Journal of Physical Sciences, Vol. 23, 2018, 11-22
ISSN: 2350-0352 (print), www.vidyasagar.ac.in/publication/journal
Published on 24 December 2018

On the Class of Homogeneous Cubic Finsler Metrics Admitting (α,β)-Types

M. Ebrahimi, D. Latifi and A. Tayebi

University of Mohaghegh Ardabili University of Mohaghegh Ardabili
Faculty of Sciences Ardabil, Iran
E-mail: m.ebrahimi@uma.ac.ir; latifi@uma.ac.ir; akbar.tayebi@gmail.com

Received 1 July 2018; accepted 3 November 2018

ABSTRACT

In this paper, we study the class of cubic metrics which are used in the theory of space-time structure and general relativity. We consider the homogeneous geodesics in the homogeneous cubic space. Let (M,F) be a homogeneous cubic space and F defined by the Riemannian metric $\tilde{a}$ and the vector field X . First, we show that X is a geodesic vector of (M,F) if and only if it is a geodesic vector of $(M,\tilde{a})$ Also, we find a condition under which an arbitrary vector is a geodesic vector of cubic metric if and only if it is a geodesic vector of Riemannian metric. Then we show that, for Berwald type cubic metric, if the underlying Riemannian metric is naturally reductive, then the cubic metric is naturally reductive. Finally, we find the formula of the flag curvature of the class of cubic metrics.

Keywords: Homogeneous Finsler spaces, homogeneous geodesic, left invariant metric, cubic metric, (α,β) metric, Berwald metric, flag curvature.

1. Introduction

There are two important classes of Finsler metrics, namely, the class of m-th root metrics and the class of (α,β)-metrics. Let (M,F) be an n-dimensional Finsler manifold, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM . Let F be a scalar function on TM defined by $F=\sqrt[m]{A}$, where A is given by $A=a_{i_1\dots i_m}(x)y^{i_1}y^{i_2}\dots y^{i_m}$ such that $a_{i_1\dots i_m}$ is symmetric in all its indices. F is called an m-th root Finsler metric. The theory of m-th root metrics has been developed by Shimada [10, 11, 14], and applied to Biology as an ecological metric by Antonelli [1]. It is regarded as a direct generalization of Riemannian metric in the sense that the second root metric is a Riemannian metric $F=\sqrt{a_{ij}(x)y^iy^j}$. The third and fourth root metrics are called the cubic metric $F=\sqrt[3]{a_{ijk}y^iy^jy^k}$ and quartic metric

$F=\sqrt[4]{a_{ijkl}y^iy^jy^ky^l}$ respectively [17,18]. The special m-th root metric $F=\sqrt[m]{y^{i_1}y^{i_2}...y^{i_m}}$ is called Berwald-Moor metric which plays a very important role in theory of space-time structure, gravitation and general relativity. Recently studies show that the theory of m-th root Finsler metrics plays a very important role in Gravitation, Cancer and Seismic Ray Theory [13, 15, 17, 19]. An (α,β)-metric is a Finsler metric of the form $F=\alpha\Phi(s)$, $s=\dfrac{\beta}{\alpha}$ where $\Phi=\Phi(s)$ is a C^∞ function on $(-b_0,b_0)$, $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on M. These metrics have important applications in Physics, Mechanics and Seismology, etc, see for instance [1, 19] .

In [20], Wegener studied cubic Finsler metrics of dimension two and three. The first half of the third section of Wegeners paper [20] is devoted to making a list of Berwald spaces and locally Minkowski spaces with cubic metric of the normal form. He has proved that every two and three-dimensional cubic Finsler metric with vanishing Landsberg curvature is Berwaldian. Wegener's paper is only an abstract of his PhD thesis without almost all calculations. In [8], Matsumoto studied cubic Finsler metrics and gave an improved version of Wegener's results. There is a interesting relation between an m-th root metric and an (α,β)-metric. In [9], Matsumoto-Numata studied the class of cubic metrics and proved the following.

