezoidal fuzzy numbers, so we have

(i) $d(A+B,B+C)=d(A,C)$,

(ii) $d(A+B,0)\le d(A,0)+d(B,0)$,

(iii) $d(\lambda A,\lambda B)\le\lambda d(A,B)$, $\lambda\in R$.

Solution:

$g_A^L=0.07+0.18y$, $g_B^L=0.06+0.03y$, $g_C^L=-0.13+0.53y$,

and

$g_A^R=0.43-0.18y$, $g_B^R=0.83-0.38y$, $g_C^R=0.93-0.53y$.

(i).

$$d(A+B,B+C)=[\int_0^1([0.07+0.18y+0.06+0.03]-[0.06+0.03y-0.13+0.53y])^2$$

$$+\int_0^1([0.43-0.18y+0.83-0.38y]-[0.83-0.38y+0.93-0.53y])^2dy]^{1/2}$$

$$=0.36,$$

and

$$d\left(A,C\right)=[\int_0^1([0.07+0.18y]-[-0.13+0.53y])^2+\int_0^1([0.43-0.18y]-[0.93-0.53y])^2dy]^{1/2}$$

$$=0.36,$$

so $d\left(A+B,B+C\right)=d\left(A,C\right)$.

(ii).

$$d\left(A+B,0\right)=[\int_0^1([0.07+0.18y+0.06+0.03y]-[0])^2+\int_0^1([0.43-0.18y+0.83-0.38y]-[0])^2dy]^{1/2}$$

$$=1.02,$$

and

$$d\left(A,0\right)=[\int_0^1([0.07+0.18y]-[0])^2+\int_0^1([0.43-0.18y]-[0])^2dy]^{1/2}=0.38,$$

and

$$d\left(B,0\right)=[\int_0^1([0.06+0.03y]-[0])^2+\int_0^1([0.83-0.38y]-[0])^2dy]^{1/2}=0.65,$$

So

$d\left(A+B,0\right)=1.02\le d\left(A,0\right)+d\left(B,0\right)=0.38+0.65=1.03$.

6. Ranking exponential trapezoidal fuzzy numbers with distance method

In this section, we can define the ranking exponential trapezoidal fuzzy numbers by distance method.

Theorem 3. If $A=(a_1,b_1,c_1,d_1)_E$ and $B=(a_2,b_2,c_2,d_2)_E$ are two exponential trapezoidal fuzzy numbers and *D* is a distance method of them, so

(i) If $D\left(A,0\right)\le D\left(B,0\right)$ then $A\le B$,

(ii) If $D\left(A,0\right)\ge D\left(B,0\right)$ then $A\ge B$,

(iii) If $D\left(A,0\right)=D\left(B,0\right)$ then $A=B$.

Example 3. Let $A=(0.2,0.4,0.6,0.8;0.35)$, $B=(0.1,\ 0.3,\ 0.3,\ 0.5;\ 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L=0.11+0.39y,\ g_A^R=0.89-0.39y$.

$$D(A,0)=\left[\int_0^1 (g_A^L)^2\,dy+\int_0^1 (g_A^R)^2\,dy\right]^{1/2}$$

$$=\left[\int_0^1 (0.11+0.39y)^2\,dy+\int_0^1 (0.89-0.39y)^2\,dy\right]=[0.6]^{1/2}=0.77,$$

and

$g_B^L=-0.01+0.26y,\ \ g_B^R=0.51-0.26y$.

$$D(B,0)=\left[\int_0^1 (g_B^L)^2\,dy+\int_0^1 (g_B^R)^2\,dy\right]^{1/2}$$

$$=\left[\int_0^1 (-0.01+0.26y)^2\,dy+\int_0^1 (0.51-0.26y)^2\,dy\right]=[0.17]^{1/2}=0.42,$$

Of Theorem 3. we have $D(A,0)=0.77>D(B,0)=0.42,$ then $A>B.$

Example 4. Let $A=(0.1,0.2,0.4,0.5;1),$ $B=(0.1,\ 0.3,\ 0.3,\ 0.5;\ 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L=-0.7+0.37y,\ \ g_A^R=0.67-0.37y$.

