A COMPREHENSIVE STUDY ON LATTICE UNDER UNCERTAIN ENVIRONMENTS

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Dedicated to

my

Parents, wife and my beloved daughter aishi

CERTIFICATE

This is to certify that the thesis entitled **"A Comprehensive Study on Lattice Under Uncertain Environments"**, submitted by **Susanta Bera** for the award of the degree of Doctor of Philosophy to the Vidyasagar University is an authentic record of bonanza original research work carried out under my supervision and guidance. The thesis is worthy of consideration for awarding the degree of Doctor of Philosophy and it satisfies and fulfil the requirements in accordance with regulation of the institution.
It is also certified that this original thesis has not been submitted to any other University or Institution for any degree/diploma.

Date:

Supervisor
(Sankar Kumar Roy)

DECLARATION

I certify that

a. The work contained in the thesis is original and has been done by myself under the general supervision of my supervisor.

b. The work has not been submitted to any other Institute/University for any degree or diploma.

c. I have followed the guidelines provided by the University in writing the thesis.

d. I have conformed to the norms and guidelines given in the Ethical Code of Conduct of the University.

e. Whenever I have used materials (data, theoretical analysis, and text) from other sources, I have given due credit to them by citing them in the text of the thesis and giving their details in the references.

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Signature of the Student

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Abstract

Many situations occur where exact solution cannot be found for real world problems and hence we solved these approximately. Probability theory, fuzzy set theory (FST), rough set theory (RST), soft set theory (SST) are novel mathematical tools to solve real world problems including uncertainty. All these theories more or less can help to understand and make dealt to imperfect knowledge. The basic philosophy of rough set is based on the assumption that human knowledge about a universe depends upon their capability to classify its objects. Every object of the universe there is associated a certain amount of information (data). Classifications of a universe and equivalence relations defined on the universe are known to be interchangeable notions. So, for mathematical point of view, equivalence relations are considered to define rough set. For every rough set, we associate two crisp sets, called lower and upper approximations and viewed as the sets of elements which certainly and possibly belong to the set. The advantage of rough set method is that it does not need any additional information about data, like probability in statistics, or membership in fuzzy set theory.

Lattice is a simple algebraic structure since the basic concepts of the theory which include only orders, least upper bound and greatest lower bound. It is widely discussed and studied in classical algebraic theory. Lattice theory is used to formulate some types of generalised rough sets. Lattice theory extensively used in computer science and engineering. Based on the existing works about the algebraic structure of rough sets and soft sets we say that lattice structures of rough set and soft set have wide field of applications. Despite its novelty, the theory and its extensions have been widely applied to many problems, including decision analysis, data-mining, intelligent control and pattern recognition.

In this thesis, we incorporate a detailed study on rough set in lattice. That is, we are trying to make an algebraic connection between the rough set and the lattice. We introduce rough modular lattice and rough distributive lattice in rough set environment by considering Pawlak's approximation space. Beside this, concept of rough ideal and rough homomorphism are established in rough set environment.

The equivalence classes are the basic building blocks for the construction of the lower and upper approximations. In real-life situation, it is difficult to find an equivalence relation. To over come this difficulty, many researchers approached to use tolerance relation, similarity relations, dominance relation to partition a universe and introduced generalized rough set. Soft set theory (SST) is introduced by Molodtsov (67), is a novel concept to deal with uncertainty. One main advantage of SST is that it has enough parameterization tools. It is well known that each equivalence relation on a set partitions the set into disjoint classes and each partition of the set provide us an equivalence relation on the set. In this thesis, we use a soft set instead of an equivalence relation to partition a set and then introduce the notion of soft approximation space. We attempt to study on lattice theory in the soft approximation space. In this thesis, we present the concepts of soft rough lattice and soft rough ideal in the soft rough approximation space.

Soft rough set is presented by Feng et al. (32) which is a hybridization of soft set and rough set. Modified soft rough (MSR) set is an another approach to hybridization of soft set and rough set. Considering modified soft rough approximation space, we approximate a soft set and hereby introduce the notion of rough soft set. We discuss lattice theory in the frame-work of rough soft theory.

In this research content, we introduce soft set relation in a new way. We incorporate lattice theory in soft set using this soft set relation. We consider the study of congruence relation on soft set. Considering lattice as universal set, we introduce soft congruence relation in soft set. The concepts of upper and lower approximations of a subset in a lattice are depicted based on this soft congruence relation.

At last, we try to make a fusion between fuzzy set and rough soft set. Here we measure roughness of rough soft set and introduce the concept of fuzzy rough soft set in MSR-approximation space. Some properties of fuzzy rough soft set are derived. Moreover, lattice theory is studied on the fuzzy rough soft set. In the whole thesis, we are trying to incorporate lattice theory in uncertain environments. Lattice theory in different approximation spaces is studied. Finally, conclusion and direction of future works of the study are presented.

Keywords: Rough set, Soft set, Fuzzy set, Approximation space, Rough lattice, Rough modular lattice, Rough distributive lattice, Rough ideal, Rough homomorphism, Soft approximation space, Soft rough set, MSR-approximation

space, Soft rough lattice, Soft rough ideal, Cartesian product, Soft set relation, Soft congruence relations, Rough soft set, Fuzzy rough soft set, Lattice.

Abbreviations

DM	Decision Making
RML	Rough Modular Lattice
RDL	Rough Distributive Lattice
MSR	Modified Soft Rough
RST	Rough Set Theory
FST	Fuzzy Set Theory
SST	Soft Set Theory
Bnd	Boundary Region
Pos	Positive Region
Neg	Negative Region
IND	Indiscernibility
min	Minimum
max	Maximum
lcm	Least common multiple
gcd	Greatest common divisor
sup	Supremum
inf	Infimum

List of Symbols

$A^\star$	Upper approximation
$A_\star$	Lower approximation
$\overline{apr}$	Soft upper approximation
$\underline{apr}$	Soft lower approximation
$\sqcup$	Soft rough union
$\sqcap$	Soft rough intersection
$\cup$	Set union
$\cap$	Set intersection
$\emptyset$	Null set
U	Universal set
$\triangle$	Infimum between soft sets
∇	Supremum between soft sets
$\leq$	Less than or equal to
$\geq$	Greater than or equal to
$\wedge$	Meet
$\vee$	Join
$\bigwedge$	AND
$\bigvee$	OR
$\cap_{\Re}$	Restricted intersection between two soft sets
$\cup_{\Re}$	Restricted union between two soft sets
$\sqcap_{\varepsilon}$	Extended intersection between two soft sets
R_ρ	Soft set relation induced by ρ
$\in$	Belong to
$\neq$	Not equal
$\subseteq$	Subset
$\sqsubseteq$	Soft rough inclusion relation
$\cong$	Equivalence
$\times$	Cartesian product
$\Rightarrow$	Implies
$\Leftrightarrow$	Implies and Implied by
$I(L)$	Set of all ideals of a lattice L
$S_R(U)$	Set of all soft rough sets over U
$S_r(X)$	Soft rough set of X

Contents

Chapter 1

Introduction and Literature Survey

The chapter describes the introduction of Rough Set Theory, Soft Set Theory, Fuzzy Set Theory, some preliminary definitions on Rough Set, Soft Set, Lattice, and Literature Survey of the proposed work, objective, scope and organization of the thesis.

1.1 Introduction

Rough Set Theory (RST), proposed by Pawlak (81) is a mathematical tool for dealing with uncertainty i.e., to vagueness (or imprecision). It provides a simple approach for the study of indiscernibility of objects. Basically, in rough set theory, it is assumed that our knowledge is restricted by an indiscernibility relation. When objects of a universe are described by a set of attributes, one may define the indiscernibility of objects based on their attribute values. When two objects have the same value over a certain group of attributes, we say they are indiscernible with respect to this group of attributes, or have the same description with respect to the indiscernibility relation. Indiscernibility relation is an equivalence relation. By this equivalence relation, we form equivalence class and all the equivalence classes form a partition of the universe, which are the basic building block of universal set called granules. Any subset of objects of the universe is approximated by two sets, called the lower and upper approximations and can be viewed as the sets of elements which certainly and possibly belong to the set. Pair of these two approximations is called Rough Set. Pawlak's rough set is mainly based on equivalence relation. But, in practical, it is very difficult to find an equivalence relation among the elements of a set. So, some other general relations such as tolerance ones and dominance ones are considered to define rough set models [(132), (135)].

In the original rough set approach it has considered that all objects in an information system have precise attribute values. Problem arises when some of the values are unknown, which sometimes happen in the real world. Therefore it is necessary to develop theory which enables classifications of objects even if there is only partial information available. The rough set model proposed by Kryszkiewicz [(51), (52)] introduced indiscernibility based on tolerance relation to deal with missing values in the information system. In these approaches, a missing value was considered as a special value that may take any possible value. However, tolerance relation sometimes lead to a poor result with respect to approximation. Stefanowski and Tsoukiàs [(111), (112)] discussed the limitation and introduced similarity relation to refine the results obtained by using tolerance relation approach. Wang (118) gave some examples to prove that similarity relation may results in lost information and proposed limited tolerance relation. Yang and Hu (126) also generalized a reasonable and flexible classification in incomplete information system by "new binary relation".

Rough set theory, has been hybridized with other soft computing methods such as fuzzy sets by Zadeh (133), soft set by Molodtsov (67), genetic algorithms (GAs), neural networks, and statistical methods such as principal component analysis (PCA) by Devijver and Kittler (28), etc. Such hybridization has highlighted the value of employing rough set theory. Fuzzy-Rough attempts to take advantage of the complementary nature of fuzzy sets and rough sets. The significance of this work is reflected in the level of research carried out in this area and also to the number of applications of fuzzy-rough set theory.

The rough set theory is of fundamental importance in artificial intelligence (AI) and cognitive science especially in the area of machine learning, knowledge acquisition and decision analysis, knowledge discovery, inductive reasoning and pattern recognition in databases, expert systems, decision support systems. It has been applied to the analysis of many issues, including medical diagnosis, engineering reliability, empirical study of material data, machine diagnosis, travel demand analysis, data mining. In the areas of machine learning, data mining, pattern recognition, and intelligent control, the ability to handle imperfect knowledge is of primary importance both in terms of theoretical advancement and practical applications. The work in the area of rough set theory (RST) by Pawlak (84) offered one of the most distinct and recent approaches in this respect. Such is the worldwide nature of the attention that RST has attracted since its inception by Pawlak (81) that much research and development have been carried out not only in applying the theory to many and various problem domains, but also to extend it theoretically. This has resulted in a significant breadth and depth of work in the area.

Both fuzzy set theory and rough set theory deal with the indescribable and perception knowledge. The most difference between them is that rough set theory is not required to be considered membership function so that it can avoid pre-assumption and subjective information in analysis. Rough set theory provides a new different mathematical approach to analyze the uncertainty, and with rough sets we can classify imperfect data or information easily. We can discover the results in terms of decision rules. Formally, a rough set is the approximation of a vague concept (set) by a pair of precise concepts, called lower and upper approximations, which are a classification of the domain of interest into disjoint categories. The lower approximation is a description of the domain objects which are known with certainty to belong to the concept of interest, whereas the upper approximation is a description of the objects which possibly belong to the concept. The approximations are constructed with regard to a particular subset of attributes or features. One of the primary drawbacks of RST lies in its inability to deal with real world data. Owing mainly to the granular approach that RST uses to handle data, and the strict structure of equivalence imposed, it does not allow any flexibility when dealing with measurement noise or imperfection that is prevalent in real world data. However, most data sets contain real-valued features and so it becomes necessary to perform a discretization step before employing RST for knowledge discovery. Take for instance a weather forecasting system which records a number of meteorological attributes, with one in particular that might be average rainfall. In reality, this is a continuous and real-valued measurement. However, in order to apply RST to such a problem, this attribute must be discretized with a set of labels such as light, medium, and heavy. This imposes subjective human judgement on what is otherwise an objective measurement. In particular, the more recent extensions of RST and its hybridization with soft set theory are examined in lattice theoretical approach. To solve complicated problems in economics, engineering, environmental science, medical science, and social science, methods in classical mathematics are not always successfully used because various uncertainties are typical for these problems. Therefore, there has been a great deal of alternative research and applications in the literature concerning some special tools such as probability theory, fuzzy set theory [(133), (134)] , rough set theory [(81), (84)], vague set theory (40). However, all of these theories have their own difficulties which are pointed out in [(67), (68)]. In 1999, Molodtsov (67) introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainties. Molodtsov (68) further pointed out that the reason for these limitations is, possibly, the inadequacy of the parameterization tool of the theory. The soft set theory (SST) introduced by Molodstov (67) is quite different from these theories in this context. The absence of any restriction on the approximate description in Soft Set Theory makes this theory very convenient and easily applicable. Fuzzy set theory was proposed by Zadeh (133) in 1965 is considered as a special case of the soft sets. Fuzzy set theory, being generalization of crisp sets, should satisfy the axioms of exclusion and contradiction. The definition of complement has defined by Zadeh which does not meet these requirements. But it has been proved recently by Baruah [(7), (8)] that the set theoretic axioms of exclusion and contradiction in the fuzzy sets satisfied.

In order to solidify the theory of soft set, Maji et al. (60) defined some basic terms of the theory such as equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set, and absolute soft set with examples. Binary operations like AND, OR, union and intersection were also defined. De Morgan's laws and a number of results are verified in soft set theory context. Sezgin and Atagun (106) proved that certain De Morgan's laws holds in soft set theory with respect to different operations on soft sets. Ali et al. (2) introduced some new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets. They improved the notion of complement in soft set and also proved that certain De Morgan's law hold in soft set theory. Soft set is a parameterized general mathematical tool which deals with a collection of approximate descriptions of objects. Each approximate description has two parts, a predicate and an approximate value set. In classical mathematics, a mathematical model of an object is constructed and is defined the notion of exact solution of this model. Usually the mathematical model is too complicated and the exact solution is not easily obtained. So, the notion of approximate solution is introduced and the solution is calculated. In the soft set theory, we have the opposite approach to this problem. The initial description of the object has an approximate nature, and we do not need to introduce the notion of exact solution. The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable in practice. Any parameterizations can be used with the help of words and sentences, real numbers, functions, mappings and so on. Maji et al. (59) applied Soft set theory to solve a decision making problem using rough set theory; and an algorithm to select the optimal choice of an object was provided. This algorithm uses fewer parameters to select the optimal object for a decision problem. However, in decision making problem in (59), there is a straightforward relationship between the decision values of objects and the conditional parameters. That is, the decision values is computed with respect to the conditional parameters. This is quite different in the case of rough sets. In rough set theory, the decision attributes are not computed according to the conditional attributes. Soft set theory, fuzzy set theory and rough set theory are all mathematical tools to handle with uncertainty. But soft set, fuzzy set and rough set are closely related concepts (2). Feng et al. (31) provided a framework to combine fuzzy sets, rough sets and soft sets all together, which provides the several interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. The study of the algebraic structure of the mathematical theory proves itself effective in making the applications more efficient. Studying on rough set, soft set and rough soft set combined with lattice is an interesting topic to the researchers.

1.2 Preliminaries

Here we give some preliminaries on rough set, soft set, and lattice which are very much essential in the sequel. Also we discuss in short about information system and indiscernibility relation which are two key elements related to uncertainty.

1.2.1 Information Systems

The basic tool for data representation in the rough set framework is an information system. An information system (or a knowledge representation system) is a pair, $I = (U, A)$ of non-empty finite sets U and A where U is the set of objects that are named as universal set and A is a set of attributes. Every attribute $a \in A$ is a function $a : U \to V_a$, where V_a is called set of values of attribute a.

An information system is represented by a table where each row represents a condition, an event, or simply, an object. Each column represents a measurability characteristic of each object. Another type of information systems is called the Decision Systems. A decision systems is a specific type of pair, $I = (U, A \cup \{d\})$ of any other information system where $d \in A$ is decision attribute. Other attributes are called as $a \in A - \{d\}$ conditional attributes. Decision attributes can receive many values, but in general, they will earn values as true or false. It may happen that some of attribute values for an object are missing. To indicate such a situation when some of attribute values for an object are missing a distinguished value, so-called null value, is usually assigned to those attributes. If V_a contains null value for at least one attribute $a \in A$, then the information system is called an incomplete information system, otherwise it is complete.

1.2.2 Indiscernibility Relation

Two objects are considered to be indiscernible or equivalent if and only if they have the same value for all attributes in the set. As a dual relation to indiscernibility, two objects are considered to be discernible if and only if they have different values for at least one attribute. Since the pair of indiscernibility and discernibility relations are defined with respect to the set of all attributes and at least one attribute, respectively, they may be viewed as strong indiscernibility and weak discernibility. The strong indiscernibility is indeed the strongest type of similarity between objects and is characterized by an equivalent. The rough set theory was introduced by Pawlak (81) dealt with situations in which the objects of a certain universe of discourse U can be identified only within the limits determined by the knowledge represented by a given indiscernibility relation. Based on such indiscernibility relation, the lower and the upper approximations of subsets of U may be defined. The lower and the upper approximations of a subset X of U can be viewed as the sets of elements which certainly and possibly belong to X, respectively. Usually it is pre- assumed that indiscernibility relations are equivalence.

1.2.3 Rough set

Definition 1.2.1 *Let U be a non-empty set of universe and ρ be an equivalence relation on U. The pair (U, ρ) is called Pawlak's approximation space.*

The equivalence relation R is often called indiscernibility relation and related to an information system.

Definition 1.2.2 *An equivalence class of $x(\in U)$ is denoted by $[x]_\rho$ and defined as follows: $[x]_\rho = \{y \in U : x\rho y\}$, where $x\rho y$ imply $(x, y) \in \rho$.*

Each equivalence class is partitioned the universe which are called granules. A partition of the universe is used to approximate a subset of the universal set in rough set theory. Using only the indiscernibility relation, in general, we are not able to observe individual objects from U but only the accessible granules of knowledge described by this relation.

Definition 1.2.3 *Lower and upper approximations of $X \subseteq U$ in an approximation space (U, ρ) are denoted by $A_\star(X)$ and $A^\star(X)$ respectively and defined as follows:*

$$A_\star(X) = \{x \in U : [x]_\rho \subseteq X\} \textit{ and } A^\star(X) = \{x \in U : [x]_\rho \cap X \neq \phi\}.$$

If $A_\star(X) \neq A^\star(X)$, then X is called rough, otherwise X is crisp.
The difference $A^\star(X) - A_\star(X)$ is called boundary region. A rough set of X is presented by the pair $(A_\star(X), A^\star(X))$. The pair (U, ρ) is called Pawlak's approximation space.

Therefore lower approximation, upper approximation and boundary region can be explained as follows:

• Lower approximation of a set X is the set of all objects which can be with certainty classified as members of X.

• Upper approximation of a set X is the set of all objects which can be only classified as possibly members of X.

• Boundary region of a set X is the set of all objects which can be decisively classified neither as members of X nor as members of X^c (X^c is the complement of X).

The definitions of set approximations presented above can be expressed in terms of granules of knowledge in the following way:

• The lower approximation of a set is union of all granules which are entirely included in the set.

• The upper approximation of a set is union of all granules which have non-empty intersection with the set.

• The boundary region of a set is the difference between the upper and the lower approximation of the set.

Graphical illustration of the set approximations is given in Figure 1.2.1

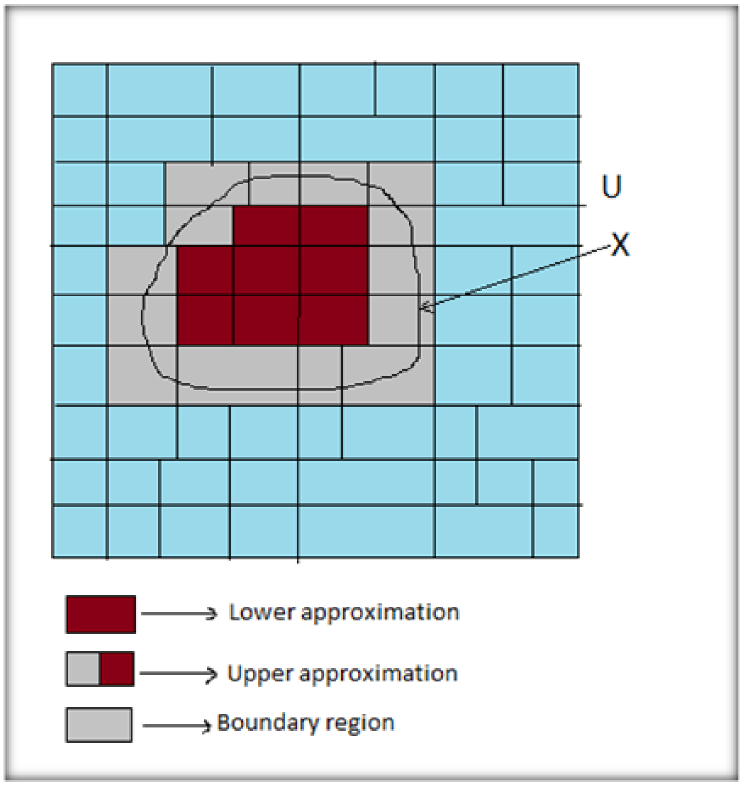

Figure 1.2.1: Upper and Lower Approximations of a Rough Set.