Lemma 1.1. ([9]) Let $F=\sqrt[3]{a_{ijk}y^iy^jy^k}$ be a cubic Finsler metric on a manifold M of dimension dim(M) ≥ 3. If F is a function of a non-degenerate quadratic form $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and a one-form $\beta=b_i(x)y^i$ which is homogeneous in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of degree one, then it is written in the following form

$$F=\sqrt[3]{c_1\alpha^2\beta+c_2\beta^3}$$

where c_1 and c_2 are real constants.

Lemma 1.1 shows that the class of Finsler metrics in the form (1.1) is very important in order to studying of the class of cubic metrics.

A connected Riemannian manifold (M,g) is said to be homogeneous if a connected group of isometries G acts transitively on it [6]. Such M can be identified with $(\dfrac{G}{H},g)$ where H is the isotropy group at a fixed point o of M. The Lie algebra g of

G admits a reductive decomposition $g = m \oplus h$, where $m \subset g$ is a subspace of g isomorphic to the tangent space T_0M and h is the Lie algebra of H [4]. In general, such a decomposition is not unique. A homogeneous geodesic through the origin $o \in M = \frac{G}{H}$ is a geodesic $\gamma(t)$ which is an orbit of a one-parameter subgroup of G, that is $\gamma(t) = \exp(tz)(o)$, $t \in R$, where Z is a nonzero vector of *g*.

A Finsler space (M,F) is called homogeneous Finsler space if the group of isometries of (M,F) i.e., $\mathrm{I}(M,F)$ acts transitively on M. A vector $X \in g - \{0\}$ will be called a geodesic vector if the curve $\gamma(t) = \exp(tX)(p)$ is a constant speed geodesic of (M,F) Let (M,F) be a Finsler space. As in the Riemannian case, we have two kinds of definitions of isometry on (M,F) in terms of Finsler function in the tangent space and the induced non-reversible distance function on the base manifold M. For the distance function induced by Finsler function, see [16]. The equivalence of these two definitions in the Finsler case is a result of Deng-Hou [3]. They also prove that the group of isometries of a Finsler space is a Lie transformation group of the underlying manifold which can be used to study homogeneous Finsler spaces.

In [5], Kowalski-Vanhecke studied Riemannian manifolds with homogeneous geodesics and proved that a vector $X \in g - \{0\}$ is a geodesic vector if and only if holds, where $\langle , \rangle$ denotes the $Ad(H)$ -invariant scalar product on m induced by the Riemannian scalar product on T_0M and the subscripts m indicates the projection into m. In this paper, we study geodesics vectors in homogeneous cubic space and prove the following.

Theorem 1.2. Let F be the cubic metric (1.1) on a homogeneous manifold M which is defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then X is a geodesic vector of $(M,\tilde{a})$ if and only if it is a geodesic vector of (M,F).

Let $(M = \frac{G}{H}, g)$ be a Riemannian homogeneous space, and $g = m \oplus h$ be a reductive decomposition. In [4], Kowalski-Vanhecke proved that $X \in g$ is a geodesic vector if and only if $g([X,y]_m, X_m) = 0, \quad \forall y \in m$. In [6], the second author proved a similar theorem for Finslerian case as follows. More precisely, he proved that a vector $X \in g - \{0\}$ is geodesic vector if and only if $g_{x_m}([X,Z]_m, X_m) = 0, \quad \forall Z \in m$ holds.

For other progress, see [6], [7] . We find a condition under which an arbitrary vector is a geodesic vector of cubic metric if and only if it is a geodesic vector of Riemannian metric.

Theorem 1.3. Let F be the cubic metric (1.1) on a homogeneous manifold $\frac{G}{H}$ which is defined by the Riemannian metric $\tilde{a}$ and the vector field X . Let $y \in g - \{0\}$ such that $\tilde{a}(X, y_m) \neq 0$ and for any $Z \in m$, $\tilde{a}(X, [y, z]_m) = 0$ .Then y is a geodesic vector of $(\frac{G}{H}, F)$ if and only if y is ageodesic vector of $(\frac{G}{H}, \tilde{a})$.