$$D(A,0)=\left[\int_0^1 (g_A^L)^2\,dy+\int_0^1 (g_A^R)^2\,dy\right]^{1/2}$$

$$=\left[\int_0^1 (-0.7+0.37y)^2\,dy+\int_0^1 (0.67-0.37y)^2\,dy\right]=[0.27]^{1/2}=0.52,$$

and

$g_B^L=-0.15+0.45y,\ \ g_B^R=0.75-0.45y$.

$$D(B,0)=\left[\int_0^1 (g_B^L)^2\,dy+\int_0^1 (g_B^R)^2\,dy\right]^{1/2}$$

$$=\left[\int_0^1 (-0.15+0.45y)^2\,dy+\int_0^1 (0.75-0.45y)^2\,dy\right]=[0.32]^{1/2}=0.56,$$

Of Theorem 3. we have $D(A,0)=0.52<D(B,0)=0.56,$ then $A<B.$

Example 5. Let $A=(0.1,0.2,0.4,0.5;1)$, $B=(1,1,1,1;\ 1)$ are two trapezoidal fuzzy numbers, then

$$g_A^L=-0.7+0.37y,\ \ g_A^R=0.67-0.37y\ .$$

$$D(A,0)=\left[\int_0^1\left(g_A^L\right)^2dy+\int_0^1\left(g_A^R\right)^2dy\right]^{1/2}$$
$$=\left[\int_0^1(-0.7+0.37y)^2dy+\int_0^1(0.67-0.37y)^2dy\right]=[0.27]^{1/2}=0.52,$$

and

$$g_B^L=1,\ \ g_B^R=-1\ .$$

$$D(B,0)=\left[\int_0^1\left(g_B^L\right)^2dy+\int_0^1\left(g_B^R\right)^2dy\right]^{1/2}$$
$$=\left[\int_0^1(1)^2dy+\int_0^1(-1)^2dy\right]=[2]^{1/2}=1.4,$$

Of Theorem 3. we have $D(A,0)=0.52<D(B,0)=1.4$, then $A<B$.

Example 6. Let $A=(-0.5,-0.3,-0.3,-0.1;1)$, $B=(0.1,\ 0.3,\ 0.3,\ 0.5;\ 1)$ are two trapezoidal fuzzy numbers, then

$$g_A^L=-0.75+0.45y,\ \ g_A^R=0.15-0.45y\ .$$

$$D(A,0)=\left[\int_0^1\left(g_A^L\right)^2dy+\int_0^1\left(g_A^R\right)^2dy\right]^{1/2}$$
$$=\left[\int_0^1(-0.7+0.37y)^2dy+\int_0^1(0.15-0.45y)^2dy\right]=[0.32]^{1/2}=0.56,$$

and

$$g_B^L=-0.15+0.45y,\ \ g_B^R=0.75-0.45y\ .$$

$$D(B,0)=\left[\int_0^1\left(g_B^L\right)^2dy+\int_0^1\left(g_B^R\right)^2dy\right]^{1/2}$$
$$=\left[\int_0^1(-0.15+0.45y)^2dy+\int_0^1(0.75-0.45y)^2dy\right]=[0.32]^{1/2}=0.56,$$

Of Theorem 3. we have $D(A,0)=0.56=D(B,0)=0.56$, then $A=B$.

Example 7. Let $A=(0.3,0.5,0.5,1;1)$, $B=(0.1,0.6,0.6,0.8;\ 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L=-0.14+0.79y,\ \ g_A^R=1.44-0.79y$.

$$D(A,0)=\left[\int_0^1 \left(g_A^L\right)^2 dy+\int_0^1 \left(g_A^R\right)^2 dy\right]^{1/2}$$

$$=\left[\int_0^1 (-0.14+0.79y)^2 dy+\int_0^1 (1.44-0.79y)^2 dy\right]=[1.26]^{1/2}=1.12,$$

and

$g_B^L=-0.34+0.79y,\ \ g_B^R=1.24-0.79y$.