There are four types of rough set in U:

[1] If $A_\star(X) \neq \phi$ and $A^\star(X) \neq U$, X is called roughly definable in U;

[2] If $A_\star(X) = \phi$ and $A^\star(X) \neq U$, X is called internally undefinable in U;

[3] If $A_\star(X) \neq \phi$ and $A^\star(X) = U$, X is called externally undefinable in U;

[4] If $A_\star(X) = \phi$, and $A^\star(X) = U$, X is called totally undefinable in U, where ϕ denotes an empty set.

Proposition 1.2.1 *Let $X, Y \in U$, then in an approximation space (U, ρ), One can easily show the following properties of approximations:*

(1) $A_\star(X) \subseteq X \subseteq A^\star(X)$

(2) $A_\star(\phi) = A^\star(\phi) = \phi$, $A_\star(U) = A^\star(U) = U$

(3) $A^\star(X \cup Y) = A^\star(X) \cup A^\star(Y)$

(4) $A_\star(X \cap Y) = A_\star(X) \cap A_\star(Y)$

(5) $X \subseteq Y \Rightarrow A_\star(X) \subseteq A_\star(Y)$ *and* $A^\star(X) \subseteq A^\star(Y)$

(6) $A_\star(X \cup Y) \supseteq A_\star(X) \cup A_\star(Y)$

(7) $A^\star(X \cap Y) \subseteq A^\star(X) \cap A^\star(Y)$

(8) $A_\star(X^c) = (A^\star(X))^c$

(9) $A^\star(X^c) = (A_\star(X))^c$

(10) $A_\star(A_\star(X)) = A^\star(A_\star(X)) = A_\star(X)$

(11) $A^\star(A^\star(X)) = A_\star(A^\star(X)) = A^\star(X)$.

1.2.4 Soft set

Rough set and fuzzy set are two novel concept to dealing with uncertainty. But there exists an inherent difficulty in case of fuzzy set: how to set the membership function in each particular case. We should not impose only one way to set the membership function. The nature of the membership function is extremely individual. So, fuzzy set operations based on the arithmetic operations with membership functions do not look natural. It may occur that these operations are similar to the addition of weights and lengths. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theory. Molodtsov (67) introduced a mathematical tool namely soft set for dealing with uncertainties which is free from the difficulties mentioned above. Here we present some basic notions about soft set.

Definition 1.2.4 *Let U be an universal set and E be a set of parameters. Let $P(U)$ denote the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a set valued function given by $F : A \to P(U)$.*

In other words, the soft set is a parameterized family of subsets of the set U. Every set $F(e)$, $e \in A$, from this family may be considered as the set of e approximate elements of the soft set (F, E).

Example 1.2.1 *Let (F, A) describe the group of students interested in different subjects. Suppose there are four students in the universe U which is given by $U = \{u_1, u_2, u_3, u_4\}$ and the attribute set $A = \{e_1, e_2, e_3\}$ where e_1 stands for Mathematics, e_2 stands for Physics, e_3 stands for English. Suppose*

$$F(e_1) = \{u_1, u_2, u_3\}, F(e_2) = \{u_1, u_2\}, F(e_3) = \{u_2, u_3, u_4\}.$$

Thus the soft set over U is given by

$$(F, A) = \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_1, u_2\}), (e_3, \{u_2, u_3, u_4\})\}.$$

Definition 1.2.5 *A soft set over U is said to be null soft set denoted by F_ϕ if $\forall\ e \in E,\ F(e) = \phi$.*

Definition 1.2.6 *A soft set over U is said to be absolute soft set denoted by F_U if $\forall\ \ e \in E,\ F(e) = U$.*

Definition 1.2.7 *A soft set (F, A) is said to be soft subset of a soft set (G, B), if* *(i)* $A \subseteq B$,

(ii) $\forall\ a \in A,\ F(a) = G(a)$.

Definition 1.2.8 *(60) The union of two soft sets* (F, A) *and* (G, B) *over a common universe* U *is a soft set* (H, C) *where* $C = A \cup B$ *and* $\forall\ e \in C$, H *is defined as follows:*

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(H, C) = (F, A) \sqcup (G, B)$ *where the symbol '*$\sqcup$*' stands for union of two soft sets.*

Ali et al. (2) denoted this union as extended union between two soft sets and they used the symbol $\cup_{\varepsilon}$.

Definition 1.2.9 *(2) The intersection of two soft sets* (γ, A) *and* (δ, B) *over a common universe* U *is a soft set* (Y, D) *where* $D = A \cup B$, *and* $\forall\ g \in D$, *and* Y *is described as follows:*

$$Y(g) = \begin{cases} \gamma(e), & \text{if } g \in A - B, \\ \delta(g), & \text{if } g \in B - A, \\ \gamma(g) \cap \delta(g), & \text{if } g \in A \cap B. \end{cases}$$

We write $(Y, D) = (\gamma, A) \sqcap_{\varepsilon} (\delta, B)$ *where the notation '*$\sqcap_{\varepsilon}$*' defines for intersection of two soft sets. Ali et al. (2) denoted this intersection as extended intersection between two soft sets.*

Definition 1.2.10 *(60) The intersection of two soft sets* (F, A) *and* (G, B) *over a common universe* U, *denoted by* $(F, A) \cap (G, B)$ *is the soft set* (H, C) *where* $C = A \cap B$ *and for all* $e \in C$, $H(e) = F(e) \cap G(e)$.
This intersection is also called restricted intersection by Ali et al. (2) and they used the symbol $\cap_{\Re}$.

Definition 1.2.11 *(2) The restricted union of two soft sets (F, A) and (G, B) over a common universe U, denoted by $(F, A) \cup_{\Re} (G, B)$ is the soft set (H, C) where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cup G(e)$.*

Definition 1.2.12 *(60) Let $E = \{e_1, e_2, e_3, ..., e_n\}$ be a set of parameters. The NOT set of E denoted by $\neg E$ is defined by, $\neg E = \{\neg e_1, \neg e_2, \neg e_3, ..., \neg e_n\}$, where $\neg e_i = not\ e_i$; $\forall i$.*

Definition 1.2.13 *(60) The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \neg A)$, where $F^c : \neg A \to P(U)$ is a mapping given by $F^c(\neg\alpha) = U - F(\alpha)$, $\forall\ \neg\alpha \in \neg A$.*

Definition 1.2.14 *(2) The relative complement of a soft set (F, A) is denoted by $(F, A)^r$ and is defined by $(F, A)^r = (F^r, A)$, where $F^r : A \to P(U)$ is a mapping given by $F^r(e) = U - F(e)$ for all $e \in A$.*

Ali et al., (2) gave the following De Morgan's laws with respect to the relative complement of a soft set in soft set theory.

Proposition 1.2.2 *Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$, then*
(1) $((F, A) \cap_{\Re} (G, B))^r = (F, A)^r \cup_{\Re} (G, B)^r$ and
(2) $((F, A) \cup_{\Re} (G, B))^r = (F, A)^r \cap_{\Re} (G, B)^r$ hold true.

The following De Morgan's laws hold in soft set theory for the extended intersection, the extended union and the complement.

Proposition 1.2.3 *Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$, then*
(1) $((F, A) \cap_{\varepsilon} (G, B))^r = (F, A)^r \cup_{\varepsilon} (G, B)^r$ and
(2) $((F, A) \cup_{\varepsilon} (G, B))^r = (F, A)^r \cap_{\varepsilon} (G, B)^r$ hold.

Definition 1.2.15 *(5) Let* (F, A) *and* (G, B) *be two soft sets over* U*, then the Cartesian product of* (F, A) *and* (G, B) *is defined as,* $(F, A) \times (G, B) = (H, A \times B)$*, where* $H : A \times B \to P(U \times U)$ *and* $H(a, b) = F(a) \times G(b)$*, where* $(a, b) \in A \times B$*. i.e.,* $H(a, b) = \{(h_i, h_j) : h_i \in F(a), h_j \in G(b)\}$*.*
The Cartesian product of three or more non-empty soft sets can be defined by generalizing the definition of the Cartesian product of two soft sets. The Cartesian product $(F_1, A) \times (F_2, A) \times (F_3, A) \times ... \times (F_n, A)$ *of the non-empty soft sets* $(F_1, A), (F_2, A), (F_3, A), ..., (F_n, A)$ *is the soft set of all ordered n-tuple* $(h_1, h_2, h_3, ..., h_n)$ *where* $h_i \in F_i(a)$*.*

Definition 1.2.16 *(60) Let* (F, A) *and* (G, B) *be two soft sets over the common universe* U*. Then* (F, A) *AND* (G, B) *denoted by* $(F, A) \bigwedge (G, B)$ *and is defined by* $(F, A) \bigwedge (G, B) = (H, A \times B)$ *where* $H(a, b) = F(a) \cap G(b)$*, for all* $(a, b) \in A \times B$*.*

Definition 1.2.17 *(16) Let* (F, A) *and* (G, B) *be two soft sets over the common universe* U*. Then* (F, A) *OR* (G, B) *denoted by* $(F, A) \bigvee (G, B)$ *and is defined by* $(F, A) \bigvee (G, B) = (H, A \times B)$ *where* $H((a, b)) = F(a) \cup G(b)$*, for all* $(a, b) \in A \times B$*.*

Proposition 1.2.4 *If* (F, A)*,* (G, B) *and* (H, C) *are three soft sets over* U*, then*

(i) $(F, A) \bigvee ((G, B) \bigvee (H, C)) = ((F, A) \bigvee (G, B)) \bigvee (H, C)$,

(ii) $(F, A) \bigwedge ((G, B) \bigwedge (H, C)) = ((F, A) \bigwedge (G, B)) \bigwedge (H, C)$,

(iii) $(F, A) \bigvee ((G, B) \bigwedge (H, C)) = ((F, A) \bigvee (G, B)) \bigwedge ((F, A) \bigvee (H, C))$,

(iv) $(F, A) \bigwedge ((G, B) \bigvee (H, C)) = ((F, A) \bigwedge (G, B)) \bigvee ((F, A) \bigwedge (H, C))$.

1.2.5 Posets and Lattices

Lattice is a simple algebraic structure whose basic philosophy includes only orders, least upper bound and greatest lower bound. It is widely discussed and studied in classical algebraic theory. Lattice is very much useful in dealing with different structures in theoretical computer science. Partial order relation and lattice theory now play an important role in many disciplines of computer science and engineering. For example, they have various applications in distributed computing (vector clocks, global predicate detection), concurrency theory, programming language semantics (fixed-point semantics), and data mining (concept analysis). They are also useful in other disciplines of mathematics such as combinatorics, number theory and group theory. A partially ordered set (or a poset for short), is a non-empty set B equipped with a transitive, reflexive and antisymmetric relation $\leq$. B is totally ordered, or a chain, if all elements of B are comparable under $\leq$ (that is, $x \leq y$ or $y \leq x$ for all $x, y \in B$). The lattice is one of the most widely discussed and studied structure in the classical algebraic theory, both as a specific algebra with a carrier and two binary operations, and as a relational structure a specific ordered set (37). In mathematics, a lattice is a partially ordered set in which any two elements have a unique supremum and an infimum. Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities. Since the two definitions are equivalent, lattice theory draws on both order theory and universal algebra. Here we present some definitions and properties related to lattice which are taken from [(21), (37)].

Definition 1.2.18 *Let P be a non-empty set. An order (or a partial order) on P is a binary relation $\leq$ such that for all $a, b, c \in P$,*
(i) $a \leq a$ (reflexive)
(ii) $a \leq b$ and $b \leq a \Rightarrow a = b$ (anti-symmetric) and

(iii) $a \leq b$ *and* $b \leq c \Rightarrow a \leq c$, *(transitive).*

Definition 1.2.19 *A set* P *together with an order relation* $\leq$ *is said to be an ordered set (or a partially ordered set or poset).*

If $a \leq b$ in a poset, we say a and b are comparable. Two elements of a poset may or may not be comparable.

Definition 1.2.20 *If* P *is a poset in which every two elements are comparable it is called a totally order set or a toset or a chain.*

Thus if P is a chain and $x, y \in P$ then either $x \leq y$ or $y \leq x$.

Definition 1.2.21 *Let* $(P, \leq)$ *be an ordered set. If there exists an element* $a \in P$ *such that,* $x \leq a$ *for all* $x \in P$*, then* a *is called greatest element of* P*.*
The least element of P *is defined dually.*

Definition 1.2.22 *An element* $x \in P$ *is said to be a maximal element of* P *if* $x < a$ *for no* $a \in P$*.*

Thus the greatest element is comparable with all elements of the poset, a maximal element need not be so.

Definition 1.2.23 *An element* $y \in P$ *is said to be a minimal element of* P *if* $b < y$ *for no* $b \in P$*.*

Definition 1.2.24 *Let* S *be a non-empty subset of a poset* P*. An element* $a \in P$ *is called an upper bound of* S*, if* $x \leq a$*, for all* $x \in S$*.*
Furthermore if a *is an upper bound of* S *such that,* $a \leq b$ *for all upper bounds* b *of* S *then* a *is called least upper bound or Supremum of* S*.*

It is clear that there can be more than one upper bound of a set. But supremum if it exists, will be unique. Again comparing with the definition of greatest element, we notice whereas the greatest element belongs to the set itself, an upper bound or the supremum can lie outside the set. In fact if the supremum lies in the set, it will be nothing but the greatest element.

Definition 1.2.25 *Let S be a non-empty subset of a poset P. An element $a \in P$ is called an lower bound of S, if $a \leq x$, for all $x \in S$ and a be called the greatest lower bound or Infimum of S, if $b \leq a$ for all lower bounds b of S.*

Let S be a non-empty subset of a poset P. If S has a least upper bound, this is called the supremum of S and is denoted by $supS$. Similarly, if S has a greatest lower bound, this is called the infimum of S and is written as $infS$. We write $a \vee b$ (read as "a join b") in place of $sup\{a, b\}$ and $a \wedge b$ (read as "a meet b") in place of $inf\{a, b\}$.

The lattice as a poset will be denoted by $(L, \ \leq)$, and the lattice as an algebra by $(L, \wedge, \vee)$. We write simply L to denote the lattice in both senses.

Definition 1.2.26 *A poset $(L, \ \leq)$ is a lattice if $sup\{a, b\}$ and $inf\{a, b\}$ exist for all $a, b \in L$.*

Lattice as an algebra is defined as follows:

Definition 1.2.27 *A non-empty set L together with a binary relation '$\leq$' is said to be a lattice if it satisfy the following conditions:*
(i) $a \leq a$ for all $a \in L$ (reflexivity)
(ii) If $a \leq b$ and $b \leq a$ then $a = b$, $\quad a, b \in L$ (anti-symmetricity)
(iii) If $a \leq b$ and $b \leq c$ then $a \leq c$, $\quad a, b, c \in L$ (transitivity).

Definition 1.2.28 *Let* $(L, \vee, \wedge)$ *be a lattice and* $A \subseteq L$. *Then* A *is a sublattice if* $a \in A$, $b \in A$ *imply* $a \vee b \in A$ *and* $a \wedge b \in A$, *where the symbols* $\vee$ *and* $\wedge$ *stand for supremum and infimum respectively.*

Definition 1.2.29 *Let* $(P, \ \leq)$ *be a non-empty ordered set. Then* $(P, \ \leq)$ *is called join-semilattice, if for all* $a, b \in P$, *the join* $a \vee b$ *exists. Similarly* $(P, \ \leq)$ *is called meet-semilattice, if for all* $a, b \in P$, *the meet* $a \wedge b$ *exists. Furthermore,* $(P, \ \leq)$ *is a lattice if it is both a join and a meet-semilattice.*

Definition 1.2.30 *A semi-lattice is a poset* $(S, \ \leq)$ *in which every non-empty finite subset has an infimum (inf). A sub-semi lattice is an ordered set* S *in which every non-empty finite subset has a supremum (sup). An ordered set which is both a semi-lattice and a sub-semi lattice is called a Lattice. The empty* inf *if it exists, is the same as* $supS$, *is called top element of* S *and is written as* 1_S. *The empty sup, if it exists, is the same as* $infS$, *is called bottom element of* S *and is written as* 0_S.

Definition 1.2.31 *A lattice* $(L, \ \leq)$ *is bounded if it has top and bottom elements.*

Definition 1.2.32 *A lattice* L *is said to be complete lattice if for every non-empty subset* X *of* L *has a least upper bound and a greatest lower bound in* X.

Definition 1.2.33 *Let* $(P, \ \leq)$ *and* $(Q, \ \leq)$ *be posets. A maping* $f : P \to Q$ *is*

(a) a join-morphism if whenever $a, b \in P$ *and* $a \vee b$ *exists in* P, *then* $f(a) \vee f(b)$ *exists in* Q *and* $f(a \vee b) = f(a) \vee f(b)$.

(b) a complete join-morphism if whenever $S \subseteq P$ *and* $\vee S$ *exists in* P, *then* $\vee f(S) = \vee f(x) : x \in S$ *exists in* Q *and* $f(\vee S) = \vee f(S)$.

The notions of a meet-morphism and a complete meet-morphism are defined dually. Further, a maping is called a morphism if it is a join-morphism and a meet-morphism. Complete morphisms are defined analogously.

Definition 1.2.34 *If P and Q are bounded, then $f : P \to Q$ is bottom-preserving if $f(1_P) = 1_Q$, and it is top-preserving if $f(0_P) = 0_Q$.*

Definition 1.2.35 *Let $(B, \ \leq)$ be a poset and let $(L, \ \leq)$ be a complete lattice. If there exists an order-embedding $\varphi : B \to L$, we say that $(L, \ \leq)$ is a completion of $(B, \ \leq)$.*

Definition 1.2.36 *A lattice $(L, \ \leq)$ is distributive if $\forall \ a, b, c \in L$, it satisfies the conditions $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.*

Definition 1.2.37 *A lattice $(L, \ \leq)$ is said to be modular lattice if $\forall \ x, y, z \in L$ with $x \geq y$ such that $x \wedge (y \vee z) = y \vee (x \wedge z)$.*

Lemma 1.2.1 *A sublattice of a modular lattice is modular.*

Proof: Let K be a sublattice of a modular lattice L. Then $K \subseteq L$, therefore $x, y, z \in K$ with $x \geq y$ which imply, $x, y, z \in L$ with $x \geq y$. Since L is modular, therefore, $x \wedge (y \vee z) = y \vee (x \wedge z)$. This completes the proof of the lemma.

Lemma 1.2.2 *A sublattice of a distributive lattice is distributive.*

Lemma 1.2.3 *A distributive lattice is always modular.*

A Stone algebra is a stone lattice together with the unary operation of pseudo implementation.

Definition 1.2.38 *Let $(L, \vee, \wedge)$ be a lattice and $A \subseteq L$. Then, A is called an ideal if* *(i)* $a \in A$ *and* $b \in A$ *imply* $a \vee b \in A$,

(ii) $b \in L$, $a \in A$ *imply* $a \wedge b \in A$.

Clearly, every ideal of a lattice L *is a sublattice and the collection of all ideals of* L *is denoted by* $I(L)$. *It is obvious that* $I(L)$ *is a lattice.*

Definition 1.2.39 *Let* $(L, \vee, \wedge)$ *and* $(K, \vee, \wedge)$ *be two lattices. A mapping* $f : L \rightarrow K$ *is called a homomorphism if for all* $x, y \in L$,

(i) $f(x \vee y) = f(x) \vee f(y)$,

(ii) $f(x \wedge y) = f(x) \wedge f(y)$.