Let G be a Lie group and H be a closed subgroup of G . The coset space $\frac{G}{H}$ has a unique smooth structure such that G is a Lie transformation group of $\frac{G}{H}$. It is called reductive if there exists a subspace m of the Lie algebra g of G such that $g = m \oplus h$, where h is the Lie algebra of H and $Ad(h)m \subset m$, $h \in H$ (see [7]). A Finsler space (M, F) is called a Berwald space if the Chern connection coefficient $\Gamma^k{}_{ij} = \Gamma^k{}_{ij}(x, y)$ in natural coordinate systems have no dependence on the vector y. In this paper, we study natural reductive homogeneous cubic space.

Theorem 1.4. *Let* $(M = \frac{G}{H}, F)$ be a homogeneous Finsler space, where F is an invariant cubic metric (1.1) defined by the Riemannian metric $\tilde{a}$ and the vector field X which is of Berwald type. If $(\frac{G}{H}, \tilde{a})$ is naturally reductive, then $(\frac{G}{H}, F)$ is naturally reductive.

Finally, we give the flag curvature formula of invariant cubic metrics which are induced by invariant Riemannian metrics and invariant vector fields on homogeneous spaces.

Theorem 1.5. Let $(M = \frac{G}{H}, F)$ be a naturally reductive cubic space with F defined by the Riemannian metric $\tilde{a}$ and the vector field X .i.e, $F(y) = \sqrt[3]{c_1\tilde{a}(X, y)\tilde{a}(y, y) + c_2\tilde{a}(X, y)^3}$ be a flag in m such that (P, y) is an orthonormal basis of p with respect to $\tilde{a}$. Then the flag curvature of the flag (P, y) in m is given by

$$K(p,y)=\frac{\tilde{a}(X,y)}{D}\{-3c_2(\tilde{a}([u,y]_h,[X,y]_m)+\frac{1}{4}\tilde{a}([u,y]_m,[X,y]_m))\tilde{a}(X,u)$$
$$+c_1(\tilde{a}(u,[[u,y]_h,y])+\frac{1}{4}\|[u,y]\|^2)\}$$
$$+\frac{2}{3DF^3}\{\tilde{a}(X,u)(\tilde{a}([u,y]_h,[X,y]_m)+\frac{1}{4}(\tilde{a}([u,y]_m,[X,y]_m))B^2\},$$

where

$$A=c_1+3c_2\tilde{a}(X,u)^2,\quad B=c_1+3c_2\tilde{a}(X,y)^2,$$
$$D=F^2\tilde{a}(X,y)A-\frac{1}{3F}\tilde{a}(X,u)^2B^2 \qquad (1.2)$$

2. Preliminaries

Let M be a n-dimensional C^∞ manifold and $TM=\bigcup_{x\in M}T_xM$ the tangent bundle. Let (M,F) be a Finsler manifold. The following quadratic form g_y on T_xM is called fundamental tensor

$$g_y(u,v)=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}[F^2(y+su+tv)]\Big|_{s=t=0,}\ u,v\in T_xM$$

Let $x\in M$ and $F_x=F\big|_{T_xM}$ To measure the non-Euclidean feature of F_x, define trilinear form $C_y:T_xM\otimes T_xM\otimes T_xM\to R$ by

$$C_y(u,v,w)=\frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}[F^2(y+ru+sv+tw)]\Big|_{r=s=t=0},$$

where $u,v,w\in T_xM$ The family $C=\{C_y\}_{y\in TM_0}$ is called the Cartan torsion [6]. It is well known that $C=0$ if and only if F is Riemannian.

Given a Finsler manifold (M,F), then a global vector field G is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by $G=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i}$ where

$$G^i:=\frac{1}{4}g^{il}\left[\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k-\frac{\partial F^2}{\partial x^l}\right],y\in T_xM.$$

G is called the spray associated to (M,F).

For a tangent vector $y \in TM_0$ define $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $B_y(u,v,w) = B^i{}_{jkl}(y)u^j v^k w \frac{\partial}{\partial x^i}|_x$ where

$$B^i{}_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = \frac{\partial^3 \Gamma^i{}_{jk}}{\partial y^l},$$

where $\Gamma^k{}_{ij} = \Gamma^k{}_{ij}(x,y)$ denotes the Christoffel symbols of the Chern (Berwald) connection of F.