$$D(B,0)=\left[\int_0^1 \left(g_B^L\right)^2 dy+\int_0^1 \left(g_B^R\right)^2 dy\right]^{1/2}$$

$$=\left[\int_0^1 (-0.34+0.79y)^2 dy+\int_0^1 (1.24-0.79y)^2 dy\right]=[0.82]^{1/2}=0.9,$$

Of Theorem 3. we have $D(A,0)=1.12>D(B,0)=0.9,$ then $A>B.$

Example 8. Let $A=(0,0.4,0.6,0.8;1)$, $B=(0.2,0.5,0.5,0.9;\ 1)$ and $C=(0.1,0.6,0.7,0.8;\ 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L=-0.42+0.82y,\ \ g_A^R=1.22-0.82y$.

$$D(A,0)=\left[\int_0^1 \left(g_A^L\right)^2 dy+\int_0^1 \left(g_A^R\right)^2 dy\right]^{1/2}$$

$$=\left[\int_0^1 (-0.42+0.82y)^2 dy+\int_0^1 (1.22-0.82y)^2 dy\right]=[0.81]^{1/2}=0.9,$$

and

$g_B^L=-0.24+0.79y,\ \ g_B^R=1.34-0.79y$.

$$D(B,0)=\left[\int_0^1 \left(g_B^L\right)^2 dy+\int_0^1 \left(g_B^R\right)^2 dy\right]^{1/2}$$

$$=\left[\int_0^1 (-0.24+0.79y)^2 dy+\int_0^1 (1.34-0.79y)^2 dy\right]=[1.01]^{1/2}=1,$$

and

$g_C^L=-0.3+0.75y,\ \ g_C^R=1.2-0.75y$.

$$D(C,0)=\left[\int_0^1 \left(g_C^L\right)^2 dy+\int_0^1 \left(g_C^R\right)^2 dy\right]^{1/2}$$

$$=\left[\int_0^1 (-0.3+0.75y)^2 dy+\int_0^1 (1.2-0.75y)^2 dy\right]=[0.78]^{1/2}=0.88,$$

Of Theorem 3. we have $D(C,0)=0.88<D(A,0)=0.9<D(B,0)=1,$ then $C<A<B.$

Example 9. Let $A=(0.1,0.2,0.4,0.5;1)$, $B=$ (-2,0,0,2; 1) are two trapezoidal fuzzy numbers, then

$g_A^L=-0.07+0.37y,\ g_A^R=0.67-0.37y$.

$$D(A,0)=\left[\int_0^1 \left(g_A^L\right)^2 dy+\int_0^1 \left(g_A^R\right)^2 dy\right]^{1/2}$$

$$=\left[\int_0^1 (-0.07+0.37y)^2 dy+\int_0^1 (0.67-0.37y)^2 dy\right]=[0.27]^{1/2}=0.52,$$

and

$g_B^L=-4.5+4.5y,\ g_B^R=4.5-4.5y$.

$$D(B,0)=\left[\int_0^1 \left(g_B^L\right)^2 dy+\int_0^1 \left(g_B^R\right)^2 dy\right]^{1/2}$$

$$=\left[\int_0^1 (-4.5+4.5y)^2 dy+\int_0^1 (4.5-4.5y)^2 dy\right]=[13.42]^{1/2}=3.66,$$

Of Theorem 3. we have $D(A,0)=0.52<D(B,0)=3.66,$ then $A<B.$

Table 1: A comparison of the ranking results for different approaches

Approches	**Ex. 1**	**Ex. 2**	**Ex. 3**	**Ex. 4**	**Ex. 5**	**Ex. 6**	**Ex. 7**
Cheng [7]	A<B	A~B	Error	A~B	A>B	A<B<C	Error
Chu [12]	A<B	A~B	Error	A<B	A>B	A<B<C	Error
Chen [9]	A<B	A<B	A<B	A<B	A>B	A<C<B	A>B
Abbasbandy [3]	Error	A~B	A<B	A~B	A<B	A<B<C	A>B