1.3 Literature Survey

Rough set theory has been attracted attention of many researchers and practitioners all over the world, who have contributed essentially to its development and applications. Pawlak (81) introduced the theory of rough set as an extension of set theory for the study of incomplete information. A key concept in Pawlak's rough set is an equivalence relation. Similar studies have been made by different researchers. As for example, Skowron and Stepaniuk (109) discussed the tolerance approximation spaces in their paper. Slowiniski and Vanderpooten (110) presented a generalized definition of rough approximations based on similarity. Greco et al. (38) proposed the rough approximation by dominance relations. Xiao et al. (122) presented a relationship between rough sets and lattice theory. Pomykala and Pomykala (87) showed that the set of rough sets forms a stone algebra. Gehrke and Walker (35) introduced a precise structure theorem for the stone algebra of rough sets and a characterization of them in the category of all stone algebras. Yao [(129),(130)] described the notion of the formal concept analysis to rough set theory. Iwinski (43) defined rough lattice and rough order and he noticed that rough set can be viewed as a pair of approximations. In this definition, Iwinski defined rough lattice without using any indiscernibility concept of rough set. Liao et al. (54) introduced lattice theory in rough set and defined rough lattice.

The generalized rough sets over fuzzy lattices have been explored by Liu (55). Rana and Roy (96) established set valued homomorphism in rough lattice. Rana and Roy (92) presented a unique approach to form lattice by choice function in rough set. Järvinen (46) proposed the lattice structure on rough sets which played an important role in rough set and Pawlak's information system. Biswas and Nanda (15) discussed the notion of rough substructure in groups. Davvaz (24) studied roughness based on fuzzy ideals. Rough set theory overlaps with many other theories. Rasouli and Davvaz (99) considered a relationship between rough sets and MV-algebra theory and introduced the notion of rough ideal with respect to an ideal of an MV-algebra, which is an extended notion of ideal in an MV-algebra. Estaji et al. (29) introduced the concept of upper and lower rough ideals in a lattice and they studied some properties on prime ideals. Samanta and Chakraborty (104) categorized the various generalized approaches for lower and upper approximations of a set in term of implication lattice. Yang (127) formulated a new lattice structure named as rough concept lattice. Kong et al. (50) introduced the method construction of rough lattice based on compressed matrix which can solve the redundancy of construction process and obtain corresponding rule. Wang et al. (119) studied object oriented concept lattice and property oriented concept lattice. Thillaigovindan and Subha (115) have discussed the relationship between rough set theory and near-ring. They considered a near-ring as the universal set and introduced the notions of rough right (left, two-sided) ideal and rough sub near-ring with respect to an ideal of a near-ring. Jun (47) has studied roughness in G-semigroup and discussed the properties of sub-semigroup/ideals in G-semigroup. Despite this overlap, rough set theory may be considered as an independent discipline in its own right. A wide range of applications of methods based on rough set theory alone or in combination with other approaches have been discovered in the following areas as computer engineering, decision analysis and systems, economics, electrical engineering (e.g., control, signal analysis, power systems), environmental studies, digital image processing, informatics, medicine, molecular biology, robotics, social science, software engineering etc.

Molodtsov (67) incorporated the concept of soft set as a completely new mathematical tool with adequate parameterization for dealing with uncertainties. This concept is free from the parameterization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory. Maji et al. (60) studied on the theory of soft sets initiated by Molodtsov and developed several basic notions of soft set theory. Ali et al. (2) introduced the notion of restricted union, restricted intersection, restricted difference, and extended intersection between two soft sets. They established the notion of complement in soft set and also proved that certain De Morgan's laws hold in soft set theory. Ozturk and Inan (78) explained the interconnections between the various operations in soft set and defined the notion of restricted symmetric difference of soft sets and investigated its properties. Furthermore, Babitha and Sunil (5) gave definitions for the soft set relation as a subset of cartesian product of two soft sets. After that, Babitha and Sunil (6) defined the partially ordered soft set by introducing ordering on soft sets. Park et al. (80) focused the discussion on equivalence relation and they established that complete lattice is defined on the poset of equivalence soft set relations under a soft set. Qin and Hong (90) initiated a theoretical study of the algebraic structures of soft sets with lattice structures and introduced the notion of soft equality relation and also investigated its related properties. It was proved that soft equality relation is a congruence relation with respect to some operations. Aktas and Cağman (1) found an algebraic connection between soft set and algebraic system and introduced soft groups. Also, Manemaran (66) discussed fuzzy soft sets algebraic structures and defined fuzzy soft group. He discussed some operations on fuzzy soft groups and established related results. Furthermore, definitions of fuzzy soft functions and fuzzy soft homomorphism are also defined. Finally, the theorems on homomorphic image and homomorphic pre-image were discussed in detail. Feng et al. (30) studied soft set in semirings and investigated the notions of soft semirings, soft subsemirings, soft ideals, idealistic soft semirings and soft semiring homomorphism. Soft set theory is extended to the theory of BCK/BCI - algebras (48). Majumdar and Samanta (63) proposed two types of similarity measure between soft sets and made a comparative study of two techniques. The lattice structure of soft set has been found in [(55), (75)]. Shabir and Naz (108) initiated the study of soft topological spaces. Topological structure of fuzzy soft set began with the work of Tanay and Kandemir (114). Applications of soft set theory in other disciplines and real life problems are now catching momentum. Molodtsov has successfully applied the soft set theory in many different fields such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, and probability. Cagman et al. (18) extended soft set to fuzzy soft sets and applied it in decision making method. Maji et al. (59) gave first practical application of soft sets in decision making problems using the notion of knowledge reduction in rough set theory. Soft set theory has potential applications in many different fields due to its no necessity to describe the membership function. As a result, this makes that the soft set theory is so simple and popular in applications of various areas. Soft set theory based classification algorithm can be applied to texture classification. Cagman et el. (17) defined soft matrices and their operations to construct a soft max-min decision making method which can be successfully applied to the problems that contain uncertainties. Applications of soft set theory in various fields have been found in [(16), (18), (20), (33), (59)].

1.4 Objective and Scope of the Thesis

The main objective of the thesis has been defined after an extension literature survey based on the status in problems of mathematical extension and associated with some specific framework such as lattice structure of rough set in case of equivalence indiscernibility relation, homomorphism, rough lattice and rough ideal, hybridization of soft set, rough set and fuzzy set, lattice structure of soft set, soft set relation and approximation of soft set in modified soft rough approximation space. The novel and significant contribution in the present research work under report are summarized as follows:

(i) In order to study the properties of lattice in an approximation space based on Pawlak's notion of indiscernibility relation among the objects in a set. Rough modular lattice and rough distributive lattice are defined.

(ii) In order to study an algebraic connection between soft set and algebraic system like lattice theory in a soft approximation space. Notions of soft rough lattice are introduced.

(iii) To study the lattice theory in the framework of rough set. Rough ideal and rough homomorphism are studied in rough set environment.

(iv) To study rough set and soft set in different types of approximation spaces. Modified soft rough approximation space is defined using soft set. The concept of rough soft set is also introduced in modified soft rough approximation space.

(v) To study soft set relation in a new way; and based on this relation, lattice theory on soft sets is discussed.

(vi) To form a hybridization structure between rough-soft set and fuzzy set. Fuzzy rough soft set is also introduced in modified soft rough approximation space.

(vii) To study congruence relation on soft set and to enrich the theoretical development of lattice theory under soft set environment.

1.5 Organization of the Thesis

The research work under report and evaluation are organized ten chapters. First Chapter presents a brief introduction related to my research work, the brief history and some preliminaries of rough set theory and soft set theory. Finally some definitions on order and lattices are also discussed. Chapter-2 is devoted to study the lattice theory under rough set environments and defined rough lattice, rough sublattice and complete rough lattice. Moreover, the notions of rough modular lattice and rough distributive lattice are established in Pawlak's approximation space. Study of rough ideal and homomorphism and their applications to lattice are initiated in Chapter-3. We provide the definition of rough ideal of rough lattice and the properties of lattice under an approximation space are studied. Besides, the concept of rough homomorphism in rough lattice under an approximation space is introduced. In Chapter-4, we study an algebraic connection between soft-rough set and algebraic system and thereby introduce the notion of soft rough lattice in a soft approximation space. The concept of a soft rough lattice, soft rough sublattice, modular soft rough lattice and distributive soft rough lattice are defined. Chapter-5 describes the lattice theory in the framework of soft rough set. We consider the soft approximation space by means of soft set and define the notions of upper and lower soft rough ideals in a lattice. In Chapter-6, we present approximation of soft set in modified soft rough (MSR) approximation space i.e., approximation of an information system with respect to another information one. Besides, the concept of rough soft set is introduced in a modified soft rough approximation space. Moreover, the measure of roughness of soft set is defined in MSR approximation space and the order relation is introduced on soft set. Furthermore, lattice theory is studied in the MSR-approximation space under a modified rough soft environment. In Chapter-7, we incorporate an another approach for Cartesian product on soft set relation. Lattice theory on soft sets considering with soft set relation is studied. Soft congruence relation over lattice is depicted in Chapter-8. The concept of congruence relation on soft sets is studied over lattice and hereby defined the notions of soft congruence relation. In Chapter-9, we implement the concept of fuzzy rough soft set in MSR approximation space which can be viewed as a pair of soft set and its roughness. Also lattice theory is studied on fuzzy rough soft set. The last chapter contains the conclusions and scope of future works.

The chapters wise summary of the proposed works are given below:

In **Chapter 2**, rough modular lattice and rough distributive lattice are introduced based on Pawlak's indiscernibility relation. At first, the rough lattice is constructed and interpreted based on the equivalence relation. Then the different types of lattice under the rough set environment are established by incorporating a pair of sets in an approximation space. It is seen that the distributive property of lattice is extended to the area of uncertainty according to our defined Rough Distributive Lattice (RDL). We also show that modularity property of ordinary lattice in crisp set is extended to area of uncertainty for rough set which is the generalization of lattice theory. We make a connection between the rough set and the lattice theory both of which have wide fields of application in the areas of computer science. [One part of this chapter has been published in ***Journal of Uncertain Systems (World Academic***

***Union)*, SCOPUS, Vol. 7, No. 4 (2013), pp. 289-293** and another part in ***Malaya Journal of Matematik (University Press, Singapore)*, Vol. 2, No. 3 (2014), pp. 273-276**].

In **Chapter 3**, we introduce the notion of rough ideals which is a generalized notion of ideals of a lattice. Important properties of rough ideals are also developed in this chapter. We consider the approximation space by means of an equivalence relation and also describe the rough set as pair of sets (lower and upper approximation sets). The objective of this chapter is to study the properties of lattice under an approximation space based on Pawlak's notion of indiscernibility relation between the objects in a set. Several important results are established. Also this chapter is devoted to study of homomorphism of rough lattice. **[A part of this chapter is Communicated to International Journal]**.

In **Chapter 4**, an algebraic connection between the soft rough set and the algebraic structure named as lattice is established. As a result, lattice structure is developed on soft rough set; and the concept of soft rough lattice based on soft approximation space is defined. After that we investigate the several properties and theorems on soft rough lattice. Finally we justify our proposed soft rough lattice with supporting examples by Hasse diagram. [A part of this chapter has been published in ***Kragujevac Journal of Mathematics (University of Kragujevac, Serbia)*, SCOPUS, Vol. 39, No. 1 (2015), pp. 15-20**].

Rough and soft sets are two different mathematical tools for dealing with uncertainty. Soft rough set, proposed by Feng et al. (32) is a study on roughness through soft set. In **Chapter 5**, we formulate a general mathematical concept defined on lattices in the framework of soft rough set. This chapter is devoted to study the lattice theory in the framework of soft rough set. The set from lattice structure is treated here as universal set and defined soft rough set on it. We construct the soft rough ideal and study their properties in soft approximation space. [A part of this chapter has been published in ***The Journal of Fuzzy Mathematics (International Fuzzy Mathematics Institute, USA)*, Vol. 24, No. 1 (2016), pp. 49-56**].

In **Chapter 6**, we introduce the concept of approximation on an information system with respect to another information one based on an MSR-approximation space. We construct the rough soft set and study their properties in MSR-approximation space. Besides, we establish the connection between a rough set and a lattice theory by measuring the roughness of a soft set. We endeavor to establish a link between soft set and rough set in connection with an application in lattice. [A part of this chapter has been published in ***Fuzzy Information and Engineering (Elsevier)*, Vol. 7, No. 3 (2015), pp. 379-387**].

In **Chapter 7**, the concept of cartesian product on soft sets is introduced in an another way. Besides this, based on this cartesian product, a soft set relation is defined. Soft set relation is also constructed based on the induced binary relation in the set of parameters of soft sets. A connection between the relations is also established. Moreover, lattice theory is studied on soft sets considering with soft set relation. Based on the ideas of the Cartesian product and soft set relation, we newly formulate the soft lattice, soft modular lattice, soft distributive lattice and soft equivalent relation which are the unique characteristic of this chapter. **[A part of this chapter is Communicated to International Journal].**

In **Chapter 8**, we establish the soft congruence relation over lattice. Several properties of soft congruence relation are studied. Approximations of subset of a lattice are studied with respect to soft congruence relation. That is the roughness of a subset of lattice is discussed using the soft set relation. We also discuss the properties of lattice ideal with respect to the soft congruence relation. [A part of this chapter has been published in ***Hacettepe Journal of Mathematics and Statistics (Hacettepe University, Turkey), SCIE, IF: 0.277, 2017, DOI: 10.15672/HJMS.2017.436***].

Soft set theory, rough set theory and fuzzy set theory are all treated as mathematical tools to deal with uncertainty for variety of problems. A possible hybridization of these theories is an interesting topic to the researchers. In **Chapter 9**, we propose the concept of fuzzy rough soft set in MSR approximation space which can be viewed as a pair of soft set and its roughness. We define the union and the intersection of fuzzy rough soft set with several examples. Also we establish the important properties of fuzzy rough soft set with respect to fuzzy rough soft union and intersection. **[A part of this chapter is Communicated to International Journal].**

Chapter 10 contains conclusion of the whole study presented in this thesis and direction of the future work emerging from this thesis.

Chapter 2

Rough Modular and Distributive Lattice*

In this chapter, rough modular lattice and rough distributive lattice are introduced based on Pawlak's indiscernibility relation. At first, the rough lattice is constructed and interpreted based on the equivalence relation. Then the different types of lattice under the rough set environment are established by incorporating a pair of sets in an approximation space. We consider the approximation space by means of an equivalence relation and also we present the rough set as pair of set (lower and upper approximation sets). The aim of this chapter is to study the properties of lattice in an approximation space based on Pawlak's notion of indiscernibility relation among the objects in a set. It is seen that the distributivity property of lattice is extended to the area of uncertainty according to our defined Rough Distributive Lattice (RDL). We also show that modularity property of ordinary lattice in crisp set is extended to area of uncertainty for rough set which is the generalization of lattice theory.

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*One part of this chapter has appeared in ***Journal of Uncertain Systems,*** SCOPUS, 7, 289-293, (2013), and another in ***Malaya Journal of Matematik*** 2, 273-276, (2014)

2.1 Introduction

Rough set was first introduced by Pawlak (81) which is a framework for systematic study of incomplete knowledge. Thereafter, researchers have put their attention to incorporate lattice into rough set. In (43), Iwinski defined rough lattice and rough order and he described that rough set can be viewed as a pair of approximations. Again he defined the rough lattice without using any indiscernibility concept of rough set. Järvinen (46) proposed the lattice structure on rough sets which played an important role in rough set and Pawlak's information system. Rana and Roy (92) introduced a unique approach to form lattice by choice function in rough set. Estaji et al. (29) introduced the concept of upper and lower rough ideals in a lattice and they studied some properties on prime ideals. Samanta and Chakraborty (104) categorized the various generalized approaches for lower and upper approximations of a set in term of implication lattice. Yang (127) formulated a new lattice structure named as rough concept lattice. Kong et al. (50) introduced the method construction of rough lattice based on compressed matrix which can solve the redundancy of construction process and obtain corresponding rule. Wang et al. (119) studied object oriented concept lattice and property oriented concept lattice.

Several concepts have been proposed to form lattice by rough set and most of these are based on tolerance relation. We construct the rough modular lattice and rough distributive lattice based on equivalence relation which is the main motivation of this chapter and to establish a connection between rough set and lattice structure. Also we define the modular lattice and distributive lattice in the rough set environment using indiscernibility relation.

Here we recall some basic properties and definitions related to rough set.

Definition 2.1.1 *Let U be the set of universe and ρ be an equivalence relation on U. An equivalence class of $x(\in U)$ is denoted by $[x]_\rho$ and defined as follows: $[x]_\rho = \{y \in U : x\rho y\}$, where $x\rho y$ imply $(x, y) \in \rho$.*

Definition 2.1.2 *The sets $A_\star(X) = \{x \in U : [x]_\rho \subseteq X\}$ and $A^\star(X) = \{x \in U : [x]_\rho \cap X \neq \phi\}$ are respectively called lower and upper approximations of $X \subseteq U$. The pair $S = (U, \rho)$ is called an approximation space and the pair $(A_\star(X), A^\star(X))$ is called the rough set of X in S and is denoted by $A(X)$. The difference $B(X) = A^\star(X) - A_\star(X)$ is called boundary region of X and treated as the area of uncertainty.*

Theorem 2.1.1 *Let $A(X) = (A_\star(X), A^\star(X))$ and $A(Y) = (A_\star(Y), A^\star(Y))$ be two rough sets under the approximation space $S = (U, \rho)$, then*

(i) $A(X) \cup A(Y) = (A_\star(X) \cup A_\star(Y), A^\star(X) \cup A^\star(Y))$

(ii) $A(X) \cap A(Y) = (A_\star(X) \cap A_\star(Y), A^\star(X) \cap A^\star(Y))$.

Proof: Straightforward.

Now we define the Cartesian product of two rough sets.

Definition 2.1.3 *The Cartesian product of two rough sets $A(X) = (A_\star(X), A^\star(X))$ and $A(Y) = (A_\star(Y), A^\star(Y))$ is defined as follows:*

$$A(X) \times A(Y) = \{(x, y) : x \in A^\star(X) \text{ and } y \in A^\star(Y)\}.$$

Definition 2.1.4 *A rough set $A(Y)$ is said to be rough subset of a rough set $A(X)$ if $A_\star(Y) \subseteq A_\star(X)$ and $A^\star(Y) \subseteq A^\star(X)$ and it is denoted by $A(Y) \subseteq A(X)$.*

2.2 Rough lattice

In this section, we introduce the rough lattice and rough modular lattice and some properties of them. Here we describe the rough lattice based on Pawlak's notion of roughness. Let $(L, \vee, \wedge)$ be a lattice and also let $S = (L, \rho)$ be an approximation space. Let $X \subseteq U$ and $A(X) = (A_\star(X), A^\star(X))$ be the rough set of X in S.

Definition 2.2.1 *$A(X)$ is said to be rough join-semi lattice if $x \vee y \in A^\star(X)$, $\forall\ x, y \in X$, and $A(X)$ is said to be rough meet-semi lattice if $x \wedge y \in A^\star(X)$, $\forall\ x, y \in X$.*

Example 2.2.1 *Consider the lattice $(L, \vee, \wedge)$ of all positive integers where $x \vee y$ = lcm of $\{x, y\}$ and $x \wedge y$ = gcd of $\{x, y\}$, $\forall\ x, y \in L$, where lcm and gcd stand for least common multiple and greatest common divisor respectively. Let ρ be an equivalence relation on L defined by $x\rho y$ iff $x = y$, $\forall\ x, y \in L$. Let $X = \{2, 3, 4, 6, 12\}$ then $A_\star(X) = X$ and $A^\star(X) = X$ under the approximation space (L, ρ). Clearly $x \vee y \in A^\star(X)$, $\forall\ x, y \in X$. But $4 \wedge 3 = 1 \notin A^\star(X)$. Therefore $A(X)$ is a rough join semi-lattice. On the other hand if we take $Z = \{1, 2, 3, 4, 5, 6\}$. Then $x \wedge y \in A^\star(Z)$, $\forall\ x, y \in Z$. But $4 \vee 6 = 12 \notin A^\star(Z)$. Therefore $A(Z)$ is a rough meet-semi lattice.*

Definition 2.2.2 *$A(X)$ is said to be rough lattice in S if it is rough join-semi* *lattice as well as rough meet-semi lattice i.e.,* $\forall\ x, y \in X$

(i) $x \vee y \in A^\star(X)$

(ii) $x \wedge y \in A^\star(X)$.

Theorem 2.2.1 *Let* $x, y, z \in X$*, then the rough lattice* $(A(X), \vee, \wedge)$ *satisfies the following properties:*

(i) $x \vee x = x$ *and* $x \wedge x = x$ *(Idempotency)*

(ii) $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$ *(Commutativity)*

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ *and* $x \vee (y \vee z) = (x \vee y) \vee z$ *(Associativity)*

(iv) $x \vee (x \wedge y) = x$ *and* $x \wedge (x \vee y) = x$ *(Absorption)*

(v) $x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$ *(Consistency).*

Definition 2.2.3 *A rough lattice* $A(X)$ *is said to be bounded rough lattice if* X *has zero and unit in* $A^\star(X)$.