B is called the Berwald curvature. Then, F is called a Berwald metric if $B = 0$.

Let $\phi:(M,F) \to (M,F)$ be a diffeomorphism. Then ϕ is called an isometry of (M,F) if $F(\phi(x), d\phi_x(x)) = F(x,X)$, $\forall x \in M$ and $x \in T_xM$. The group of isometries $\mathrm{I}(M,F)$ of a manifold M is a Lie transformation group of M. (M,F) is called Finslerian homogeneous space if the group of isometries, i.e., $\mathrm{I}(M,F)$ acts transitively on M. Hence, in homogeneous Finsler space the tangent Minkowski spaces (T_xM, F_x) are all linearly isometric to each other.

A homogeneous space $\frac{G}{H}$ of a connected Lie group G is called reductive if the following conditions are satisfied:

(1) In the Lie algebra g of G there exists a subspace m such that $g = m + h$ (direct sum of vector subspaces),

(2) $ad(h)m \subset m$ for all $h \in H$, where h is the subalgebra of g corresponding to the identity component H_0 of H and $ad(h)$ denotes the adjoint representation of H in g.

It is remarkable that, condition (2) implies $(2)'$ $[h,m] \subset m$. Conversely, if H is connected, then $(2)'$ implies (2) (For more details, see [4] and [12]).

A homogeneous manifold $\frac{G}{H}$ with an invariant Finsler metric F is called naturally reductive if there exists an $Ad(H)$ - invariant decomposition $g = m + h$ such that

$$g_y([x,u]_m, v) + g_y([x,v]_m, u) + 2C_y([x,y]_m, u, v) = 0, \qquad (2.1)$$

where $y \neq 0$ and $x,u,v \in m$ (see[5]).

For $y \in TM_0$, the Riemann curvature is a family of linear transformations $R_y : T_x M \to T_x M$ with homogeneity $R_{\lambda y} = \lambda^2 R_y$, $\forall \lambda \rangle 0$ defined by $R_y(u) = R^i_k(y) u^k \dfrac{\partial}{\partial x^i}$ where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k} \tag{2.2}$$

The family $R = \{R(y)\}_{y \in TM_0}$ is called the Riemann curvature.

For a flag $P = span\{y, u\} \subset T_x M$. With flagpole *y*, the flag curvature $K = K(p, y)$ is defined by

$$K(x,y,p) = \frac{g_y(u, R_y(u))}{g_y(y,y) g_y(u,u) - g_y(u,y)^2} \tag{2.3}$$

The flag curvature $K(x, y, p)$ is a function of tangent planes. $P = span\{y, u\} \subset T_x M$. This quantity tells us how curved the space is at a point. If F is a Riemannian metric, $K(x, y, p) = K(x, p)$ is independent of $y \in p \setminus \{0\}$. Thus the flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry.

3. Proof of main theorems

Proof of Theorem 1.2. According to Lemma 1.1, we have

$$F(y) = \sqrt[3]{c_1 \tilde{a}(X, y)\tilde{a}(y, y) + c_2 \tilde{a}(X, y)^3} \tag{3.1}$$

where c_1 and c_2 are real constants. For $r, s \in R$, (3.1) can be written as follows

$$F(y+ru+sv) = [c_1\tilde{a}(X, y+ru+sv)\tilde{a}(y+ru+sv, y+ru+sv)$$
$$+ c_2\tilde{a}(X, y+ru+sv)^3]^{\frac{2}{3}} \tag{3.2}$$

By (3.2), we get

$$g_y(u,v) = \frac{2}{3F}\{c_1\tilde{a}(X,u)\tilde{a}(y,v) + c_1\tilde{a}(X,v)\tilde{a}(y,u) + 3c_2\tilde{a}(X,u)\tilde{a}(X,v)\tilde{a}(X,y)$$
$$+ c_1\tilde{a}(u,v)\tilde{a}(X,y)\} - \frac{1}{9F^4}\{c_1\tilde{a}(X,v)\tilde{a}(y,y) + 2c_1\tilde{a}(X,y)\tilde{a}(y,v)$$
$$+ 3c_2\tilde{a}(X,v)\tilde{a}(X,y)^2\}\{c_1\tilde{a}(X,u)\tilde{a}(y,y) + 2c_1\tilde{a}(X,y)\tilde{a}(y,u) + 3c_2\tilde{a}(X,u)\tilde{a}(X,y)^2\} \tag{3.3}$$