Chen [10]	A<B	A<B	A<B	A<B	A>B	A<B<C	A>B
Kumar [16]	A>B	A∼B	A<B	A<B	A>B	A<B<C	A>B
Singh [24]	A<B	A<B	A<B	A<B	A>B	A<B<C	A>B
Rezvani [21]	A>B	A>B	A<B	A<B	A<B	A<B<C	A<B
Proposed approach	A>B	A<B	A<B	A∼B	A>B	C<A<B	A<B

7. A Similarity Measure between Exponential Trapezoidal Fuzzy Numbers

In this section, a method is presented to calculate the degree of similarity between fuzzy numbers based on S. M. Chen method and proposed similarity measure is compared with seven existing methods (Chens method, Lees method, Chen and Chens method, Wei and Chens method, Wen and Zhous method, Hejazi's method and Rezvani's method) using 17 sets of generalized fuzzy numbers shown in Figure 1. The proposed method combines concepts of the exponential trapezoidal fuzzy number and distance method.
The similarity measure $S\left(\tilde{A},\tilde{B}\right)$ between $\tilde{A}$ and $\tilde{B}$ is calculated as follows:

$$S\left(\tilde{A},\tilde{B}\right)=1-\frac{d\left(\tilde{A},\tilde{B}\right)}{4} \qquad (18)$$

where

$$D\left(A,0\right)=\left[\int_0^1\left(g_A^L-g_B^L\right)^2 dy+\int_0^1\left(g_A^R-g_B^R\right)^2 dy\right]^{1/2}$$

The proposed similarity measure between generalized fuzzy numbers has the following properties.

Theorem 4. Let $A=(a_1,b_1,c_1,d_1)_E$ and $B=(a_2,b_2,c_2,d_2)_E$ are two exponential trapezoidal fuzzy numbers, so $S\left(\tilde{A},\tilde{B}\right)\in[0,1]$.

Proof. $\tilde{A},\tilde{B}\in[0,1]$, so $d\left(\tilde{A},\tilde{B}\right)\in[0,1]$, we can obtain $S\left(\tilde{A},\tilde{B}\right)\in[0,1]$. □

Theorem 5. $S\left(\tilde{A},\tilde{B}\right)=1\Leftrightarrow \tilde{A}=\tilde{B}.$

Proof. By theorem.1 part (ii) we know $d\left(\tilde{A},\tilde{A}\right)=0,$ so

If $\tilde{A}=\tilde{B}\Leftrightarrow d\left(\tilde{A},\tilde{A}\right)=0,$ then

$$S\left(\tilde{A},\tilde{B}\right)=1-\frac{d\left(\tilde{A},\tilde{B}\right)}{4}\Leftrightarrow S\left(\tilde{A},\tilde{B}\right)=1-\frac{0}{4}=1\Leftrightarrow S\left(\tilde{A},\tilde{B}\right)=1. \qquad \square$$

Theorem 6. $S\left(\tilde{A},\tilde{B}\right)=S\left(\tilde{B},\tilde{A}\right).$

Proof.

$$S\left(\tilde{A},\tilde{B}\right)=1-\frac{d\left(\tilde{A},\tilde{B}\right)}{4}=1-\frac{\left[\int_0^1\left(g_A^L-g_B^L\right)^2dy+\int_0^1\left(g_A^R-g_B^R\right)^2dy\right]^{1/2}}{4}$$

$$=1-\frac{\left[\int_0^1\left[\left(g_A^L(y)\right)^2+\left(g_B^L(y)\right)^2-2g_A^L(y)\,g_B^L(y)\right]dy+\int_0^1\left[\left(g_A^R(y)\right)^2+\left(g_B^R(y)\right)^2-2g_A^R(y)\,g_B^R(y)\right]dy\right]^{1/2}}{4}$$

$$=1-\frac{\left[\int_0^1\left[\left(g_B^L(y)\right)^2+\left(g_A^L(y)\right)^2-2g_B^L(y)\,g_A^L(y)\right]dy+\int_0^1\left[\left(g_B^R(y)\right)^2+\left(g_A^R(y)\right)^2-2g_B^R(y)\,g_A^R(y)\right]dy\right]^{1/2}}{4}$$

$$=1-\frac{\left[\int_0^1\left(g_B^L-g_A^L\right)^2dy+\int_0^1\left(g_B^R-g_A^R\right)^2dy\right]^{1/2}}{4}=1-\frac{d\left(\tilde{B},\tilde{A}\right)}{4}=S\left(\tilde{B},\tilde{A}\right),$$