Definition 2.2.4 *A rough subset* $A(Y)$ *of a rough lattice* $A(X)$ *in an approximation space* $S = (L, \rho)$ *is said to be rough sublattice if* $A(Y)$ *itself forms a rough lattice with respect to the same operation.*

Proposition 2.2.1 *If* $A(X) = (A_\star(X), A^\star(X))$ *is a rough lattice in an approximation space* $S = (L, \rho)$ *such that* $A^\star(X) = X$*, then* $A^\star(X)$ *is a sublattice of* L.

Proof: Since $A(X) = (A_\star(X), A^\star(X))$ is a rough lattice in the approximation space $S = (L, \rho)$, then clearly, $x \vee y \in A^\star(X)$, and $x \wedge y \in A^\star(X)$, $\forall\ x, y \in A^\star(X)$. This completes the proof of the proposition.

Proposition 2.2.2 *If L is a modular lattice and $A(X)$ is a rough lattice in $S = (L, \rho)$ such that $A^{\star}(X) = X$ then $A^{\star}(X)$ is a modular lattice.*

Proof: Since $A^{\star}(X) = X$ and $A(X)$ is a rough lattice, therefore by Proposition 2.2.1 and Lemma 1.2.1, $A^{\star}(X)$ is a sublattice of L and hence $A^{\star}(X)$ is a modular lattice.

Proposition 2.2.3 *If L is a distributive lattice and $A(X)$ is a rough lattice in $S = (L, \rho)$ such that $A^{\star}(X) = X$ then $A^{\star}(X)$ is a distributive lattice.*

Proof: Since $A^{\star}(X) = X$ and $A(X)$ is a rough lattice, therefore by Proposition 2.2.1, $A^{\star}(X)$ is a sublattice of L. and hence by Lemma 1.2.2, $A^{\star}(X)$ is a distributive lattice.

Definition 2.2.5 *A rough lattice $A(X)$ under an approximation space $S = (L, \rho)$ is said to be a complete rough lattice if every non-empty subset of X has least upper bound and greatest lower bound in $A^{\star}(X)$.*

Proposition 2.2.4 *A rough lattice $A(X)$ under an approximation space $S = (L, \rho)$ is complete rough lattice if $A^{\star}(X)$ is a complete sublattice of L.*

Proof: Let $A^{\star}(X)$ is a complete sublattice of L. Then every non-empty subset of $A^{\star}(X)$ has least upper bound and greatest lower bound in $A^{\star}(X)$. Since X is a non-empty subset of $A^{\star}(X)$, therefore X has least upper bound and greatest lower bound in $A^{\star}(X)$. Therefore, $A(X)$ is a complete rough lattice.

2.3 Rough modular lattice

Here we define Rough Modular Lattice (RML) in Pawlak's approximation space. Let $(L, \vee, \wedge)$ be a lattice and $S = (L, \rho)$ be an approximation space.

Definition 2.3.1 *Let* $(A(X), \vee, \wedge)$ *is a rough lattice under an approximation space* $S = (L, \rho)$*, then* $(A(X), \vee, \wedge)$ *is said to be rough modular lattice (RML) if* $\forall\ x, y, z \in A^{\star}(X)$ *with* $x \geq y$*,* $x \wedge (y \vee z) = y \vee (x \wedge z)$*.*

Example 2.3.1 Let $U = \{a, b, c\}$. The power set of U, $P(U) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, U\}$ forms a lattice where the operators $\vee$ and $\wedge$ are defined as $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ and the order relation is set inclusion. Consider an equivalence relation ρ on $P(U)$ by $A\rho B$ iff $O(A) = O(B)$ $\forall$ $A, B \in P(U)$, where $O(A)$ and $O(B)$ denote the number of elements in the set A and B respectively. Let $X = \{\phi, \{a\}, \{a, b, c\}\}$. Then $A_{\star}(X) = \{\phi, \{a, b, c\}\}$ and $A^{\star}(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$. Clearly $A(X)$ is a rough lattice. Also $\forall$ $A, B, C \in A^{\star}(X)$ with $A \supseteq B$, $A \wedge (B \vee C) = B \vee (A \wedge C)$ i.e., $A(X)$ is a RML.

Proposition 2.3.1 *Rough sublattice of a RML is RML.*

Proof: Let $A(M)$ is a RML and $A(N)$ is a rough sublattice of $A(M)$. Therefore, $A^{\star}(N) \subseteq A^{\star}(M)$ and hence if $x, y, z \in A^{\star}(N)$ with $x \geq y$ imply $x, y, z \in A^{\star}(M)$ with $x \geq y$. Since $A(M)$ is modular rough lattice, therefore, $x \wedge (y \vee z) = y \vee (x \wedge z)$. Therefore, if $x, y, z \in A^{\star}(N)$ with $x \geq y$, $x \wedge (y \vee z) = y \vee (x \wedge z)$. This completes the proof of the proposition.

Proposition 2.3.2 *If* L *is a modular lattice and* $A(X)$ *is a rough lattice then* $A(X)$ *is a RML.*

Proof: Since $A(X)$ is a rough lattice, therefore $\forall$ $x, y, z \in A^{\star}(X)$ with $x \geq y$ imply $x, y, z \in L$ and since L is modular, then $x \wedge (y \vee z) = y \vee (x \wedge z)$. So $A(X)$ is a RML.

Proposition 2.3.3 *If $A(X)$ is a RML in $S = (L, \rho)$ and if $A^\star(X) = X$ then X is modular sublattice of L and vice-versa.*

Proof: Since $A(X)$ is RML and $A^\star(X) = X$, therefore modular equality holds in X. Also by Proposition 2.2.1, $A^\star(X)$ is sublattice of L and therefore $A^\star(X) = X$ is modular sublattice of L.

Conversely, let $A^\star(X) = X$ is modular sublattice of L. Therefore $A^\star(X)$ is rough lattice and also modular equality holds in $A^\star(X)$. So $A(X)$ is RML.

Proposition 2.3.4 *Two rough lattices $A(X)$ and $A(Y)$ are RML iff $A(X) \times A(Y)$ is RML.*

Proof: Let us consider that $A(X)$ and $A(Y)$ be RML and $A(X) = (A_\star(X), A^\star(X))$ and $A(Y) = (A_\star(Y), A^\star(Y))$. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A^\star(X) \times A^\star(Y)$ with $(x_1, y_1) \geq (x_2, y_2)$. Therefore

$$\begin{aligned}(x_1, y_1) \wedge ((x_2, y_2) \vee (x_3, y_3)) &= (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) \\ &= (x_2 \vee (x_1 \wedge x_3), y_2 \vee (y_1 \wedge y_3)) \\ &= (x_2, y_2) \vee ((x_1, y_1) \wedge (x_3, y_3)).\end{aligned}$$

Therefore $A^\star(X) \times A^\star(Y)$ is RML.

Conversely, let $A^\star(X) \times A^\star(Y)$ be RML. Let $x_1, x_2, x_3 \in A^\star(X)$ with $x_1 \geq x_2$ and $y_1, y_2, y_3 \in A^\star(Y)$ with $y_1 \geq y_2$ then
$(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A^\star(X) \times A^\star(Y)$ and $(x_1, y_1) \geq (x_2, y_2)$.

Since $A^\star(X) \times A^\star(Y)$ is RML, we find

$$\begin{aligned} (x_1, y_1) \wedge ((x_2, y_2) \vee (x_3, y_3)) &= (x_2, y_2) \vee ((x_1, y_1) \wedge (x_3, y_3)) \\ \text{or,} \quad (x_1, y_1) \wedge (x_2 \vee x_3, y_2 \vee y_3) &= (x_2, y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3) \\ \text{or,} \quad (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) &= (x_2 \vee (x_1 \wedge x_3), y_2 \vee (y_1 \wedge y_3)) \end{aligned}$$

which indicates, $x_1 \wedge (x_2 \vee x_3) = x_2 \vee (x_1 \wedge x_3)$ and $y_1 \wedge (y_2 \vee y_3) = y_2 \vee (y_1 \wedge y_3)$. Hence $A(X)$ and $A(Y)$ are RML.

2.4 Rough distributive lattice

In this section, we study distributive lattice in rough set environment and define Rough Distributive Lattice (RDL) and establish some propositions which combine with rough approximation and RML.

Definition 2.4.1 *Let* $(A(X), \vee, \wedge)$ *is a rough lattice under an approximation space* $S[= (L, \rho)]$*, then* $(A(X), \vee, \wedge)$ *is said to be rough distributive lattice (RDL) if* $\forall\ x, y, z \in A^\star(X)$*,* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$*.*

Example 2.4.1 *The set* $L = \{1, 2, 4, 5, 10, 20\}$ *of factors of 20 forms a lattice where the operators '*$\vee$*' and '*$\wedge$*' are defined as* $a \vee b$ *=least common multiple of* $\{a, b\}$ *and* $a \wedge b$ *=greatest common divisor of* $\{a, b\}$ *and the order relation is divisibility. Let us consider an equivalence relation* ρ *on* L *by* $x\rho y$ *iff "*x *is congruent to* y *modulo 2"* $\forall\ x, y \in L$*. Let* $X = \{2, 4\}$*. Then* $A_\star(X) = \phi$ *and* $A^\star(X) = \{2, 4, 10, 20\}$*. Clearly* $A(X)$ *is rough lattice. Also* $\forall\ x,\ y,\ z \in A^\star(x)$*,* $x\ \wedge (y\ \vee z) = (x \wedge y) \vee (x \wedge z)$*. Therefore* $A(X)$ *is a RDL.*

Proposition 2.4.1 *Rough sublattice of a RDL is RDL.*

Proof: Let $A(X)$ is a RDL and $A(Y)$ is a rough sublattice of $A(X)$. Therefore $A^\star(Y) \subseteq A^\star(X)$ and hence if $x,\ y,\ z \in A^\star(Y)$ then $x,\ y,\ z \in A^\star(X)$ and since $A(X)$ is RDL, therefore, $x\ \wedge\ (y\ \vee\ z) = (x\ \wedge\ y)\ \vee\ (x\ \wedge\ z)$. Therefore if $x,\ y,\ z \in A^\star(Y)$, $x\ \wedge\ (y\ \vee\ z) = (x\ \wedge\ y)\ \vee\ (x\ \wedge\ z)$.

Proposition 2.4.2 *If L is a distributive lattice and $A(X)$ is a rough lattice then $A(X)$ is a RDL.*

Proof: Since $A(X)$ is a rough lattice, $A^\star(X) \subseteq L$. Therefore $\forall\, x, y, z \in A^\star(X)$ imply, $x, y, z \in L$ and since L is distributive, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Therefore $\forall\, x, y, z \in A^\star(X)$, $x\ \wedge(y\ \vee\ z) = (x\ \wedge\ y)\ \vee\ (x\ \wedge\ z)$ i.e., $A(X)$ is a RDL. This completes the proof of the proposition.

Proposition 2.4.3 *If $A(X)$ is a RDL in $S = (L, \rho)$ and if $A^\star(X) = X$ then X is distributive sublattice of L and vice-versa.*

Proof: Since $A(X)$ is RDL and $A^\star(X) = X$, therefore $\forall\, x, y, z \in A^\star(X) = X$, $x\ \wedge\ (y\ \vee\ z) = (x\ \wedge\ y)\ \vee\ (x\ \wedge\ z)$. Also by Proposition 2.2.1, $A^\star(X)$ is sublattice of L and since sublattice of a distributive lattice is distributive, therefore $A^\star(X) = X$ is distributive sublattice of L.

Conversely, let $A^\star(X) = X$ is distributive sublattice of L. Therefore $A^\star(X)$ is rough lattice with $x\ \wedge\ (y \vee z) = (x\ \wedge\ y)\ \vee\ (x\ \wedge\ z)$, $\forall\, x, y, z \in A^\star(X)$. So $A(X)$ is RDL.

Proposition 2.4.4 *Two rough lattices $A(X)$ and $A(Y)$ are RDL iff $A(X) \times A(Y)$ is RDL.*

Proof: Let $A(X)$ and $A(Y)$ be RDL and also let $A(X) = (A_\star(X), A^\star(X))$ and $A(Y) = (A_\star(Y), A^\star(Y))$. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A^\star(X) \times A^\star(Y)$.

Then $x_1, x_2, x_3 \in A^{\star}(X)$ and $y_1, y_2, y_3 \in A^{\star}(Y)$. Therefore

$$\begin{aligned}(x_1, y_1) \wedge ((x_2, y_2) \vee (x_3, y_3)) &= (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) \\ &= ((x_1 \wedge x_2) \vee (x_1 \wedge x_3), (y_1 \wedge y_2) \vee (y_1 \wedge y_3)) \\ &= (x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3) \\ &= ((x_1, y_1) \wedge (x_2, y_2)) \vee ((x_1, y_1) \wedge (x_3, y_3)).\end{aligned}$$

Hence $A(X) \times A(Y)$ is RDL.

Conversely, let $A(X) \times A(Y)$ be RDL. Let $x_1, x_2, x_3 \in A^{\star}(X)$ and $y_1, y_2, y_3 \in A^{\star}(Y)$. Then $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A^{\star}(X) \times A^{\star}(Y)$. As $A(X) \times A(Y)$ is RDL, therefore

$$(x_1, y_1) \wedge ((x_2, y_2) \vee (x_3, y_3)) = ((x_1, y_1) \wedge (x_2, y_2)\} \vee ((x_1, y_1) \wedge (x_3, y_3))$$

or, $\quad (x_1, y_1) \wedge (x_2 \vee x_3, y_2 \vee y_3) = (x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3)$

or, $(x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) = ((x_1 \wedge x_2) \vee (x_1 \wedge x_3), (y_1 \wedge y_2) g \vee (y_1 \wedge y_3))$,

which gives

$x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$ and $y_1 \wedge (y_2 \vee y_3) = y_1 \wedge y_2) \vee (y_1 \wedge y_3)$.

This implies that $A(X)$ and $A(Y)$ are RDL.

Proposition 2.4.5 *Every RDL is RML but converse is not true.*

Proof: Let $A(X)$ is a RDL. Therefore $x, y, z \in A^{\star}(Y)$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

If $x \geq y$ and $x, y, z \in A^{\star}(Y)$, then

$$\begin{aligned}x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ &= y \vee (x \wedge z).\end{aligned}$$

Hence $A(X)$ is a RDL.

The converse is not true which is illustrated by the following example:

Example 2.4.2 *Let $K_4 = \{e, a, b, c\}$ be the Klein's four group. Let L be the set of all subgroups of K_4. Then $L = \{\{e\}, \{e, a\}, \{e, b\}, \{e, c\}, K_4\}$. L forms a lattice under set inclusion and the operations '$\vee$' and '$\wedge$' are defined by $A \vee B = A \cup B$ and $A \wedge B = A \cap B$, $\forall\ A, B \in L$. Let us consider an equivalence relation ρ on L defined by $A\rho B$ iff $O(A) = O(B)$, $\forall\ A, B \in L$. Let $X = \{\{e\}, \{e, a\}, \{K_4\}\}$. Then $A_\star(X) = \{\{e\}, \{K_4\}\}$ and $A^\star(X) = \{\{e\}, \{e, a\}, \{e, b\}, \{e, c\}, \{K_4\}\}$. Clearly, $A(X)$ is a rough modular lattice but $A(X)$ is not rough distributive lattice.*

2.5 Conclusion

Lattice and ordered set play an important role in the area of computer science. Lattice and ordered set can be applied in various fields such as area of knowledge representation, text categorization and data mining order in a fundamental way. In this chapter, the concept of rough modular lattice and rough distributive lattice have been introduced based on Pawlak's indiscernibility relation. At first, the rough lattice has constructed and interpreted based on the equivalence relation and then we have studied various properties of them in compare to ordinary lattice. Then the different types of lattice under the rough set environment have been established by incorporating a pair of sets in an approximation space. The rough lattice (as we defined) is a rough set with two binary operations and it behaves in a lattice like manner within the rough boundary. We also have showed that modularity property of ordinary lattice in crisp set is extended to area of uncertainty for rough set which is the generalization of lattice theory. This concept may be useful on lattice structure when the elements are imprecise. We have addressed a connection between rough set and lattice theory both of which have wide fields of applications in the area of computer science. This chapter has made special interest for lattice structure, when the elements of the set are imprecise.

Chapter 3

Rough Ideal and Homomorphism and Their Applications to Lattice*

In this chapter, we analyze the rough ideal of rough lattice through rough set environment. The definition of rough ideal of rough lattice and the properties of lattice under an approximation space are studied. Besides, the concept of rough homomorphism in rough lattice under an approximation space is introduced. We consider the approximation space by means of an equivalence relation and also describe the rough set as pair of sets (lower and upper approximation sets). The objective of this chapter is to study the properties of lattice under an approximation space based on Pawlak's notion of indiscernibility relation between the objects in a set. Several important results are established. Finally, we include various examples to usefulness and truthfulness of the proposed study in this chapter.

3.1 Introduction

The study of the algebraic structure of the mathematical theory proves itself effective in making the applications more efficient. Many researchers have studied the algebraic structure on rough sets. A few of them are presented here with their works. Pomykala and Pomykala (87) showed that the set of rough sets forms a stone algebra. Iwinski (43) suggested a lattice theoretic

*A part of this chapter has communicated to the International Journal approach to the rough set. Thomas and Nair (116) studied the concept of intuitionistic fuzzy sublattices and intuitionistic fuzzy ideals of a lattice. Xiao et al. (122) presented a relationship between rough sets and lattice theory. Davvaz (24) studied roughness based on fuzzy ideals. The generalized rough sets over fuzzy lattices have been explored by Liu (55). Rana and Roy (96) established set valued homomorphism in rough lattice.

From the mathematical point of view, lattice (21) is a partially ordered set in which any two elements have a unique supremum and an infimum. Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities. Since the two definitions are equivalent, lattice theory draws on both order relation and universal algebra. Lattice and order set have wide fields of applications in computer science, engineering, discrete mathematics, data mining, number theory, group theory etc. In addition to the above, many applications utilize lattices and ordered set in fundamental ways. These include such areas as knowledge representation, tex categorization and data mining, where order plays an fundamental organizing principle. Also, for the application of lattice and ordered set to inductive logic programming, ordered set form basic models. On the other hand in our complex world, there are many situations occur, where we cannot use traditional methods to solve problems in economics, engineering, environment, social science, medical science etc. because of various types of uncertainties present in these problems. For that situation lattice theory under uncertain environments can be applied with the help of rough set.

In this chapter, we focus our main intention to develop the lattice theoretic foundation based on rough set theory. This chapter formulates a continuous study on rough lattice. We present a general framework for the study of approximation in lattice. We consider a lattice as a universal set and study the rough sets in a lattice. We endeavor the notion of rough ideal and to establish a link between a rough set and a lattice structure through rough ideal. Also we define the upper rough homomorphism, lower rough homomorphism and rough homomorphism of rough lattice under rough set environment.

First we recall some preliminaries from Chapters 1 and 2. Let L be the set of universe and ρ be an equivalence relation on L. The pair (L, ρ) is called an approximation space and we denote this approximation space by S throughout the chapter. The pair $A(X) = (A_\star(X), A^\star(X))$ is the rough set of X in S.

Definition 3.1.1 *(81) A rough set $A(Y)$ is said to be rough subset of a rough set $A(X)$ if $A_\star(Y) \subseteq A_\star(X)$ and $A^\star(Y) \subseteq A^\star(X)$ and it is denoted by $A(Y) \subseteq A(X)$.*

Definition 3.1.2 *(21) Let $(L, \vee, \wedge)$ be a lattice and $A \subseteq L$. Then A is a sublattice if $a \in A$, $b \in A$ imply $a \vee b \in A$ and $a \wedge b \in A$, where the symbol $\vee$ and $\wedge$ stands for supremum and infimum respectively.*

Definition 3.1.3 *(21) Let $(L, \vee, \wedge)$ be a lattice and $A \subseteq L$. Then, A is called an ideal if*
(i) $a \in A$ and $b \in A$ imply $a \vee b \in A$,
(ii) $b \in L$, $a \in A$ imply $a \wedge b \in A$.
Clearly, every ideal of a lattice L is a sublattice and the collection of all ideals of L is denoted by $I(L)$. It is obvious that $I(L)$ is a lattice.

Definition 3.1.4 *(9) $A(X)$ is said to be rough lattice in S if $\forall\ x, y \in X$,*
(i) $x \vee y \in A^\star(X)$,
(ii) $x \wedge y \in A^\star(X)$.

We denote this rough lattice as $(A(X), \vee, \wedge)$.