For all $Z \in m$, (3.3) implies that

$$g_X(X,[X,z]_m) = \frac{2}{3F}\{c_1\tilde{a}(X,X)\tilde{a}(X,[X,z]) + c_1\tilde{a}(X,[X,z])\tilde{a}(X,X) + c_1\tilde{a}(X,X)\tilde{a}(X,[X,z])$$

$$+3c_2\tilde{a}(X,X)\tilde{a}(X,X)\tilde{a}(X,[X,z])\}$$

$$-\frac{1}{9F^4}\{c_1\tilde{a}(X,[X,z])\tilde{a}(X,X)+2c_1\tilde{a}(X,X)\tilde{a}(X,[X,z])+3c_2\tilde{a}(X,[X,z])\tilde{a}(X,X)^2\}$$

$$\{c_1\tilde{a}(X,X)\tilde{a}(X,X)+2c_1\tilde{a}(X,X)\tilde{a}(X,X)+3c_2\tilde{a}(X,X)\tilde{a}(X,X)^2\}$$

Equivalently, we have

$$g_X(X,[X,z]_m)=\tilde{a}(X,[X,z])\frac{1}{F}(c_1\tilde{a}(X,X)+c_2\tilde{a}(X,X)^2)$$

Since $c_1\tilde{a}(X,X)+c_2\tilde{a}(X,X)^2\neq 0$ then it follows that $g_X(X,[X,z]_m)=\mathbf{0}$ holds if and only if $\tilde{a}(X,[X,z]_m)=0$

Proof of Theorem 1.3. Using (3.3) and after some computations, we get

$$g_{y_m}(y_m,[y,z]_m)=\frac{2}{3F}\{c_1\tilde{a}(X,y_m)\tilde{a}(y_m,[y,z]_m)+c_1\tilde{a}(X,[y,z]_m)\tilde{a}(y_m,y_m)+c_1\tilde{a}(X,y_m)\tilde{a}(y_m,[y,z]_m)$$

$$+3c_2\tilde{a}(X,y_m)\tilde{a}(X,y_m)\tilde{a}(X,[y,z]_m)\}$$

$$-\frac{1}{9F^4}\{c_1\tilde{a}(X,[y,z]_m)\tilde{a}(y_m,y_m)+2c_1\tilde{a}(X,y_m)\tilde{a}(y_m,[y,z]_m)+3c_2\tilde{a}(X,[y,z]_m)\tilde{a}(X,y_m)^2\}$$

$$\{c_1\tilde{a}(X,y_m)\tilde{a}(y_m,y_m)+2c_1\tilde{a}(X,y_m)\tilde{a}(y_m,y_m)+3c_2\tilde{a}(X,y_m)\tilde{a}(X,y_m)^2\}$$

Since for any $Z\in m$, $\tilde{a}(X,[y,z])=0$ holds, then we have

$$g_{y_m}(y_m,[y,z]_m)=\frac{-2c_1}{9F^4}\tilde{a}(X,y_m)\tilde{a}(y_m,[y,z]_m)\{3c_1\tilde{a}(X,y_m)\tilde{a}(y_m,y_m)$$

$$+3c_2\tilde{a}(X,y_m)^3\}+\frac{2}{3F}\{2c_1\tilde{a}(X,y_m)\tilde{a}(y_m,[y,z]_m)\} \qquad (3.4)$$

Simplifying (3.4) implies that

$$g_{y_m}(y_m,[y,z]_m)=\frac{2}{3F}\{2c_1\tilde{a}(X,y_m)\tilde{a}(y_m,[y,z]_m)\} \qquad (3.5)$$

By (3.5), y is a geodesic vector of (M,F) if and only if it is a geodesic vector of $(M,\tilde{a})$.