So $S\left(\tilde{A},\tilde{B}\right)=S\left(\tilde{B},\tilde{A}\right).$

Example 10. Let $A=\left(0.1,0.2,0.3,0.4;1\right)$, $B=(0.1,0.25,0.25,0.4;\ 1)$ are two trapezoidal fuzzy numbers, then

$$g_A^L=-0.05+0.3y,\ g_A^R=0.55-0.3y\ ,\ g_B^L=-0.09+0.34y,\ g_B^R=0.59-0.34y$$

$$\Rightarrow d\left(A,B\right)=0.048\Rightarrow S\left(A,B\right)=1-\frac{0.048}{4}=0.988\ .$$

Example 11. Let $A=\left(0.1,0.2,0.3,0.4;1\right)$, $B=\left(0.1,0.2,0.3,0.4;1\right)$ are two trapezoidal fuzzy numbers, then

$g_A^L = -0.05 + 0.3y,\ \ g_A^R = 0.55 - 0.3y\ \ , g_B^L = -0.05 + 0.3y,\ \ g_B^R = 0.55 - 0.3y$

$\Rightarrow d(A,B) = 0 \Rightarrow S(A,B) = 1 - \frac{0}{4} = 1$.

Example 12. Let $A = (0.1, 0.2, 0.3, 0.4; 1),\ B = (0.4, 0.55, 0.55, 0.7; 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = -0.05 + 0.3y,\ \ g_A^R = 0.55 - 0.3y\ \ , g_B^L = 0.61 + 0.34y,\ \ g_B^R = 0.89 - 0.34y$

$\Rightarrow d(A,B) = 0.73 \Rightarrow S(A,B) = 1 - \frac{0.73}{4} = 0.817$.

Example 13. Let $A = (0.1, 0.2, 0.3, 0.4; 1),\ B = (0.4, 0.5, 0.6, 0.7; 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = -0.05 + 0.3y,\ \ g_A^R = 0.55 - 0.3y\ \ , g_B^L = 0.25 + 0.3y,\ \ g_B^R = 0.85 - 0.3y$

$\Rightarrow d(A,B) = 0.42 \Rightarrow S(A,B) = 1 - \frac{0.42}{4} = 0.895$.

Example 14. Let $A = (0.1, 0.2, 0.3, 0.4; 1),\ B = (0.1, 0.2, 0.3, 0.4; 0.8)$ are two trapezoidal fuzzy numbers, then

$g_A^L = -0.05 + 0.3y,\ \ g_A^R = 0.55 - 0.3y\ \ , g_B^L = -0.028 + 0.28y,\ \ g_B^R = 0.53 - 0.28y$

$\Rightarrow d(A,B) = 0.014 \Rightarrow S(A,B) = 1 - \frac{0.014}{4} = 0.996$.

Example 15. Let $A = (0.3, 0.3, 0.3, 0.3; 1),\ B = (0.3, 0.3, 0.3, 0.3; 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = 0.3,\ \ g_A^R = 0.3\ \ , g_B^L = 0.3,\ \ g_B^R = 0.3$

$\Rightarrow d(A,B) = 0 \Rightarrow S(A,B) = 1 - \frac{0}{4} = 1$.

Example 16. Let $A = (0.2, 0.2, 0.2, 0.2; 1),\ B = (0.3, 0.3, 0.3, 0.3; 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = 0.2,\ \ g_A^R = 0.2\ \ , g_B^L = 0.3,\ \ g_B^R = 0.3$

$\Rightarrow d(A,B) = 0.14 \Rightarrow S(A,B) = 1 - \frac{0.14}{4} = 0.965$.