Definition 3.1.5 *(9) A rough subset $A(Y)$ of a rough lattice $(A(X), \vee, \wedge)$ in an approximation space S is said to be rough sublattice if $A(Y)$ itself forms a rough lattice with respect to same operation.*

Definition 3.1.6 *(9) Let $(A(X), \vee, \wedge)$ is a rough lattice under an approximation space S, then $(A(X), \vee, \wedge)$ is said to be Rough Modular Lattice (RML) if $\forall\ \ x, y, z \in A^\star(X)$ with $x \geq y$, $x \wedge (y \vee z) = y \vee (x \wedge z)$.*

Definition 3.1.7 *(100) Let* $(A(X), \vee, \wedge)$ *is a rough lattice under an approximation space* S*, then* $(A(X), \vee, \wedge)$ *is said to be Rough Distributive Lattice (RDL) if* $\forall\ x, y, z \in A^{\star}(X)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Definition 3.1.8 *(21) Let* L *and* K *be two lattices. A mapping* $f : L \to K$ *is called a homomorphism if* $\forall\ \ x, y \in L$,
(i) $f(x \vee y) = f(x) \vee f(y)$,
(ii) $f(x \wedge y) = f(x) \wedge f(y)$.

3.2 Rough ideal

Here, we introduce the rough ideal of rough lattice and to define the properties of rough ideal and rough lattice. Let $(L, \vee, \wedge)$ be a lattice and also let $S = (L, \rho)$ be an approximation space. Let $X \subseteq U$ and $A(X) = (A_{\star}(X), A^{\star}(X))$ be the rough set of X in S.

Definition 3.2.1 *A rough subset* $A(Y)$ *of the rough lattice* $A(X)$ *is said to be rough ideal of rough lattice* $A(X)$ *in an approximation space* S *if it satisfies the following conditions:*
(i) $a, b \in Y$ *imply that* $a \vee b \in A^{\star}(Y)$*, and*
(ii) if $a \in A^{\star}(Y)$ *and* $b \in A^{\star}(X)$ *then* $a \wedge b \in A^{\star}(Y)$.

We denote the set of sets whose rough sets are rough ideal by $S(I)$. Clearly $S(I)$ is non-empty because every rough lattice is its own rough ideal. Also the empty set ϕ is in $S(I)$.

Example 3.2.1 *The set* $L = \{1, 2, 4, 5, 10, 20\}$ *of factors of 20 forms a lattice where the operators* $\vee$ *and* $\wedge$ *are defined as* $a \vee b$ *=least common multiple of* $\{a, b\}$ *and* $a \wedge b$ *=greatest common divisor of* $\{a, b\}$ *and the order relation is divisibility. Let us consider an equivalence relation* ρ *on* L *by* $x\rho y$ *iff "x is congruent to y under modulo 2"* $\forall\ x, y \in L$. *Let* $X = \{1, 2, 5\}$. *Then* $A_{\star}(X) = \{1, 5\}$ *and* $A^{\star}(X) = \{1, 2, 4, 5, 10, 20\}$. *Clearly* $A(X)$ *is rough lattice. Let* $Y = \{2, 5\}$ *then,* $A_{\star}(Y) = \phi$ *and* $A^{\star}(Y) = \{1, 2, 4, 5, 10, 20\}$. *Clearly* $A(Y) \subseteq A(X)$ *and also* $2, 5 \in Y$ *imply* $2 \vee 5 = 10 \in A^{\star}(X)$ *and* $\forall\ y \in A^{\star}(Y)$

and $x \in A^{\star}(X)$ *imply* $x \wedge y \in A^{\star}(Y)$. *Therefore,* $A(Y)$ *is a rough ideal of the rough lattice* $A(X)$.

Proposition 3.2.1 *Every rough ideal is a rough sublattice.*

Proof: Let $A(Y)$ be a rough ideal of a rough lattice $A(X)$ in the approximation space S. Therefore, $\forall\ a, b \in Y$ imply, $a \vee b \in A^{\star}(Y)$ (by Definition 3.2.1). Again, $a \in Y \subseteq A^{\star}(Y)$, $b \in Y \subseteq A^{\star}(Y) \subseteq A^{\star}(X)$ imply $a \wedge b \in A^{\star}(Y)$. Hence $A(Y)$ is a rough sublattice of $A(X)$.

Proposition 3.2.2 $S(I)$ *is a lattice.*

Proof: Straightforward.

Proposition 3.2.3 *If* $A(Y)$ *and* $A(Z)$ *are two rough ideals of* $A(X)$ *then* $A(Y \cap Z)$ *is also a rough ideal of* $A(X)$.

Proof: Let $A(Y) = (A_{\star}(Y), A^{\star}(Y))$ and $A(Z) = (A_{\star}(Z), A^{\star}(Z))$ are two rough ideals of the rough lattice $A(X) = (A_{\star}(X), A^{\star}(X))$. Let $x,\ y \in (Y \cap Z)$ then $x \vee y \in A^{\star}(Y)$ and $x \vee y \in A^{\star}(Z)$. Therefore, $x \vee y \in A^{\star}(Y) \cap A^{\star}(Z) = A^{\star}(Y \cap Z)$. Also, let $x \in A^{\star}(Y \cap Z)$ and $i \in A^{\star}(X)$ and since $A(Y)$ and $A(Z)$ are rough ideals of $A(X)$. Therefore, $x \wedge i \in A^{\star}(Y)$ and $x \wedge i \in A^{\star}(Z)$ i.e., $x \wedge i \in A^{\star}(Y) \cap A^{\star}(Z) = A^{\star}(Y \cap Z)$. This completes the proof of the proposition.

Proposition 3.2.4 *If* $A(Y)$ *and* $A(Z)$ *are two rough ideals of the rough lattice* $A(X)$ *then* $A(Y \cup Z)$ *also be a rough ideal of* $A(X)$ *if* $Y \subseteq Z$ *or* $Z \subseteq Y$.

Proof: Let $x,\ y \in (Y \cup Z)$. Then $x,\ y \in Y$ or $x,\ y \in Z$ or in both. Since $A(Y)$ and $A(Z)$ are both rough ideals of $A(X)$, therefore $x \vee y \in A^{\star}(Y)$ and $x \vee y \in A^{\star}(Z)$. So, $x \vee y \in A^{\star}(Y) \cup A^{\star}(Z) = A^{\star}(Y \cup Z)$. Again let $x \in A^{\star}(Y \cup Z) = A^{\star}(Y) \cup A^{\star}(Z)$ and $y \in A^{\star}(X)$. Therefore $x \in A^{\star}(Y)$ or $x \in A^{\star}(Z)$. Since $A(Y)$ and $A(Z)$ are both rough ideals of $A(X)$, therefore, $x \wedge y \in A^{\star}(Y)$ and $x \wedge y \in A^{\star}(Z)$, this imply $x \wedge y \in A^{\star}(Y) \cup A^{\star}(Z) = A^{\star}(Y \cup Z)$.

Proposition 3.2.5 *If $A(X)$ is a rough lattice in an approximation space S and $A^\star(X) = X$. Then $A^\star(X)$ is a rough ideal of $A(X)$.*

Proof: Since $A^\star(X) = X$, $A^\star(X)$ is a rough sub-lattice of $A(X)$. Let $x, y \in A^\star(X)$ then since $A(X)$ is a rough lattice, therefore $x \vee y \in A^\star(X)$. Let $x \in A^\star(A^\star(X))$ and $y \in A^\star(X)$. Since $A^\star(A^\star(X)) = A^\star(X)$ and $A(X)$ is a rough lattice. Therefore, $x \wedge y \in A^\star(X)$. This evinces the proof of the proposition.

Remark: From Proposition 3.2.6, we conclude that if $A^\star(X) = X$ then $A(X)$ is a rough ideal itself.

3.3 Rough homomorphism in rough lattice

Here, we establish a correspondence between two rough lattices and hereby introduce the notions of rough homomorphism. Let $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ be two lattices, and $S_1 = (L_1, \rho_1)$ and $S_2 = (L_2, \rho_2)$ be two approximation spaces.

Definition 3.3.1 *Let $(A(X), \vee_1, \wedge_1)$ and $(A(Y), \vee_2, \wedge_2)$ be two rough lattices under the approximation spaces $S_1 = (L_1, \rho_1)$ and $S_2 = (L_2, \rho_2)$ respectively. A mapping $f^\star : A^\star(X) \to A^\star(Y)$ is said to be upper rough homomorphism if the following conditions are satisfied.*
(i) $f^\star(a \vee_1 b) = f^\star(a) \vee_2 f^\star(b)\ \forall\ a, b \in A^\star(X)$, and
(ii) $f^\star(a \wedge_1 b) = f^\star(a) \wedge_2 f^\star(b)\ \forall\ a, b \in A^\star(X)$.
A mapping $f_\star : A_\star(X) \to A_\star(Y)$ is said to be lower rough homomorphism, if the following conditions are satisfied.
(i) $f_\star(a \vee_1 b) = f_\star(a) \vee_2 f_\star(b)\ \forall\ a, b \in A_\star(X)$, and
(ii) $f_\star(a \wedge_1 b) = f_\star(a) \wedge_2 f_\star(b)\ \forall\ a, b \in A_\star(X)$.
The pair $(f_\star, f^\star)$ is called rough homomorphism from the rough lattice $A(X)$ to $A(Y)$ in the approximation space $S = (L, \rho)$.

That is, a rough homomorphism $(f_\star, f^\star)$ is a correspondence,
$(f_\star, f^\star) : (A_\star(X), A^\star(X)) \to (A_\star(Y), A^\star(Y))$, and is defined as $(f_\star, f^\star)(a, b) =$

$(f_\star(a), f^\star(b))$, where $a \in A_\star(X)$, $b \in A^\star(X)$ and $f_\star(a) \in A_\star(Y)$, $f^\star(b) \in A^\star(Y)$.

Example 3.3.1 *The set $L = \{1, 2, 4, 5, 10, 20\}$ forms a lattice under the operators '$\vee$' and '$\wedge$' and we define as $a \vee b$ = least common multiple of $\{a, b\}$ and $a \wedge b$ = greatest common divisor of $\{a, b\}$ and the order relation is divisibility. Let us consider an equivalence relation ρ on L by $x\rho y$ iff "x is congruent to y under modulo 4" $\forall\ x, y \in L$. Let $X = \{1, 4, 5\}$. Then $A_\star(X) = \{1, 5\}$ and $A^\star(X) = \{1, 4, 5, 20\}$. Clearly $A(X)$ is a rough lattice.*
Let $U = \{a, b\}$. The power set of U is $P(U) = \{\phi, \{a\}, \{b\}, U\}$ which forms a lattice under the operators $\vee$ and $\wedge$ and we treat as $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ and the order relation is set inclusion. Consider an equivalence relation r on $P(U)$ by ArB if $O(A) = O(B)\ \forall\ A, B \in P(U)$, where $O(A)$ and $O(B)$ denote the orders of A and B respectively. Let $Y = \{\phi, \{a, b\}\}$. Then $A_\star(Y) = \{\phi, \{a, b\}\}$ and $A^\star(Y) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Clearly $A(Y)$ is a rough lattice.
Let $f_\star : A_\star(X) \to A_\star(Y)$ and $f^\star : A^\star(X) \to A^\star(Y)$ be two set valued mappings and are defined by $f_\star(1) = \phi$, $f_\star(5) = \{a, b\}$, $f^\star(1) = \phi$, $f^\star(4) = \{a\}$, $f^\star(5) = \{b\}$, $f^\star(20) = \{a, b\}$. Then $(f_\star, f^\star)$ is a rough homomorphism from $A(X)$ to $A(Y)$.

Proposition 3.3.1 *If $f^\star : A^\star(X) \to A^\star(Y)$ is an onto homomorphism and $x, y \in A^\star(Y)$ with $x < y$, then there exist $a, b \in A^\star(X)$ with $a < b$ such that $x = f^\star(a)$ and $y = f^\star(b)$.*

Proof: Since $f^\star : A^\star(X) \to A^\star(Y)$ is an onto homomorphism and $x, y \in A^\star(Y)$ then there exist a, c in $A^\star(X)$ such that $f^\star(a) = x$ and $f^\star(c) = y$. Now, we have, $f^\star(a \vee c) = f^\star(a) \vee f^\star(c) = x \vee y = y$ as $x < y$. Also since $a \leq a \vee c$. Now if $a = a \vee c$, then $x = f^\star(a) = f^\star(a \vee c) = f^\star(a) \vee f^\star(c) = y$ i.e., $x = y$ which is impossible. Therefore, $a < a \vee c$. Now if we consider $b = a \vee c$, then proof of the proposition is completed.

Proposition 3.3.2 *Upper rough homomorphic image of a rough modular lattice is rough modular lattice.*

Proof: Let $A(X)$ be a rough modular lattice and $f^\star : A^\star(X) \to A^\star(Y)$ be an onto upper rough homomorphism. Let $x, y, z \in A^\star(Y)$ be any three elements with $x > y$. Since $f^\star$ is onto homomorphism, so, $\exists\ a, b, c \in A^\star(X)$ such that $f^\star(a) = x,\ f^\star(b) = y,\ f^\star(c) = z$ where $a > b$.
Now, $A^\star(X)$ is rough modular and $a, b, c \in A^\star(X)$ and $a > b$, thus we get $a \wedge (b \vee c) = b \vee (a \wedge c)$.

$$
\begin{aligned}
\text{Now } x \wedge (y \vee z) &= f^\star(a) \wedge (f^\star(b) \vee f^\star(c)) \\
&= f^\star(a) \wedge f^\star(b \vee c) \\
&= f^\star(a \wedge (b \vee c)) \\
&= f^\star(b \vee (a \wedge c) \\
&= f^\star(b) \vee f^\star(a \wedge c) \\
&= y \vee (x \wedge z).
\end{aligned}
$$

This shows the proof of the proposition.

Proposition 3.3.3 *Upper rough homomorphic image of a rough distributive lattice is rough distributive lattice.*

Proof: Suppose that $A(X)$ is a rough distributive lattice and $f^\star : A^\star(X) \to A^\star(Y)$ be an onto upper rough homomorphism. Let $x,\ y,\ z \in A^\star(Y)$ be any three elements. Since $f^\star$ is onto upper rough homomorphism, then there exist $a, b, c \in A^\star(X)$ such that $f^\star(a) = x,\ f^\star(b) = y,\ f^\star(c) = z$.
Now $A^\star(X)$ is rough distributive and $a,\ b,\ c \in A^\star(X)$, thus we get $a\ \wedge\ (b\ \vee\ c) = (a\ \wedge\ b)\ \vee\ (a\ \wedge\ c)$.

$$
\begin{aligned}
\text{Now } x \wedge (y \vee z) &= f^\star(a) \wedge (f^\star(b) \vee f^\star(c)) \\
&= f^\star(a) \wedge f^\star(b \vee c) \\
&= f^\star(a \wedge (b \vee c)) \\
&= f^\star((a \wedge b) \vee (a \wedge c)) \\
&= f^\star(a \wedge b) \vee f^\star(a \wedge c) \\
&= (f^\star(a) \wedge f^\star(b)) \vee (f^\star(a) \wedge f^\star(c)) \\
&= (x \wedge y) \vee (x \wedge z).
\end{aligned}
$$

Hence, $A(Y)$ is a rough distributive lattice.

3.4 Conclusion

For the first time, in this investigation, we have introduced the notion of rough ideals which is a generalized notion of ideals in a lattice. Important properties of rough ideals are also developed in this chapter. We have established a connection between the rough set and the lattice theory both of which have wide fields of applications in the areas of computer science. This chapter has made on special interest for lattice structure, when the elements of the set are imprecise.

Chapter 4

Soft Rough Lattice*

Rough and soft sets are both mathematical tools for dealing with uncertainty. But soft set theory is utilized for the first time, to generalize Pawlak's rough set model. Soft rough set is a possible fusion between these two mathematical approaches to vagueness. In this chapter, an algebraic connection between soft rough set and algebraic system is investigated and thereby introduce the notion of soft rough lattice in a soft approximation space. We define the concept of a soft rough lattice, soft rough sublattice, modular soft rough lattice and distributive soft rough lattice. Finally, we include some examples to illustrate the definitions.

4.1 Introduction

In 1999, Molodtsov (67) introduced soft set as a mathematical tool for dealing with uncertainty. Thereafter a rapid growth of applications [(59)-(67)] of soft set has been found in many fields of mathematics.

The rough set theory is often a useful and powerful approach to dealing with uncertainty but have some inherent difficulties which mentioned by Molodtsov (67). Soft set theory is a possible way to solve the difficulties of rough set. Thereafter a possible fusion of rough set and soft set has been proposed by Feng et al. (32) for the first time and introduced the concept of soft rough set.

*A part of this chapter has appeared in ***Kragujevac Journal of Mathematics***, SCOPUS, 39(9), 13-20, (2015).

In this theory generalized rough set has been studied based on soft set.

In this chapter, we find an algebraic connection between soft rough set and algebraic system and thereby introduce the notion of soft rough lattice in a soft approximation space.

4.2 Soft set and soft rough set: an overview

Let U be an initial universe of objects and E be the set of parameters and $A \subseteq E$. $P(U)$ is the power set of U.

Definition 4.2.1 *(33) Let $S = (F, A)$ be a soft set over U. Then the pair $P = (U, S)$ is called a soft approximation space. Let $X \subseteq U$. We define the following operations on P.*

$$\begin{aligned} \underline{apr}(X) &= \bigcup_{a \in A} \{F(a) : F(a) \subseteq X\} \text{ and} \\ \overline{apr}(X) &= \bigcup_{a \in A} \{F(a) : F(a) \cap X \neq \phi\}, \end{aligned}$$

which are called soft lower and upper approximations respectively of X and the pair $(\underline{apr}(X), \overline{apr}(X))$ is called soft rough set of X with respect to P and is denoted by $S_r(X)$. If $\underline{apr}(X) = \overline{apr}(X)$, X is said to be soft definable; otherwise X is called soft rough set.

The set of all soft rough sets over U is denoted by $S_R(U)$ with respect to soft approximation space P.

Suppose $S = (F, A)$ is a soft set over U and $P = (U, S)$ is the corresponding soft approximation space. Then soft approximations satisfy the following properties:

$$\begin{aligned}
\underline{apr}(\phi) &= \overline{apr}(\phi) = \phi, \\
\underline{apr}(U) &= \overline{apr}(U) = \bigcup_{a \in A} \{F(a)\}, \\
\underline{apr}(X \cap Y) &\subseteq \underline{apr}(X) \cap \underline{apr}(Y), \\
\underline{apr}(X \cup Y) &\supseteq \underline{apr}(X) \cup \underline{apr}(Y), \\
\overline{apr}(X \cup Y) &= \overline{apr}(X) \cup \overline{apr}(Y), \\
\overline{apr}(X \cap Y) &\subseteq \overline{apr}(X) \cap \overline{apr}(Y), \\
X \subseteq Y \Rightarrow \underline{apr}(X) &\subseteq \underline{apr}(Y) \text{ and } \overline{apr}(X) \subseteq \overline{apr}(Y).
\end{aligned}$$

Definition 4.2.2 *Let $S_r(X) = (\underline{apr}(X), \overline{apr}(X))$ and $S_r(Y) = (\underline{apr}(Y), \overline{apr}(Y))$ be two soft rough set. Then soft rough union and soft rough intersection of $S_r(X)$ and $S_r(Y)$ are defined by*

$$\begin{aligned}
S_r(X) \sqcup S_r(Y) &= (\underline{apr}(X) \bigcup \underline{apr}(Y), \overline{apr}(X) \bigcup \overline{apr}(Y)) \quad \text{and} \\
S_r(X) \sqcap S_r(Y) &= (\underline{apr}(X) \bigcap \underline{apr}(Y), \overline{apr}(X) \bigcap \overline{apr}(Y)),
\end{aligned}$$

respectively, where the symbols $\sqcup$ and $\sqcap$ stand for soft rough union and intersection respectively.

Definition 4.2.3 *Let $S_r(X) = (\underline{apr}(X), \overline{apr}(X))$ and $S_r(Y) = (\underline{apr}(Y), \overline{apr}(Y))$ be two soft rough sets. Then $S_r(Y)$ is said to be soft rough subset of $S_r(X)$, denoted by $S_r(Y) \sqsubseteq S_r(X)$ if $\underline{apr}(Y) \subseteq \underline{apr}(X)$ and $\overline{apr}(Y) \subseteq \overline{apr}(X)$, where $\sqsubseteq$ stands for soft rough inclusion relation.*

4.3 Soft rough lattice

Let $S = (F, A)$ be a soft set over U and $P = (U, S)$ be a soft approximation space and $S_R(U)$ be the set of all soft rough sets over U with respect to P.