Let (M,F) be a Finsler space. Then (M,F) is called a Berwald space if the Chern connection coeffcients $\Gamma^k{}_{ij}(x,y)$ in natural coordinate systems have no dependence on the vector y, or in other words, if the Chern connection defines a linear connection directly on the underlying manifold.

Proof of Theorem 1.4. Let $(M,\tilde{a})$ be a naturally reductive Riemannian manifold. We show that for all $0\neq y,z,u,v\in m$ the following holds

$$g_y([x,u]_m,v)+g_y([x,v]_m,u)+2C_y([x,y]_m,u,v)=0, \quad (3.6)$$

Since F is a Berwald metric, then (M,F) and $(M,\tilde{a})$ have the same connection. So for all $0\neq y\in m$, we have $\tilde{a}(X,[y,z])=0$, $\forall z\in m$. By (3.3), it follows that

$$g_y([z,u]_m,v)=\tilde{a}(y,[z,u])\{\frac{2c_1\tilde{a}(X,v)}{3F}-\frac{c_1\tilde{a}(X,y)}{9F^4}(c_1\tilde{a}(X,v)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,v))$$
$$+3c_2\tilde{a}(X,v)\tilde{a}(X,y)^2)\}+\frac{2c_1\tilde{a}(X,y)}{3F}\tilde{a}(v,[z,u]).$$

And

$$g_y([z,u]_m,v)=-\tilde{a}(y,[z,v])\{\frac{c_1\tilde{a}(X,y)}{9F^4}(c_1\tilde{a}(X,u)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,u))$$
$$+3c_2\tilde{a}(X,u)\tilde{a}(X,y)^2)\}-\frac{2c_1\tilde{a}(X,u)}{3F}\}+\frac{2c_1\tilde{a}(X,y)}{3F}\tilde{a}(u,[z,v]).$$

By definition, we get

$$2c_y(z,u,v)=\frac{1}{3F}\{6c_2\tilde{a}(X,z)\tilde{a}(X,u)\tilde{a}(X,v)+2c_1\tilde{a}(X,z)\tilde{a}(u,v)+2c_1\tilde{a}(X,u)\tilde{a}(v,z)$$
$$+2c_1\tilde{a}(X,v)\tilde{a}(z,u)\}-\frac{1}{9F^4}\{6c_2\tilde{a}(X,z)\tilde{a}(X,u)\tilde{a}(X,y)+2c_1\tilde{a}(X,z)\tilde{a}(y,u)$$
$$+2c_1\tilde{a}(X,u)\tilde{a}(y,z)+2c_1\tilde{a}(X,y)\tilde{a}(z,u)\}\{c_1\tilde{a}(X,v)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,v)$$
$$+3c_2\tilde{a}(X,y)^2\tilde{a}(X,v)\}-\frac{1}{9F^4}\{2c_1\tilde{a}(X,u)\tilde{a}(y,v)+2c_1\tilde{a}(X,v)\tilde{a}(y,u)$$
$$+2c_1\tilde{a}(X,y)\tilde{a}(v,u)+6c_2\tilde{a}(X,y)\tilde{a}(X,u)\tilde{a}(X,v)\}\{c_1\tilde{a}(X,z)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,z)$$
$$+3c_2\tilde{a}(X,y)^2\tilde{a}(X,z)\}-\frac{1}{9F^4}\{2c_1\tilde{a}(X,z)\tilde{a}(y,v)+2c_1\tilde{a}(X,v)\tilde{a}(y,z)+2c_1\tilde{a}(v,z)\tilde{a}(X,y)$$
$$+6c_2\tilde{a}(X,z)\tilde{a}(X,y)\tilde{a}(X,v)\}\{c_1\tilde{a}(X,u)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,u)+3c_2\tilde{a}(X,y)^2\tilde{a}(X,u)\}$$
$$-\frac{4}{27F^{15}}\{c_1\tilde{a}(X,v)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,v)+3c_2\tilde{a}(X,y)^2\tilde{a}(X,v)\}\{c_1\tilde{a}(X,u)\tilde{a}(y,y)$$
$$+2c_1\tilde{a}(X,y)\tilde{a}(y,u)+3c_2\tilde{a}(X,v)^2\tilde{a}(X,u)\}\{c_1\tilde{a}(X,z)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,z)$$
$$+3c_2\tilde{a}(X,y)^2\tilde{a}(X,z)\}$$