Example 17. Let $A = (0.1, 0.2, 0.2, 0.3; 1),\ B = (0.3, 0.3, 0.3, 0.3; 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = -0.025 + 0.225y,\ \ g_A^R = 0.425 - 0.225y\ \ , g_B^L = 0.3,\ \ g_B^R = 0.3$

$\Rightarrow d(A,B) = 0.226 \Rightarrow S(A,B) = 1 - \frac{0.226}{4} = 0.943$.

Example 18. Let $A = (0.1, 0.2, 0.2, 0.3;1),\ \ B = (0.2, 0.3, 0.3, 0.4;1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = -0.025 + 0.225y,\ \ g_A^R = 0.425 - 0.225y\ \ , g_B^L = 0.075 + 0.225y,\ \ g_B^R = 0.52 - 0.225y$

$\Rightarrow d(A,B) = 0.14 \Rightarrow S(A,B) = 1 - \frac{0.14}{4} = 0.965$.

Example 19. Let $A = (0.1, 0.4, 0.4, 0.7;1),\ \ B = (0.3, 0.4, 0.4, 0.5;1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = -0.275 + 0.675y,\ \ g_A^R = 1.07 - 0.675y\ \ , g_B^L = 0.175 + 0.225y,\ \ g_B^R = 0.625 - 0.225y$

$\Rightarrow d(A,B) = 0.36 \Rightarrow S(A,B) = 1 - \frac{0.36}{4} = 0.91$.

Example 20. Let $A = (0.2, 0.3, 0.5, 0.6;1),\ \ B = (0.3, 0.4, 0.4, 0.5;1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = 0.025 + 0.375y,\ \ g_A^R = 0.775 - 0.375y\ \ , g_B^L = 0.175 + 0.225y,\ \ g_B^R = 0.625 - 0.225y$

$\Rightarrow d(A,B) = 0.12 \Rightarrow S(A,B) = 1 - \frac{0.12}{4} = 0.97$.

Example 21. Let $A = (0.4, 0.4, 0.4, 0.8;1),\ \ B = (0.3, 0.4, 0.4, 0.5;1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = 0.15 + 0.45y,\ \ g_A^R = 1.05 - 0.45y\ \ , g_B^L = 0.175 + 0.225y,\ \ g_B^R = 0.625 - 0.225y$

$\Rightarrow d(A,B) = 0.34 \Rightarrow S(A,B) = 1 - \frac{0.34}{4} = 0.915$.

Example 22. Let $A = (0.2, 0.3, 0.4, 0.5;1),\ \ B = (0.3, 0.4, 0.5, 0.6;1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = 0.05 + 0.3y,\ \ g_A^R = 0.54 - 0.3y\ \ , g_B^L = 0.15 + 0.3y,\ \ g_B^R = 0.75 - 0.3y$

$\Rightarrow d(A,B) = 0.23 \Rightarrow S(A,B) = 1 - \frac{0.23}{4} = 0.942$.

Example 23. Let $A = (0.1, 0.2, 0.2, 0.3;1),\ \ B = (0.1, 0.2, 0.2, 0.3;0.7)$ are two trapezoidal fuzzy numbers, then

$g_A^L = -0.025 + 0.225y,\ \ g_A^R = 0.425 - 0.225y\ \ , g_B^L = 0.011 + 0.189y,\ \ g_B^R = 0.389 - 0.189y$

$\Rightarrow d(A,B) = 0.03 \Rightarrow S(A,B) = 1 - \frac{0.03}{4} = 0.992$.

Example 24. Let $A = (0.1, 0.2, 0.2, 0.3; 1),\ \ B = (0.2, 0.2, 0.2, 0.2; 0.7)$ are two trapezoidal fuzzy numbers, then

$g_A^L = -0.025 + 0.225y,\ \ g_A^R = 0.425 - 0.225y\ \ , g_B^L = 0.2,\ \ g_B^R = 0.2$

$\Rightarrow d(A,B) = 0.17 \Rightarrow S(A,B) = 1 - \frac{0.17}{4} = 0.957$.