Definition 4.3.1 *Let $L \subseteq S_R(U)$, and $\vee$ and $\wedge$ be two binary operations on L. The algebraic structure $(L, \vee, \wedge)$ is said to be soft rough lattice if*

(i) $\vee$ *and* $\wedge$ *are associative*
(ii) $\vee$ *and* $\wedge$ *are commutative*
(iii) $\vee$ *and* $\wedge$ *satisfy absorption laws.*

Example 4.3.1 *Let* $Y = \{u_1, u_2, u_3\}, A = \{e_1, e_2, e_3, e_4\}$. *Let* $S = (F, A)$ *be a soft set over* Y *given by* $F(e_1) = \{u_1\}, F(e_2) = \{u_3\}, F(e_3) = \phi, F(e_4) = \{u_1, u_3\}$. *Let* $X_1 = \phi, X_2 = \{u_1\}, X_3 = \{u_2\}, X_4 = \{u_2, u_3\}, X_5 = \{u_1, u_3\}$. *For simplicity, we denote the subset of* Y, *other than* ϕ *and* Y *by sequence of letters. For example* $\{u_1, u_3\}$ *is written as* u_1u_3. *The soft rough sets on the soft approximation space* $P = (Y, S)$ *are given by* $S_r(X_1) = (\phi, \phi)$, $S_r(X_2) = (u_1, u_1u_2)$, $S_r(X_3) = (\phi, u_1u_2)$, $S_r(X_4) = (u_3, Y)$, $S_r(X_5) = (u_1u_3, Y)$, $S_r(Y) = (Y, Y)$. *Then the set* $L = \{S_r(X_1), S_r(X_2), S_r(X_3), S_r(X_4), S_r(X_5), S_r(Y)\}$ *form soft rough lattice with the operations* $\sqcup$ *and* $\sqcap$ *which are shown in the Tables 4.3.1 and 4.3.2 as follows:*

Table 4.3.1: Soft rough union on L.

$\sqcup$	$S_r(X_1)$	$S_r(X_2)$	$S_r(X_3)$	$S_r(X_4)$	$S_r(X_5)$	$S_r(X_6)$
$S_r(X_1)$	$S_r(X_1)$	$S_r(X_2)$	$S_r(X_3)$	$S_r(X_4)$	$S_r(X_5)$	$S_r(X_6)$
$S_r(X_2)$	$S_r(X_2)$	$S_r(X_2)$	$S_r(X_2)$	$S_r(X_5)$	$S_r(X_5)$	$S_r(X_6)$
$S_r(X_3)$	$S_r(X_3)$	$S_r(X_2)$	$S_r(X_3)$	$S_r(X_4)$	$S_r(X_5)$	$S_r(X_6)$
$S_r(X_4)$	$S_r(X_4)$	$S_r(X_5)$	$S_r(X_4)$	$S_r(X_4)$	$S_r(X_5)$	$S_r(X_6)$
$S_r(X_5)$	$S_r(X_5)$	$S_r(Y)$	$S_r(Y)$	$S_r(Y)$	$S_r(Y)$	$S_r(Y)$
$S_r(X_6)$	$S_r(Y)$	$S_r(Y)$	$S_r(Y)$	$S_r(Y)$	$S_r(Y)$	$S_r(Y)$

Table 4.3.2: Soft rough intersection on L.

$\sqcap$	$S_r(X_1)$	$S_r(X_2)$	$S_r(X_3)$	$S_r(X_4)$	$S_r(X_5)$	$S_r(X_6)$
$S_r(X_1)$	$S_r(X_1)$	$S_r(X_1)$	$S_r(X_1)$	$S_r(X_1)$	$S_r(X_1)$	$S_r(X_1)$
$S_r(X_2)$	$S_r(X_1)$	$S_r(X_2)$	$S_r(X_3)$	$S_r(X_3)$	$S_r(X_2)$	$S_r(X_2)$
$S_r(X_3)$	$S_r(X_1)$	$S_r(X_3)$	$S_r(X_3)$	$S_r(X_3)$	$S_r(X_3)$	$S_r(X_3)$
$S_r(X_4)$	$S_r(X_1)$	$S_r(X_3)$	$S_r(X_3)$	$S_r(X_4)$	$S_r(X_4)$	$S_r(X_4)$
$S_r(X_5)$	$S_r(X_1)$	$S_r(X_2)$	$S_r(X_3)$	$S_r(X_4)$	$S_r(X_5)$	$S_r(X_5)$
$S_r(X_6)$	$S_r(X_1)$	$S_r(X_2)$	$S_r(X_3)$	$S_r(X_4)$	$S_r(X_5)$	$S_r(X_6)$

The Hasse diagram of this soft rough lattice L appears in Figure 4.3.1.

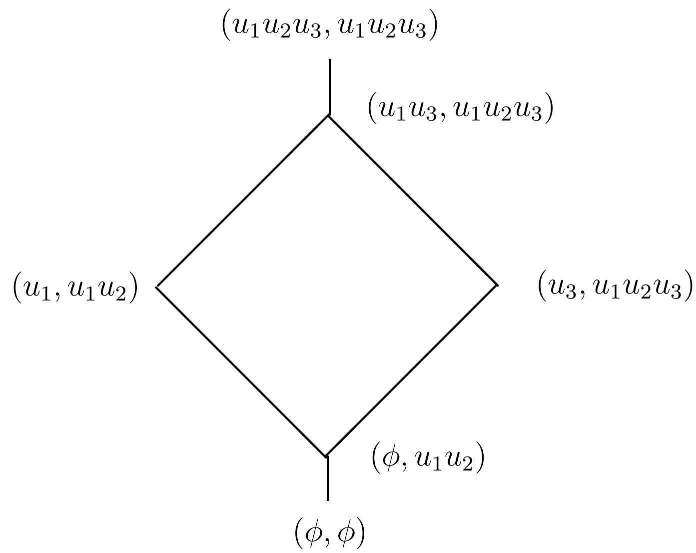

Figure 4.3.1: Soft rough lattice.

Theorem 4.3.1 *Let $(L, \vee, \wedge)$ be a soft rough lattice and $S_r(X), S_r(Y) \in L$. Then a relation $\preceq$ defined by $S_r(X) \preceq S_r(Y) \Leftrightarrow S_r(X) \vee S_r(Y) = S_r(Y)$ or $S_r(X) \wedge S_r(Y) = S_r(X)$ is an order relation on L.*

Proof: *Reflexive:* $S_r(X) \preceq S_r(X) \Leftrightarrow S_r(X) \vee S_r(X) = S_r(X)$.

Antisymmetric: Let $S_r(X) \preceq S_r(Y)$ and $S_r(Y) \preceq S_r(X)$. Then

$$\begin{aligned} S_r(X) &= S_r(X) \wedge S_r(Y) \\ &= S_r(Y) \wedge S_r(X) \\ &= S_r(Y). \end{aligned}$$

Transitive: Let $S_r(X) \preceq S_r(Y)$ and $S_r(Y) \preceq S_r(Z)$. Then

$$\begin{aligned} S_r(X) \wedge S_r(Z) &= (S_r(X) \wedge S_r(Y)) \wedge S_r(Z) \\ &= S_r(X) \wedge (S_r(Y) \wedge S_r(Z)) \\ &= S_r(X) \wedge S_r(Y) \\ &= S_r(X). \end{aligned}$$

Therefore $S_r(X) \preceq S_r(Z)$.

Lemma 4.3.1 *Let $L \in S_R(U)$. The soft rough inclusion relation '$\sqsubseteq$' is an order relation on L.*

Theorem 4.3.2 *Let $(L, \vee, \wedge)$ be a soft rough lattice and $S_r(X), S_r(Y) \in L$. Then*
(i) $S_r(X) \wedge S_r(Y) \preceq S_r(X)$ and $S_r(X) \wedge S_r(Y) \preceq S_r(Y)$
(ii) $S_r(X) \preceq S_r(Y) \vee S_r(X)$ and $S_r(Y) \preceq S_r(X) \vee S_r(Y)$.

Proof: (i) By the definition of order relation '$\preceq$',

$$(S_r(X) \wedge S_r(Y)) \vee S_r(X) = S_r(X) \vee (S_r(X) \wedge S_r(Y)) = S_r(X)$$

Therefore $S_r(X) \vee S_r(Y) \preceq S_r(X)$.
The proof of (ii) can be done in a similar way.

Theorem 4.3.3 *Let $(L, \vee, \wedge)$ be a soft rough lattice and $S_r(W),\ S_r(X),\ S_r(Y),\ S_r(Z) \in L$. Then $S_r(W) \preceq S_r(X)$ and $S_r(Y) \preceq S_r(Z)$ implies*
(i) $S_r(W) \wedge S_r(Y) \preceq S_r(X) \wedge S_r(Z)$ and
(ii) $S_r(W) \vee S_r(Y) \preceq S_r(X) \vee S_r(Z)$.

Proof: From Theorem 4.3.1, we have

$$S_r(W) \wedge S_r(X) = S_r(W) \text{ and } S_r(Y) \wedge S_r(Z) = S_r(Y).$$

Now

$$\begin{aligned} [S_r(W) \wedge S_r(Y)] \wedge [S_r(X) \wedge S_r(Z)] &= [S_r(W) \wedge S_r(X) \wedge S_r(Y)] \wedge S_r(Z) \\ &= [(S_r(W) \wedge S_r(X)) \wedge S_r(Y)] \wedge S_r(Z) \\ &= [S_r(W) \wedge S_r(Y)] \wedge S_r(Z) \\ &= S_r(W) \wedge [S_r(Y) \wedge S_r(Z)] \\ &= S_r(W) \wedge S_r(Y). \end{aligned}$$

This completes the proof of the proposition.

Theorem 4.3.4 *$S_r(X) \vee S_r(Y)$ and $S_r(X) \wedge S_r(Y)$ are the least upper bound and greatest lower bound of $S_r(X)$ and $S_r(Y)$ respectively.*

Proof: From Theorem 4.3.2, $S_r(X) \wedge S_r(Y)$ and $S_r(X) \vee S_r(Y)$ are lower bound and upper bound of $S_r(X)$ and $S_r(Y)$ respectively. Now we are to show that the lower bound and upper bound are respectively greatest lower bound and least upper bound of $S_r(X)$ and $S_r(Y)$. Assume that, $S_r(X) \wedge S_r(Y)$ is not greatest lower bound of $S_r(X)$ and $S_r(Y)$. Then there exists $S_r(Z)$ such that

$$S_r(X) \wedge S_r(Y) \preceq S_r(Z) \preceq S_r(X) \text{ and } S_r(X) \wedge S_r(Y) \preceq S_r(Z) \preceq S_r(Y)$$

Hence $S_r(Z) \wedge S_r(Z) \preceq S_r(X) \wedge S_r(Y)$
or, $S_r(Z) \preceq S_r(X) \wedge S_r(Y)$.
Therefore $S_r(Z) = S_r(X) \wedge S_r(Y)$. Hence a contradiction arises. Therefore $S_r(X) \wedge S_r(Y)$ is the greatest lower bound of $S_r(X)$ and $S_r(Y)$.
By the similar way we can show that $S_r(X) \vee S_r(Y)$ is the least upper bound of $S_r(X)$ and $S_r(Y)$.

Definition 4.3.2 *Let $(L, \vee, \wedge, \preceq)$ be a soft rough lattice and $K \subseteq L$. If K is a soft rough lattice with the operation of L then K is called soft rough sublattice of L.*

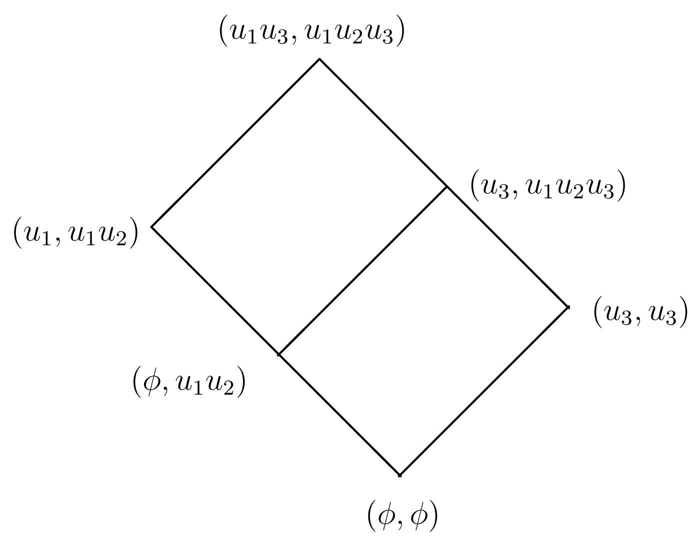

Figure 4.3.2: Distributive soft rough lattice.

Example 4.3.2 *In Example 4.3.1, let $K = \{S_r(X_2), S_r(X_3), S_r(X_4), S_r(X_5)\}$. Then K is a soft rough sublattice of L.*

Theorem 4.3.5 *Every soft rough lattice is a soft rough sublattice itself.*

Proof: Straightforward.

Definition 4.3.3 *A soft rough lattice* $(L, \vee, \wedge, \preceq)$ *is said to be distributive soft rough lattice if* $\forall\ S_r(X), S_r(Y), S_r(Z) \in L$, *then*

$$S_r(X) \wedge (S_r(Y) \vee S_r(Z)) = (S_r(X) \wedge S_r(Y)) \vee (S_r(X) \wedge S_r(Z))$$

Example 4.3.3 *For the soft set* (F, A) *given in Example 4.3.1 if we consider* $X_1 = \phi$, $X_2 = \{u_2\}$, $X_3 = \{u_3\}$, $X_4 = \{u_1, u_2\}$, $X_5 = \{u_2, u_3\}$, $X_6 = \{u_1, u_3\}$, *then* $S_r(X_1) = (\phi, \phi)$, $S_r(X_2) = (\phi, u_1u_2)$, $S_r(X_3) = (u_3, u_3)$, $S_r(X_4) = (u_1, u_1u_2)$, $S_r(X_5) = (u_3, u_1u_2u_3)$, $S_r(X_6) = (u_1u_3, u_1u_2u_3)$. *Then the set* $L = \{S_r(X_1), S_r(X_2), S_r(X_3), S_r(X_4), S_r(X_5), S_r(X_6)\}$ *is distributive soft rough lattice with the operations* $\sqcup$, $\sqcap$. *The Hasse diagram of it appears in Figure 4.3.2.*

Definition 4.3.4 *A soft rough lattice* $(L, \vee, \wedge, \preceq)$ *is said to be modular soft rough lattice if* $\forall\ S_r(X), S_r(Y), S_r(Z) \in L$, *with* $S_r(X) \succeq S_r(Y)$ *the following equality holds:*

$$S_r(X) \wedge (S_r(Y) \vee S_r(Z)) = S_r(Y) \vee (S_r(X) \wedge S_r(Z)).$$

Example 4.3.4 *Let* U *be the the set of universe and* A *be the set of parameters are defined as:* $U = \{u_1, u_2, u_3, u_4\}$ *and* $A = \{e_1, e_2, e_3, e_4\}$. *Let the soft set* $S = (U, A)$ *over* U *is given by* $F(e_1) = \{u_1, u_2\}, F(e_2) = \{u_1\}, F(e_3) = \{u_3, u_4\}, F(e_4) = \{u_2\}$. *Let* $X_1 = \phi, X_2 = \{u_1\}, X_3 = \{u_4\}, X_4 = \{u_2\}, X_5 = \{u_1, u_2, u_3\}$. *Then the soft rough sets on the soft approximation space* $P = (U, S)$ *are given by* $S_r(X_1) = (\phi, \phi)$, $S_r(X_2) = (u_1)$, $S_r(X_3) = (\phi, u_3u_4)$, $S_r(X_4) = (u_2, u_1u_2)$, $S_r(X_5) = (u_1u_2, u_1u_2u_3u_4)$. *Then the set* $L = \{S_r(X_1), S_r(X_2), S_r(X_3), S_r(X_4), S_r(X_5)\}$ *is modular soft rough lattice. Hasse diagram of it appears in Figure 4.3.3.*

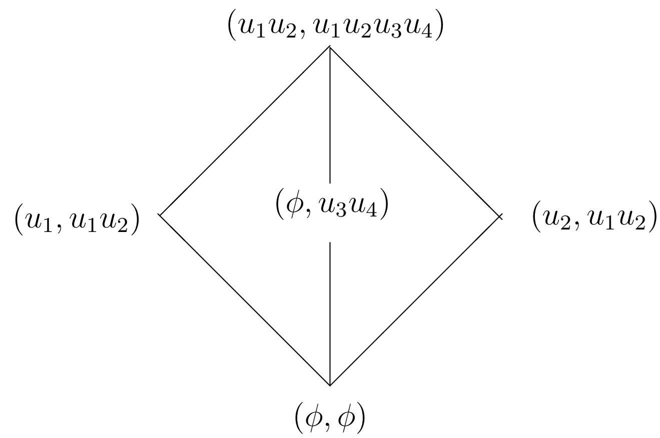

Figure 4.3.3: Modular soft rough lattice.

Theorem 4.3.6 *A distributive soft rough lattice is always modular soft rough lattice.*

Proof: Let $(L, \vee, \wedge, \preceq)$ is said to be distributive soft rough lattice, then $\forall\ S_r(X),\ S_r(Y),\ S_r(Z) \in L$, we have

$$S_r(X) \wedge (S_r(Y) \vee S_r(Z)) = (S_r(X) \wedge S_r(Y)) \vee (S_r(X) \wedge S_r(Z)).$$

Now if $S_r(X) \succeq S_r(Y)$, then $S_r(X) \wedge S_r(Y) = S_r(Y)$.
Therefore $S_r(X) \wedge (S_r(Y) \vee S_r(Z)) = S_r(Y) \vee (S_r(X) \wedge S_r(Z))$.

4.4 Conclusion

Soft rough set is generalization of rough set based on soft set. In this chapter, we have established an algebraic connection between soft rough set and algebraic structure named as lattice. As a result, lattice structure has been developed on soft rough set and defined this concept as soft rough lattice based on soft approximation space. After that we have investigated the several properties and theorems on soft rough lattice. Finally we have justified our proposed soft rough lattice with supporting examples by Hasse diagram.

Chapter 5

Soft Rough Approach to Lattice-Ideal*

Rough and soft sets are two different mathematical tools for dealing with uncertainties. Soft rough set is the study on roughness through soft set. Soft rough set is a fusion, proposed by Feng et al. (32) between these two mathematical approaches to vagueness. The aim of this chapter is to study the lattice theory in the framework of soft rough set. We consider the soft approximation space by means of soft set and define the notions of upper and lower soft rough ideals in a lattice. The numerical examples are presented to support of our study.

5.1 Introduction

Rough set consists of two key notions: rough set approximations and information systems. Rough set approximations are defined by means of an equivalence relation namely indiscernibility relation. Many interesting and meaningful applications in the field of mathematics, computer science and other related fields have been designed with the help of Pawlak's rough set model. A key concept in Pawlak's rough set is an equivalence relation. Similar studies have been made by different researchers. As for example, Skowron and Stepaniuk (109) discussed the tolerance approximation spaces in their study. Slowiniski and

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Vanderpooten (110) presented a generalized definition of rough approximations based on similarity. Greco et al. (38) proposed the rough approximation by dominance relations. Concept of rough set with cover is another generalization. Actually, different types of generalization are based on different granulation structures.
A possible fusion of rough set and soft set is proposed by Feng et al. (32) and introduced the notion of soft rough sets, where instead of equivalence classes, standard soft set model is used to form the granulation structure of the universe, namely the soft approximation space.
Several researchers have showed great interest to study the lattices in the light of rough set environment. For example, see [(24), (29), (46), (55), (77), (79), (92)]. But it is noticed that, no detail studies have been made on lattices under soft rough set. To fill up the gaps, this chapter formulates a general mathematical concept defined on lattices in the framework of soft rough set. We treat the lattice as a universal set and then study the lattices in the shade of soft rough set. We incorporate soft rough ideal and discuss their properties in the soft approximation space which is the main motivation of this chapter.

5.2 Preliminaries

Here we recall some preliminaries which are given in Chapter 4. Let U be the set of initial universe of objects and E be the set of parameters and $A \subseteq E$. $P(U)$ is the power set of U.