It follows that

$$c_y([z,y]_m,u,v)=\frac{-c_1}{9F^4}\tilde{a}(v,[z,y])\{\tilde{a}(X,y)(c_1\tilde{a}(X,u)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,u)+3c_2\tilde{a}(X,u)\tilde{a}(X,y)^2)$$
$$-3F^3\tilde{a}(X,u)\}-\frac{-c_1}{9F^4}\tilde{a}(u,[X,y]_m)\{\tilde{a}(X,y)(c_1\tilde{a}(X,v)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,v)+3c_2\tilde{a}(X,v)\tilde{a}(X,y)^2)$$
$$-3F^3\tilde{a}(X,v)\}$$

Therefore

$$g_y([x,u]_m,v)+g_y([x,v]_m,u)+2C_y([x,y]_m,u,v)=\frac{2c_1\tilde{a}(X,y)}{3F}(\tilde{a}([z,u],v)+\tilde{a}(u,[z,v]))$$
$$+(\tilde{a}([z,u],y)+\tilde{a}(u,[z,y]))\{\frac{2c_1\tilde{a}(X,v)}{3F}-\frac{c_1\tilde{a}(X,y)}{9F^4}(c_1\tilde{a}(X,v)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,v))$$
$$+3c_2\tilde{a}(X,v)\tilde{a}(X,y)^2)\}+(\tilde{a}([z,v],y)+\tilde{a}(v,[z,y]))\{\frac{2c_1\tilde{a}(X,u)}{3F}-\frac{c_1\tilde{a}(X,y)}{9F^4}$$
$$(c_1\tilde{a}(X,u)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,u))\ +3c_2\tilde{a}(X,u)\tilde{a}(X,y)^2)\}$$

Thus, we get (3.6).

Now, we are going to prove Theorem 1.5.

Proof of Theorem 1.5. Using the explicit expression for the connection of M, a straightforward lengthy calculation leads to the following expression for the curvature tensor R of $(\frac{G}{H},F)$:

$R(v,w)z=-[[v,w]_h,z]-\frac{1}{2}[[v,w]_m,z]-\frac{1}{4}[[z,v]_m,w]+\frac{1}{4}[[z,w]_m,v]_m$ for all $v,w,z\in m\cong TM_0$. The following holds

$$K(P,y)=\frac{g_y(R(u,y)y,u)}{g_y(y,y)g_y(u,u)-g_y(u,y)^2}.$$

Then for $r,s\in R$ we get

$$F^2(y+ru+sv)=[c_1\tilde{a}(X,y+ru+sv)\tilde{a}(y+ru+sv,y+ru+sv)+c_2\tilde{a}(X,y+ru+sv)^3]^{\frac{2}{3}}$$

By a direct computation, we get

$$g_y(u,v)=\frac{2}{3F}\{c_1\tilde{a}(X,u)\tilde{a}(y,v)+c_1\tilde{a}(X,v)\tilde{a}(y,u)+3c_2\tilde{a}(X,u)\tilde{a}(X,v)\tilde{a}(X,y)$$
$$+c_1\tilde{a}(u,v)\tilde{a}(X,y)\}-\frac{1}{9F^4}\{c_1\tilde{a}(X,v)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,v)$$
$$+3c_2\tilde{a}(X,v)\tilde{a}(X,y)^2\}\{c_1\tilde{a}(X,u)\tilde{a}(y,y)+2c_1\tilde{a}(X,y)\tilde{a}(y,u)+3c_2\tilde{a}(X,u)\tilde{a}(X,y)^2\}$$

According to the above formula we have

$$g_y(y,y)=F^2(y) \tag{3.8}$$

$$g_y(u,u)=\frac{2\tilde{a}(X,y)}{3F(y)}A-\frac{1}{9F^4(y)}\tilde{a}(X,u)^2B^2, \tag{3.9}$$

$$g_y(y,u)=\frac{B}{3F(y)}\tilde{a}(X,u), \tag{3.10}$$

$$g_y(R(u,y)y,u)=\frac{2\tilde{a}(X,y)}{3F(y)}(c_1\tilde{a}(R(u,y)y,u)+3c_2\tilde{a}(R(u,y)y,X)\tilde{a}(X,u))$$

$$-\frac{B^2\tilde{a}(X,u)}{3F^4(y)}\tilde{a}(R(u,y)y,X), \tag{3.11}$$

where *A*, *B* and *C* defined by (1.2). Substituting (3.8), (3.9), (3.10) and (3.11) in (3.7) completes the proof.