Example 25. Let $A = (0.1, 0.4, 0.4, 0.7; 0.825),\ \ B = (0.3, 0.4, 0.4, 0.5; 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = -0.22 + 0.62y,\ \ g_A^R = 1.02 - 0.62y\ \ , g_B^L = 0.175 + 0.225y,\ \ g_B^R = 0.625 - 0.225y$

$\Rightarrow d(A,B) = 0.32 \Rightarrow S(A,B) = 1 - \frac{0.32}{4} = 0.92$.

Example 26. Let $A = (0.2, 0.3, 0.5, 0.6; 0.79205),\ \ B = (0.3, 0.4, 0.4, 0.5; 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = 0.048 + 0.33y,\ \ g_A^R = 0.75 - 0.33y\ \ , g_B^L = 0.175 + 0.225y,\ \ g_B^R = 0.625 - 0.225y$

$\Rightarrow d(A,B) = 0.11 \Rightarrow S(A,B) = 1 - \frac{0.11}{4} = 0.972$.

Example 27. Let $A = (0.2, 0.3, 0.5, 0.6; 1),\ \ B = (0.2, 0.3, 0.3, 0.4; 1)$ are two trapezoidal fuzzy numbers, then

$g_A^L = 0.025 + 0.375y,\ \ g_A^R = 0.775 - 0.373y\ \ , g_B^L = 0.075 + 0.225yy,\ \ g_B^R = 0.52 - 0.225y$

$\Rightarrow d(A,B) = 0.19 \Rightarrow S(A,B) = 1 - \frac{0.19}{4} = 0.95$.

Table 2: A comparison of the ranking results for different approaches

Sets	Chen. S. M	Lee. H. S	Chen. S. J	Wei. S. H	Hejazi. S.R	Wen. J	Rezvani. S	proposed method
Set 1	0.975	0.9617	0.8357	0.95	0.9004	0.9473	0.9464	0.988
Set 2	1	1	1	1	1	1	1	1
Set 3	0.7	0.5	0.42	0.682	0.6465	0.6631	0.6625	0.817
Set 4	0.7	0.5	0.49	0.7	0.7	0.7	0.7	0.8951
Set 5	1	1	0.8	0.8248	0.6681	0.6659	0.6308	0.996
Set 6	1	*	1	1	1	1	1	1
Set 7	0.9	0	0.81	0.9	0.9	0.9	0.9	0.965
Set 8	0.9	0.6	0.54	0.8411	0.37	0.3896	0.7794	0.943
Set 9	0.9	0.6667	0.81	0.9	0.9	0.9	0.9	0.965
Set 10	0.9	0.8333	0.9	0.7833	0.6261	0.7731	0.7752	0.91
Set 11	0.9	0.75	0.72	0.8003	0.6448	0.7938	0.7938	0.97
Set 12	0.9	0.8	0.8325	0.8289	0.7361	0.7478	0.5707	0.915
Set 13	0.9	0.75	0.81	0.9	0.9	0.9	0.92	0.942
Set 14	1	1	0.7	0.7209	0.5113	0.5104	0.4817	0.992
Set 15	0.95	0.7	0.9048	0.6215	0.383	0.4242	0.3928	0.957
Set 16	0.9	0.8333	0.7425	0.814	0.6261	0.7321	0.6752	0.92
Set 17	0.9	0.75	0.8911	0.838	0.6448	0.7432	0.4445	0.972
Set 18	0.9	0.75	0.6976	0.8003	0.6448	0.7144	0.711	0.95

8. Conclusion

Similarity and distance measure between two fuzzy numbers are closely related concepts. So it is possible to express similarity measure and distance measure between fuzzy numbers by a functional relationship. In this paper, we define the metric distance between two exponential trapezoidal fuzzy numbers and we use this metric for similarity measure. The basic properties of the above-mentioned similarity measure are proved in detail. Finally, we rank two generalized fuzzy numbers using distance measure and the similarity measure between them.