Definition 5.2.1 *Let $S = (F, A)$ be a soft set over U. Then the pair $P = (U, S)$ is called a soft approximation space. Let $X \subseteq U$, we define the following operations on P.*

$$\begin{aligned} \underline{apr}(X) &= \bigcup_{a \in A} \{F(a) : F(a) \subseteq X\} \text{ and} \\ \overline{apr}(X) &= \bigcup_{a \in A} \{F(a) : F(a) \cap X \neq \phi\}, \end{aligned}$$

which are called soft lower and upper approximations respectively of X and the pair $(\underline{apr}(X), \overline{apr}(X))$ is called soft rough set of X with respect to P and *is denoted by* $S_r(X)$. *If* $\underline{apr}(X) = \overline{apr}(X)$, X *is said to be soft definable; otherwise* X *is called soft rough set.*

The set of all soft rough sets over U is denoted by $S_R(U)$ with respect to some soft approximation space P.

Proposition 5.2.1 *Suppose* $S = (F, A)$ *is a soft set over* U *and* $P = (U, S)$ *is the corresponding soft approximation space. Then soft approximations satisfy the following properties:*

$$\underline{apr}(\phi) = \overline{apr}(\phi) = \phi$$
$$\underline{apr}(U) = \overline{apr}(U) = \bigcup_{a\in A}\{F(a)\}$$
$$\underline{apr}(X \cap Y) \subseteq \underline{apr}(X) \cap \underline{apr}(Y)$$
$$\underline{apr}(X \cup Y) \supseteq \underline{apr}(X) \cup \underline{apr}(Y)$$
$$\overline{apr}(X \cup Y) = \overline{apr}(X) \cup \overline{apr}(Y)$$
$$\overline{apr}(X \cap Y) \subseteq \overline{apr}(X) \cap \overline{apr}(Y)$$
$$X \subseteq Y \Rightarrow \underline{apr}(X) \subseteq \underline{apr}(Y) \text{ and } \overline{apr}(X) \subseteq \overline{apr}(Y).$$

Definition 5.2.2 *(32) Let* $S = (F, A)$ *be a soft set over* U*. If* $\bigcup_{a\in A}\{F(a)\} = U$*, then* S *is said to be a full soft set.*

Definition 5.2.3 *(32) Let* $S = (F, A)$ *is a soft set over* U*. If for any* $a_1, a_2 \in A$*, there exists* $a_3 \in A$ *such that* $F(a_3) = F(a_1) \cap F(a_2)$*, whenever* $F(a_1) \cap F(a_2) \neq \phi$*, then* S *is said to be intersection complete soft set.*

Proposition 5.2.2 *(32) Let* $S = (F, A)$ *be the intersection complete soft set over* U*. Then we have* $\underline{apr}(U) = \overline{apr}(U) = U$*.*

Proposition 5.2.3 *(32) Let* $S = (F, A)$ *be an intersection complete soft set over* U*. Then we have* $\underline{apr}(X \cap Y) = \underline{apr}(X) \cap \underline{apr}(Y)$ *for all* $X, Y \subseteq U$*.*

Definition 5.2.4 *(32) A soft set* $S = (F, A)$ *over* U *is called a partition soft set if* $\{F(a) : a \in A\}$ *forms a partition of* U*.*

By definition, we immediately have that every partition soft set is a full soft set.

Proposition 5.2.4 *(32) Suppose that $S = (F, A)$ is a partition soft set over U and $P = (U, S)$ is the corresponding soft approximation space. Then soft rough approximation satisfies the following properties:*
(i) $\underline{apr}(X) \subseteq X \subseteq \overline{apr}(X)$
(ii) $\underline{apr}(X \cap Y) = \underline{apr}(X) \cap \underline{apr}(Y)$
(iii) $\overline{apr}(X \cup Y) = \overline{apr}(X) \cup \overline{apr}(Y)$.

Proposition 5.2.5 *(32) Let $S = (F, A)$ be a partition soft set over U. Then $\overline{apr}(\overline{apr}(X)) = \overline{apr}(X)$ for all $X \subseteq U$.*

5.3 Soft rough ideal

Let $S = (F, A)$ be a soft set over a lattice $(L, \leq, \vee, \wedge)$ and $P = (L, S)$ a soft approximation space. Also let X be sub-set of L. Through out the chapter, we denote the notation L for a lattice.

Definition 5.3.1 *X is called an upper soft rough ideal of L in the approximation space P if $\overline{apr}(X)$ is an ideal of L. X is called a lower soft rough ideal of L in the approximation space P if $\underline{apr}(X)$ is an ideal of L. X is said to be soft rough ideal of L in P if it is both lower and upper soft rough ideals.*

Empty set ϕ is always a soft rough ideal and if $S = (L, F)$ is a full soft set then L is trivial soft rough ideal in the approximation space $P = (L, F)$.

Example 5.3.1 *Let $L = \{u_1, u_2, u_3, u_4\}$ be a lattice. The partial order relation over L is defined as shown in Figure 5.3.1. The soft set $S = (F, A)$ over L is defined as follows: $F(e_1) = \{u_1\}, F(e_2) = \{u_2, u_3\}, F(e_3) = \{u_4\}, F(e_4) = \{u_1, u_3\}$. Here $X = \{u_1, u_3\}$ is a soft rough ideal of L in the approximation space $P = (L, S)$ where $\underline{apr}(X) = \{u_1, u_3\}$ and $\overline{apr}(X) = \{u_1, u_2, u_3, \}$.*

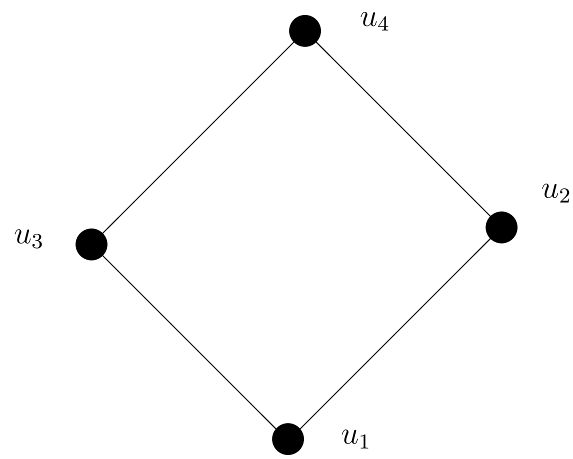

Figure 5.3.1: Lattice.

Proposition 5.3.1 *Let X be soft rough ideal of L in the soft approximation space $P = (L, S)$. Then $\underline{apr}(X)$ and $\overline{apr}(X)$ are sublattices of L.*

Proof: Let $a, b \in \underline{apr}(X)$. Then by definition $a \vee b \in \underline{apr}(X)$. Let $a, l \in \underline{apr}(X)$ this implies $a \in \underline{apr}(X)$ and $l \in L$ and by definition of lower soft rough ideal $a \wedge l \in \underline{apr}(X)$. Therefore $\underline{apr}(X)$ is a sublattice of L. Similarly, we can prove $\overline{apr}(X)$ is a sublattice of L.

Proposition 5.3.2 *Let $S = (F, A)$ be an intersection complete soft set over L and $P = (L, S)$ be a soft approximation space. Then intersection of two lower soft rough ideals is an ideal.*

Proof: Let X, Y be two lower soft rough ideals of lattice L. Since S is an intersection complete soft set, $\underline{apr}(X \cap Y) = \underline{apr}(X) \cap \underline{apr}(X)$. Now $\underline{apr}(X)$ and $\underline{apr}(Y)$ are ideals of L and since intersection of two ideals is an ideal. This completes the proof of the proposition.

Proposition 5.3.3 *Let $S = (F, A)$ be a soft set over L and $P = (L, S)$ a soft approximation space. Then union of two upper soft rough ideals is an upper soft rough ideal if one is contained in the other.*

Proof: Let X, Y be two upper soft rough ideals of L and also let $X \subseteq Y$. Then $\overline{apr}(X) \subseteq \overline{apr}(Y)$. Now $\overline{apr}(X \bigcup Y) = \overline{apr}(X) \bigcup \overline{apr}(Y) = \overline{apr}(Y)$, which is an ideal.

Proposition 5.3.4 *Let $S = (F, A)$ be a soft set over L and $P = (L, S)$ a soft approximation space. If X is a lower soft rough ideal, then $\underline{apr}(X)$ is a lower soft rough ideal.*

Proof: Let X be a lower soft rough ideal of L in P. Therefore $\underline{apr}(X)$ is an ideal. Let $x, y \in \underline{apr}(\underline{apr}(X))$. Since $\underline{apr}(\underline{apr}(X)) \subseteq \underline{apr}(X)$ for any $X \subseteq L$. Therefore $x, y \in \underline{apr}(X)$. Since $\underline{apr}(X)$ is an ideal, therefore $x \vee y \in \underline{apr}(X)$. Then $x \vee y \in F(a) \subseteq X$ for some $a \in A$. But $\underline{apr}(X) = \bigcup_{a\in A}\{F(a) : F(a) \subseteq X\}$. Hence, we have $x \vee y \in F(a) \subseteq \underline{apr}(X)$ for some $a \in A$. Thus $x \vee y \in \underline{apr}(\underline{apr}(X))$. Again let $x \in \underline{apr}(\underline{apr}(X))$ and $l \in L$, this imply $X \in \underline{apr}(X)$, $l \in L$ and since $\underline{apr}(X)$ is an ideal, therefore $x \wedge l \in \underline{apr}(X)$. Similarly, we can prove $x \wedge l \in \underline{apr}(\underline{apr}(X))$. Therefore the proof of the proposition is completed.

Proposition 5.3.5 *Let $S = (F, A)$ be a soft set over L and $P = (L, S)$ a soft approximation space. If X is an upper soft rough ideal then $\overline{apr}(X)$ is a lower soft rough ideal.*

Proof: Let X be an upper soft rough ideal of L in P. Therefore $\overline{apr}(X)$ is an ideal of L. We have to prove that $\underline{apr}(\overline{apr}(X))$ is an ideal of L. Let $x, y \in \underline{apr}(\overline{apr}(X))$. Since $\underline{apr}(\overline{apr}(X)) \subseteq \overline{apr}(X)$ for any $X \subseteq L$. Therefore $x, y \in \overline{apr}(X)$. Since $\overline{apr}(X)$ is an ideal, therefore $x \vee y \in \overline{apr}(X)$. Then $x \vee y \in F(a)$ and $F(a) \cap X \neq \phi$ for some $a \in A$. But $\overline{apr}(X) = \bigcup_{a\in A}\{F(a) : F(a) \cap X \neq \phi\}$. Hence, we have $x \vee y \in F(a) \subseteq \overline{apr}(X)$ for some $a \in A$. So $x \vee y \in \underline{apr}(\overline{apr}(X))$. Again, let $x \in \underline{apr}(\overline{apr}(X))$, $l \in L$, then $X \in \overline{apr}(X))$, $l \in L$ and since $\overline{apr}(X)$ is an ideal, therefore $x \wedge l \in \overline{apr}(X)$. Similarly, we can establish $x \wedge l \in \underline{apr}(\overline{apr}(X))$. Hence $\overline{apr}(X)$ is an lower soft rough ideal of L as required.

Proposition 5.3.6 *Let $S = (F, A)$ be a partition soft set over L and $P =$*

(L, S) is a soft approximation space. If X is a upper soft rough ideal then $\overline{apr}(X)$ is also soft rough ideal.

Proof: Let X be an upper soft rough ideal of L in P. Therefore $\overline{apr}(X)$ is an ideal of L. We have to prove that $\overline{apr}(X)$ is an ideal of L. We know that for a partition soft set $\overline{apr}(\overline{apr}(X)) = \overline{apr}(X)$. Since $\overline{apr}(X)$ is an ideal, therefore, $\overline{apr}(\overline{apr}(X))$ is an ideal. This completes the proof of the proposition.

5.4 Conclusion

The set from lattice structure is treated here as universal set and defined soft rough set on it. We have constructed the soft rough ideal and studied their properties in a soft approximation space. We have established the connection between soft rough set and lattice theory both of which have wide field of applications in the area of computer science and information sciences. The contents of this chapter may be extended to more results on lattices under soft rough environment.

Chapter 6

Approximation of Rough Soft Set and Its Application to Lattice*

The approximation of soft set is presented in modified soft rough (MSR) approximation space in this chapter, i.e., approximation of an information system with respect to another information one. Besides, the concept of rough soft set is introduced in a modified soft rough approximation space. Various properties are studied like subset, union, intersection on rough soft set with some propositions presented on rough soft set. Moreover, the measure of roughness of soft set is defined in MSR-approximation space and the order relation is introduced on soft set. Furthermore, lattice theory is studied in the MSR-approximation space under a modified rough soft environment. Finally, some realistic examples are considered to usefulness and illustrate of the chapter.

6.1 Introduction

Soft set theory (SST) and rough set theory (RST) are treated as mathematical tools to deal with uncertainty. A connection between these two has been made by Feng et al. (32) and introduced the notion of soft rough set. In their model, they described the parameterize subset on the universe of discourse. As a result, some unusual situations have occurred, like upper approximation of a non-empty set may be empty. Upper approximation of a subset may not

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*A part of this chapter has appeared in ***Fuzzy Information and Engineering***, Elsevier, 7, 379-387, (2015).

contain the set which does not occur in classical rough set theory. To overcome these difficulties, Shabir et al. (107) redefined a soft rough set model called MSR set.

In this chapter, we study the approximations of an information system with respect to another information ones. We approximate a soft set with respect a modified soft rough approximation space and introduce the notion of rough soft set. Here, we endeavor to establish link between soft set and rough set in connection with an application in lattice. Also, we introduce the concept of measure of roughness in a soft set and consequently some propositions and examples are presented here.

6.2 Preliminaries

Here, we present some preliminary definitions on MSR set and also recall the concept of information system and indiscernibility relation which are very much essential in the sequel.

Definition 6.2.1 *An information system (or a knowledge representation system) is a pair (U, A) of non-empty finite sets U and A where U is a set of objects and A is a set of attributes; each attribute $a \in A$ is a function $a : U \to V_a$, where V_a is called set of values of attribute a.*

Let U be a non-empty set of universe and R be an equivalence relation on U. The pair (U, R) is called Pawlak's approximation space. The equivalence relation R is often called indiscernibility relation and related to an information system. An indiscernibility relation $R = I(B)$, $B \subseteq A$ is defined as:

$$(x, y) \in I(B) \Leftrightarrow a(x) = a(y), \quad \forall\, a \in B,$$

where $x, y \in U$, and $a(x)$ denotes the value of attribute a for object x.

Using this indiscernibility relation, one can define the following operations as:

$$A_\star(X) = \{x \in U : [x]_R \subseteq X\} \text{ and } A^\star(X) = \{x \in U \mid [x]_R \cap X \neq \phi\}.$$

For any $X \subseteq U$, $A_\star(X)$ and $A^\star(X)$ are called lower and upper approximations of X respectively. If $A_\star(X) \neq A^\star(X)$, then X is called the rough set in the approximation space (U, R). The difference $A^\star(X) - A_\star(X)$ is called boundary region of X and is treated as the area of uncertainty.

Let U be an initial universe of objects and E be the set of parameters and $A \subseteq E$. $P(U)$ is the power set of U.

Definition 6.2.2 *(107) Let (F, A) be a soft set over U, where F is a mapping from A to $P(U)$, i.e., $F : A \to P(U)$, where $P(U)$ is the power set of U. Let $\psi : U \to P(A)$ be another mapping defined as $\psi(x) = \{a : x \in F(a)\}$. Then the pair (U, ψ) is called MSR approximation space and for any $X \subseteq U$, lower MSR-approximation and upper MSR-approximation respectively are defined as follows:*

$\underline{X}_\psi = \{x \in U : \psi(x) \neq \psi(y)\ \forall\ y \in X^c\},\ \ \textit{where}\ X^c = U - X,$

$\overline{X}_\psi = \{x \in U : \psi(x) = \psi(y)\ \textit{for}\ \ \textit{some}\ y \in X\}.$

If $\underline{X}_\psi \neq \overline{X}_\psi$, then X is said to be modified soft rough set.

Example 6.2.1 *Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of schools considered as universal set and an attribute set $A = \{e_1, e_2, e_3, e_4\}$. Here e_1 denotes good location, e_2 denotes sufficient teachers, e_3 denotes good maintenance of discipline, e_4 denotes good relation in teacher-student. Let the soft set (F, A) over U be given by the following table:*

Table 6.2.1: Table represents the soft set.

	e_1	e_2	e_3	e_4
u_1	1	1	0	0
u_2	0	0	1	1
u_3	1	1	1	0
u_4	0	1	1	1
u_5	0	1	0	1

Here 1 and 0 denote 'yes' and 'no' respectively. Then from the definition of MSR set, $\psi : U \to P(A)$ is defined as follows:

$$\psi(u_1) = \{e_1, e_2\};\ \psi(u_2) = \{e_2\};\ \psi(u_3) = \{e_4\};\ \psi(u_4) = \{e_1, e_3\} = \psi(u_5).$$

Let $X = \{u_1, u_3, u_5\}$. *Therefore for the MSR-approximation space* (U, ψ), *we can write*

$$\underline{X}_\psi = \{u_1, u_3\} \text{ and } \overline{X}_\psi = \{u_1, u_3, u_4, u_5\}.$$

Clearly, $\underline{X}_\psi \neq \overline{X}_\psi$, *so* X *is a modified soft rough set.*

6.3 Rough soft set

In this section, the notion of rough soft set in modified soft rough approximation space is introduced.

Definition 6.3.1 *Let* (F, A) *be a soft set over* U *and* (U, ψ) *be an MSR-approximation space with respect to* A. *Let* (G, B) *be another soft set over* U. (G, B) *is said to be rough soft set with respect to a parameter* $e \in B$ *if* $\underline{G(e)}_\psi \neq \overline{G(e)}_\psi$, (G, B) *is said to be a full rough soft set or a simply rough one if* $\underline{G(e)}_\psi \neq \overline{G(e)}_\psi$ $\forall\ e \in B$ *and we denote it by* $RsG(e_B)$. *We denote rough soft set with respect to* e *by* $RsG(e) = (\underline{G(e)}_\psi, \overline{G(e)}_\psi)$.

Example 6.3.1 *Considering a universal set of batsman* $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ *and an attribute set* $A = \{e_1, e_2, e_3\}$ *where* e_1 *denotes Bold out,* e_2 *denotes Catch out,* e_3 *denotes LBW. Let* (F, A) *be a soft set representing the record of the player given by the following table:*

Table 6.3.1: Table represents the data on information system.

	e_1	e_2	e_3
u_1	1	1	1
u_2	1	0	1
u_3	0	1	1
u_4	1	0	1
u_5	0	0	1
u_6	1	0	0
u_7	1	1	0
u_8	0	0	1

Here 1 *and* 0 *denotes 'yes' and 'no' respectively. Then from the definition of MSR set,* $\psi : U \to P(A)$ *is defined as follows:*

$\psi(u_1) = \{e_1, e_2, e_3\}$; $\psi(u_2) = \{e_1, e_3\}$; $\psi(u_3) = \{e_2, e_3\}$; $\psi(u_4) = \{e_1, e_3\}$; $\psi(u_5) = \{e_3\}$; $\psi(u_6) = \{e_1\}$; $\psi(u_7) = \{e_1, e_2\}$; $\psi(u_8) = \{e_2\}$.
Let (G, B) be another soft set defined as
$G(e_1) = \{u_1, u_2, u_4, u_6, u_7\}$; $G(e_2) = \{u_1, u_3, u_4, u_6\}$;
$G(e_3) = \{u_2, u_3, u_5, u_6, u_7, u_8\}$; $G(e_4) = \{u_1, u_2, u_3, u_5, u_6, u_7\}$, *where e_1 denotes Bold out, e_2 denotes Catch out, e_3 denotes LBW and e_4 denotes Run out. Lower MSR-approximation set and upper MSR-approximation set of (G, B) are*

$\underline{G(e_1)}_\psi = \{u_1, u_2, u_4, u_6, u_7\}$; $\overline{G(e_1)}_\psi = \{u_1, u_2, u_4, u_6, u_7\}$;
$\underline{G(e_2)}_\psi = \{u_1, u_3, u_6\}$; $\overline{G(e_2)}_\psi = \{u_1, u_2, u_3, u_4, u_6\}$;
$\underline{G(e_3)}_\psi = \{u_3, u_5, u_6, u_7, u_8\}$; $\overline{G(e_3)}_\psi = \{u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$;
$\underline{G(e_4)}_\psi = \{u_1, u_3, u_5, u_6, u_7\}$; $\overline{G(e_4)}_\psi = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$.

Clearly, (G, B) is a rough soft set with respect to parameters e_2, e_3 and e_4.