Acknowledgements. The author gratefully acknowledges the helpful comments and suggestions of the reviewers, which have improved the presentation.

REFERENCES

1. P.L.Antonelli, R.Ingarden and M.Matsumoto, The Theory of Sprays and Finsler Spaces. with Applications in Physics and Biology, Kluwer Acad. Publ., Netherlands (1993).
2. X.Bai, Y.Shi and Y.Liu, Reliability Analysis of Competing Risks with Masked Failure Causes Based on Progressive Type-II Censoring with Random Removals, Journal of Physical Sciences, 22 (2017) 1-11.
3. S.Deng and Z.Hou, The group of isometries of a Finsler space, Pacific. J. Math, 207(1) (2002) 149-155.
4. S.Kobayashi and K.Nomizu, Foundations of Differential Geometry II, Interscience Publishers, New York, 1969.
5. O.Kowalski and L.Vanhecke, Riemannian manifolds with homogeneous geodesics, Boll.Un. Mat. Ital. 5 (1991) 189-246.
6. D.Latifi, Homogeneous geodesics in homogeneous Finsler spaces, J. Geom. Phys. 57(5) (2007) 1421-1433.
7. D.Latifi, Naturally reductive homogeneous Randers spaces. J. Geom. Phys. 60(12) (2010) 1968-1973.
8. M.Matsumoto, Theory of Finsler spaces with m-th root metric. II, Publ. Math. Debrecen.49 (1996) 135-155.
9. M.Matsumoto and S.Numata, On Finsler spaces with a cubic metric, Tensor (N.S.) 33(2) (1979) 153-162.
10. M.Matsumoto and H.Shimada, On Finsler spaces with 1-form metric. II. Berwald-Moor's metric $L=(y^1y^2...y^n)^{\frac{1}{n}}$ Tensor (N.S.) 32(3) (1978) 275-278.
11. M.Matsumoto and H.Shimada, On Finsler spaces with 1-form metric. Tensor (N.S.) 32(2) (1978) 161-169.
12. K.Nomizu, Invariant Affne Connections on Homogeneous Spaces, Am. J. Math. 76 (1954) 33-65.
13. S.Rezvani, Transformations in Fuzzy Planes, Journal of Physical Sciences, Vol. 14, 1-11, 2010.
14. H.Shimada, On Finsler spaces with metric $L=\sqrt[m]{a_{i_1i_2...i_m}y^{i_1}y^{i_2}...y^{i_m}}$, Tensor, N.S.33 (1979) 365-372.
15. A.Tayebi, On generalized 4-th root metrics of isotropic scalar curvature, Mathematica Slovaca, 68 (2018) 1-22.

16. A.Tayebi and A.Alipour, On distance functions induces by Finsler metrics, Publ Math Debrecen. 90 (2017) 333-357.
17. A.Tayebi and B.Najafi, On m-th root Finsler metrics, J. Geom. Phys. 61 (2011) 1479-1484.
18. A.Tayebi and B.Najafi, On m-th root metrics with special curvature properties, C. R. Acad. Sci. Paris, Ser. I, 349 (2011) 691-693.
19. A.Tayebi and H.Sadeghi, Generalized P-reducible (α,β) -metrics with vanishing S-curvature, Ann. Polon. Math., 114(1) (2015) 67-79.
20. J.M.Wegener, Untersuchungen der zwei- und dreidimensionalen Finslerschen RÄaume mitder Grundform $L=\sqrt[3]{a_{ikl}x'^{i}\ x'^{k}x'^{l}}$, Koninkl. Akad.Wetensch., Amsterdam, Proc, 38 (1935) 949-955.