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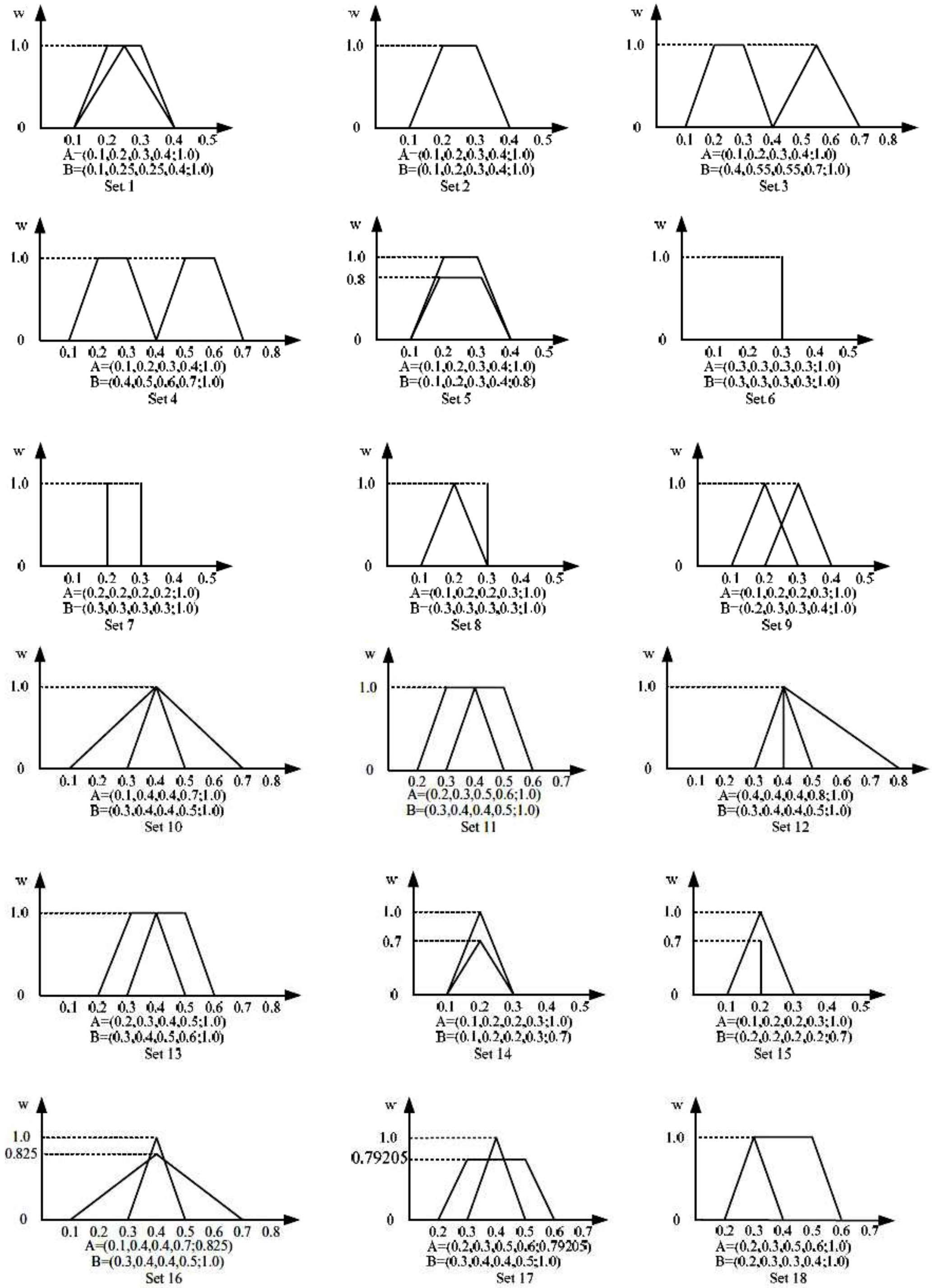
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B=(0.1,0.2,0.3,0.4;1.0)
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B=(0.4,0.55,0.55,0.7;1.0)
Set 3
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