Proposition 6.3.1 *Let (F, A) be a soft set over U and (U, ψ) be an MSR approximation space with respect to A. Let (G_1, B_1), (G_2, B_2) be two rough soft sets. Then*

$$H(e) = \begin{cases} G_1(e), & \text{if } e \in B_1 - B_2, \\ G_2(e), & \text{if } e \in B_2 - B_1, \\ G_1(e) \cup G_2(e), & \text{if } e \in B_1 \cap B_2 \end{cases}$$

is a rough soft set if $B_1 \cap B_2 = \phi$.

Proposition 6.3.2 *Let (F, A) be a soft set over U and (U, ψ) be an MSR approximation space with respect to A. Let (G_1, B_1), (G_2, B_2) be two rough soft sets. Then $\forall\, e \in B_1 \cap B_2$, $H(e) = G_1(e) \cap G_2(e)$ is a rough soft set.*

Definition 6.3.2 *Let (F, A) be a soft set over U and (U, ψ) be an MSR approximation space with respect to A. Let (G_1, B_1), (G_2, B_2) be two rough soft sets. (G_1, B_1) is said to be rough soft subset of (G_2, B_2) if*

(i) *$B_1 \subseteq B_2$, and*

(ii) *$\forall\, e \in B_1$, $\underline{G_1(e)}_\psi = \underline{G_2(e)}_\psi$ and $\overline{G_1(e)}_\psi = \overline{G_2(e)}_\psi$.*

We write $(G_1, B_1) \sqsubseteq (G_2, B_2)$, where $\sqsubseteq$ denotes soft rough subset.

Definition 6.3.3 *The union of rough soft sets $RsG(e_1)$ and $RsG(e_2)$ with respect to the parameters e_1 and e_2 respectively in MSR-approximation space (U, ψ) is denoted by $RsG(e_1) \sqcup RsG(e_2)$ and is defined as $RsG(e_1) \sqcup RsG(e_2) = (\underline{G(e_1)}_\psi \cup \underline{G(e_2)}_\psi, \overline{G(e_1)}_\psi \cup \overline{G(e_2)}_\psi)$.*
The union of rough soft sets $RsG(e_A)$ and $RsG(e_B)$ is defined as $RsG(e_A) \sqcup RsG(e_B) = (\underline{G(e)}_\psi \cup \underline{G(f)}_\psi, \overline{G(e)}_\psi \cup \overline{G(f)}_\psi)$ for all $e \in A$ and $f \in B$.

Definition 6.3.4 *The intersection of rough soft sets $RsG(e_1)$ and $RsG(e_2)$ with respect to parameters e_1 and e_2 respectively in MSR-approximation space (U, ψ) is denoted by $RsG(e_1) \sqcap RsG(e_2)$ and is defined as*
$RsG(e_1) \sqcap RsG(e_2) = (\underline{G(e_1)}_\psi \cap \underline{G(e_2)}_\psi, \overline{G(e_1)}_\psi \cap \overline{G(e_2)}_\psi)$.
The intersection of rough soft sets $RsG(e_A)$ and $RsG(e_B)$ is defined as $RsG(e_A) \sqcap RsG(e_B) = (\underline{G(e)}_\psi \cap \underline{G(f)}_\psi, \overline{G(e)}_\psi \cap \overline{G(f)}_\psi)$ for all $e \in A$ and $f \in B$.

Proposition 6.3.3 *Let (G, B) be a soft set over U and (U, ψ) be a MSR-approximation space. Then set $(RsG(e), \sqcup, \sqcap)$, $\forall\ e \in B$ together with (U, U) and (ϕ, ϕ) form a lattice where the order relation $\subseteq$ is defined as $RsG(e_1) \subseteq RsG(e_2) \Rightarrow \underline{G(e_1)}_\psi \subseteq \underline{G(e_2)}_\psi$ and $\overline{G(e_1)}_\psi \subseteq \overline{G(e_2)}_\psi$.*

Example 6.3.2 *Consider the Example 6.3.1, for simplicity, we denote the subset of U, other than ϕ and U by sequence of numeric suffices. For example u_1, u_2, u_4, u_6, u_7 is written as 12467. The Hasse diagram of lattice under the soft rough set is given in Figure 6.3.1.*

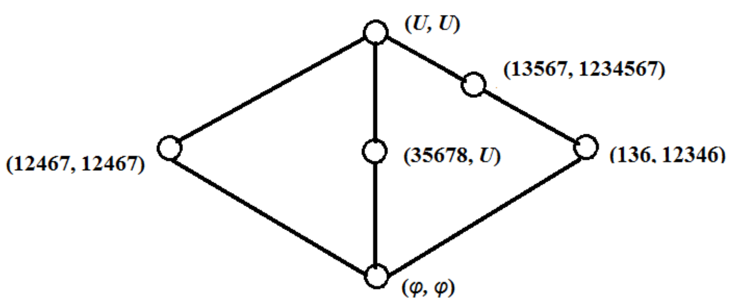

Figure 6.3.1: Lattice under rough soft set.

Proposition 6.3.4 *Let (G_1, B_1) be a soft subset of (G_2, B_2). If (G_2, B_2) is a rough soft set, then (G_1, B_1) is rough soft subset of (G_2, B_2).*

Proof: Since (G_1, B_1) is soft subset of (G_2, B_2), therefore $G_1(e) = G_2(e)$ for all $e \in B_1$. Therefore $\underline{G_1(e)}_\psi = \underline{G_2(e)}_\psi$ and $\overline{G_1(e)}_\psi = \overline{G_2(e)}_\psi$ for all $e \in B_1$. Hence (G_1, B_1) is rough soft subset.

Definition 6.3.5 *Let (F, A) be a soft set over U and (U, ψ) be an MSR approximation space with respect to A. Let (G, B) be another soft set over U. We define*
$\underline{G(e_1)}_\psi = \underline{G(e_2)}_\psi \Leftrightarrow G(e_1) \simeq G(e_2)$,
$\overline{G(e_1)}_\psi = \overline{G(e_2)}_\psi \Leftrightarrow G(e_1) \simeq G(e_2)$,
$\underline{G(e_1)}_\psi = \underline{G(e_2)}_\psi$ and $\overline{G(e_1)}_\psi = \overline{G(e_2)}_\psi \Leftrightarrow G(e_1) \approx G(e_2)$.
These binary relations are called lower rough soft, upper rough soft and rough soft equal relations respectively.

Proposition 6.3.5 *The rough soft equal relation is an equivalence one.*

Proof: Straightforward.

6.4 Measure of roughness of soft set

In this section, we study the measure of roughness of a soft set with respect to an MSR-approximation space.

Definition 6.4.1 *Let (F, A) be a soft set over U and (U, ψ) be an MSR-approximation space. Let (G, B) be another soft set over U. Measure of roughness of (G, B) with respect to parameter $e \in B$ is denoted by $R_{G(e)}$ and is defined as follows:*

$$R_{G(e)} = \frac{|\underline{G(e)}_\psi|}{|\overline{G(e)}_\psi|}.$$

Clearly, $0 \leq R_{G(e)} \leq 1$. Now, we define binary relation '$\equiv$' on soft set (G, B) as $G(e_1) \equiv G(e_2)$ if and only if $R_{G(e_1)} = R_{G(e_2)}$ for $e_1, e_2 \in B$.

Proposition 6.4.2 *'$\equiv$' is an equivalence relation on (G, B). The partition $[G(e)]_{\equiv}$ has a strict order in its element.*

Proof: The measure of roughness of all members of a class is the same. Therefore, each class is characterized by a unique number belonging to interval $[0, 1]$. So there is a strict order among these classes.

Proposition 6.4.3 *(G, B) forms a chain by the order relation $\equiv$.*

Proof: Since relation '$\equiv$' partitions the soft set and the partition has a strict order relation, therefore the soft set forms a chain.

Example 6.4.1 *Suppose (F, A) is a soft set over $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and the set of parameters is $A = \{e_1, e_2, e_3, e_4\}$, where e_1 denotes under stress, e_2 denotes young age, e_3 denotes drug addicted, e_4 denotes healthy. The soft set (F, A) is given in Table 6.4.1.*

Table 6.4.1: Table represents the data on information system.

	e_1	e_2	e_3	e_4
u_1	0	1	0	1
u_2	0	1	0	1
u_3	0	0	0	1
u_4	0	0	0	0
u_5	1	0	0	0
u_6	0	0	0	1

Now (U, ψ) is the modified soft rough approximation space where $\psi : U \to P(A)$ is defined as $\psi(u_1) = \{e_2, e_4\}$; $\psi(u_2) = \{e_2, e_4\}$; $\psi(u_3) = \{e_4\}$; $\psi(u_4) = \phi$; $\psi(u_5) = \{e_1\}$; $\psi(u_6) = \{e_4\}$. Let (G, B) be another soft set over U and the set of parameters $B = \{e_1, e_2, e_3, e_4\}$, where e_1 denotes smokers; e_2 denotes smokers and drinkers; e_3 denotes men; e_4 denotes people live in city. The soft set (G, B) is given in Table 6.4.2.
Lower MSR-approximation and upper MSR-approximation of (G, B) are $\underline{G(e_1)}_{\psi} = \phi$; $\overline{G(e_1)}_{\psi} = \{u_1, u_2\}$; $\underline{G(e_2)}_{\psi} = \{u_4\}$; $\overline{G(e_2)}_{\psi} = \{u_4\}$; $\underline{G(e_3)}_{\psi} = \{u_3, u_4, u_5, u_6\}$; $\overline{G(e_3)}_{\psi} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$; $\underline{G(e_4)}_{\psi} = \{u_1, u_2\}$; $\overline{G(e_4)}_{\psi} = \{u_1, u_2, u_3, u_6\}$. Clearly, (G, B) is a rough soft set with respect to parameters e_1, e_3, e_4. Now $R_{G(e_1)} = 0$, $R_{G(e_2)} = 1$, $R_{G(e_3)} = 2/3$, $R_{G(e_4)} = 1/2$.

Table 6.4.2: Table represents the another data on information system.

	e_1	e_2	e_3	e_4
u_1	1	0	1	1
u_2	0	0	1	1
u_3	0	0	1	1
u_4	0	1	0	0
u_5	0	0	0	0
u_6	0	0	0	0

The Hasse diagram of this chain is given in Figure 6.4.1.

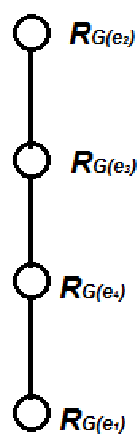

Figure 6.4.1: Chain form by rough soft set.

6.5 Conclusion

Soft and rough sets are two different approaches to uncertainty and rough soft set is a fusion of these two theories. We have introduced the concept of approximation on an information system with respect to another information one based on an MSR-approximation space. We have constructed the rough soft set and studied their properties in MSR-approximation space. Besides, we have established the connection between a rough soft set and a lattice theory by measuring the roughness of a soft set. The theme of this chapter may be extended to further results in lattices under different environments of soft-rough relations.

Chapter 7

An Another Approach for Cartesian Product on Soft Set Relation and Its Application to Lattice*

In this chapter, the concept of cartesian product on soft sets is introduced in a new way. Besides this, based on this cartesian product, a soft set relation is defined. Soft set relation is also constructed based on the induced binary relation in the set of parameters of soft sets. A connection between the relations is also established. Moreover, lattice theory is studied on soft sets by considering with soft set relation.

7.1 Introduction

In this chapter we study lattice structure of soft set based on the Cartesian product of soft sets. Up-to-date, many researchers have shown great interest to study on soft set and they established the various properties on it. However, some studies [(5), (6), (90)] are available to define the Cartesian product and soft set relation on a soft set, but in this chapter, we describe the Cartesian product for soft set relation on soft set in new way. Moreover, based on the ideas of the Cartesian product and soft set relation, we newly formulate the

*A part of this chapter has communicated for publication to the International Journal.

soft lattice, soft modular lattice, soft distributive lattice and soft equivalent relation.

7.2 Preliminaries

Here, we present some basic definitions on soft sets, which are the most useful to design of our proposed study.
Throughout this chapter, U is an universal set, $P(U)$ is the power set of U and E is a set of the parameters. A soft set over U is a parameterized family of subsets of U. Every set $F(e), \forall\, e \in A$ from this family may be considered as the set of e-elements of the soft set (F, A) or considered as the set of e-approximate elements of the soft set. According to this manner, we can view a soft set (F, A) as consisting of collection of approximations: $(F, A) = \{(e, F(e)) : e \in A\}$. Without loss of generality, we write simply $F(e)$ for the pair $(e, F(e))$ and if $F(e) = V$, then we write V_e for the pair (e, V).

Example 7.2.1 *Let (F, A) describe the family of students interested in different subjects. Suppose there are four students in the universe U which is given by $U = \{u_1, u_2, u_3, u_4\}$ and $A = \{e_1, e_2, e_3\}$, where e_1 stands for Mathematics, e_2 stands for Science, e_3 stands for English. Suppose $F(e_1) = \{u_1, u_2, u_3\}$, $F(e_2) = \{u_1, u_2\}$, $F(e_3) = \{u_2, u_3, u_4\}$. Thus the soft set (F, A) over U is given by*

$$\begin{aligned}(F, A) &= \{(e_1, F(e_1)), (e_2, F(e_2)), (e_3, F(e_3))\}\\ &= \{F(e_1), F(e_2), F(e_3)\}\\ &= \{\{u_1, u_2, u_3\}_{e_1}, \{u_1, u_2\}_{e_2}, \{u_2, u_3, u_4\}_{e_3}\}.\end{aligned}$$

Definition 7.2.1 *(60) Let (F, A) and (G, B) be two soft sets over U. (G, B) is said to be a soft subset of (F, A), if the following conditions are satisfied;*
(i) $B \subseteq A$, and
(ii) $F(e) = G(e)$, $\forall\, e \in B$.
We denote $(G, B) \sqsubseteq (F, A)$ where "$\sqsubseteq$" stands for subset over a soft set.
Two soft sets (F, A) and (G, B) over a common universe U are said to be equal if $(G, B) \sqsubseteq (F, A)$ and $(F, A) \sqsubseteq (G, B)$.

Definition 7.2.2 *(60) Union of two soft sets* (F, A) *and* (G, B) *over a common universe* U *is a soft set* (H, C)*, where* $C = A \cup B$*, and* $\forall\ e \in C$*,*

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(H, C) = (F, A) \sqcup (G, B)$ *where the symbol "*$\sqcup$*" stands for union over the soft sets.*

Definition 7.2.3 *(60) Intersection of two soft sets* (F, A) *and* (G, B) *over a common universe* U *is a soft set* (H, C)*, where* $C = A \cap B$*, and* $\forall\ e \in C$*,* $H(e) = F(e) \cap G(e)$*.*
We write $(H, C) = (F, A) \sqcap (G, B)$ *where the notation "*$\sqcap$*" defines for intersection over the soft sets.*

7.3 Cartesian product and relation

In this section, we introduce a new concept on Cartesian product between the two soft sets and then we define soft relation which is a subset of this Cartesian product.

Definition 7.3.1 *Cartesian product between the two soft sets* (F, A) *and* (G, B) *over* U *is defined as*

$$(F, A) \times (G, B) = \{(F(a), G(b)) : F(a) \in (F, A) \text{ and } G(b) \in (G, B)\}.$$

The elements of Cartesian product of two soft sets are considered as the pair of set of parameter-approximate elements.

An example is incorporated to understand Definition 7.3.1

Example 7.3.1 *Let* $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ *be a set of people and let* $A = \{$*red, white, indigo, pink*$\}$ *describe the different colors of shirts that people like to wear i.e.,* $A = \{r, w, i, p\}$*, where* r*,* w*,* i *and* p *represent for red, white, indigo and pink respectively. Let* $B = \{$*engineer, teacher, doctor*$\}$

denote the different professions of people, i.e., $B = \{e, t, d\}$, *where* e, t *and* d *imply for engineer, teacher and doctor respectively. Then the soft set* $(F, A) = \{F(r), F(w), F(i), F(p)\}$ *defines the set of people choose to wear different color shirts and the soft set* $(G, B) = \{G(e), G(t), G(d)\}$ *indicates the people having different professions, where* $F(r) = \{u_1, u_2, u_4, u_6\}$, $F(w) = \{u_2, u_3, u_5, u_4\}$, $F(i) = \{u_5, u_6, u_8, u_4\}$, $F(p) = \{u_3, u_5, u_8\}$ *and*
$G(e) = \{u_1, u_2, u_6, u_7\}$, $G(t) = \{u_3, u_5, u_8\}$, $G(d) = \{u_3, u_5, u_6, u_8, u_4\}$.
The Cartesian product of (F, A) *and* (G, B) *is considered as follows:*
$(F, A) \times (G, B) = \{(F(r), G(e)), (F(r), G(t)), (F(r), G(d)), (F(w), G(e)), (F(w), G(t)), (F(w), G(d)), (F(i), G(e)), (F(i), G(t)), (F(i), G(d)), (F(p), G(e)), (F(p), G(t)), (F(p), G(d))\}$.

Definition 7.3.2 *Let* (F, A) *and* (G, B) *be two soft sets over a common universe* U. *A soft set relation from* (F, A) *to* (G, B) *is a subset of* $(F, A) \times (G, B)$. *If* R *is a relation from* (F, A) *to* (G, B), *then* $R \subseteq (F, A) \times (G, B)$. *We write* $(F(r), G(t)) \in R$ *if and only if* $F(r) R\, G(t)$ *for* $r \in A$ *and* $t \in B$.

Definition 7.3.3 *A soft set relation on* (F, A) *is a subset of* $(F, A) \times (F, A)$.

Example 7.3.2 *Consider the Example 7.3.1 and also define the following.*
Let $R = \{(F(r), G(e)), (F(r), G(t)), (F(p), G(t)), (F(p), G(d))\}$ *which is a relation from* (F, A) *to* (G, B) *and* $\rho = \{(G(e), G(e)), (G(t), G(t)), (G(d), G(d)), (G(e), G(t)), (G(d), G(t))\}$ *which is a soft set relation on* (G, B).

Definition 7.3.4 *Let* R *be a soft set relation on* (F, A). *Then* R *is said to be*
(i) Reflexive if $(F(a), F(a)) \in R, \forall\ a \in A$;
(ii) Symmetric if $(F(a), F(b)) \in R$ *imply* $(F(b), F(a)) \in R$, $a, b \in A$;
(iii) Transitive if $(F(a), F(b)) \in R$ *and* $(F(b), F(c)) \in R$ *imply* $(F(a), F(c)) \in R$, $a, b, c \in A$;
(iv) Antisymmetric if $(F(a), F(b)) \in R$ *and* $(F(b), F(a)) \in R$ *imply* $F(a) = F(b)$.

Definition 7.3.5 *A soft set relation R on (F, A), which is reflexive, antisymmetric and transitive is called partial order relation on (F, A).*
A soft set together with a partial order relation is called a partial ordered soft set or po-soft set. For the soft set (F, A) with order relation '$\preceq$'; we denote po-soft set as $((F, A), \preceq)$.

Definition 7.3.6 *The Cartesian product of two po-soft sets $((F, A), \preceq_1)$ and $((G, B), \preceq_2)$ is a set which is defined as follows:*

$$(F, A) \times (G, B) = \{(F(a), G(b)) : F(a) \in (F, A), G(b) \in (G, B)\}.$$

Proposition 7.3.1 *The Cartesian product of two po-soft sets $((F, A), \preceq_1)$ and $((G, B), \preceq_2)$ is a po-soft set under the relation $\preceq_3$ and is defined by*
$(F(a_1), G(b_1)) \preceq_3 (F(a_2), G(b_2)) \Leftrightarrow F(a_1) \preceq_1 F(a_2)$ in $((F, A), \preceq_1)$ and
$G(b_1) \preceq_2 G(b_2)$ in $((G, B), \preceq_2)$.

Proof: *Reflexivity:* $(F(a), F(b)) \preceq_3 (F(a), F(b))$, $\forall$ $(F(a), F(b)) \in (F, A) \times (G, B)$, as $F(a) \preceq_1 F(a)$ in (F, A) and $G(b) \preceq_2 G(b)$ in G.
Antisymmetry: Assume that the two relations are
$(F(a_1), G(b_1)) \preceq_3 (F(a_2), G(b_2))$ and $(F(a_2), G(b_2)) \preceq_3 (F(a_1), G(b_1))$. Then $F(a_1) \preceq_1 F(a_2)$, $G(b_1) \preceq_2 G(b_2)$ and $F(a_2) \preceq_1 F(a_1)$, $G(b_2) \preceq_2 G(b_1)$. This gives $F(a_1) = F(a_2)$ and $G(b_1) = G(b_2)$, hence $(F(a_1), G(b_1)) = (F(a_2), G(b_2))$.
Transitivity: Let $(F(a_1), G(b_1)) \preceq_3 (F(a_2), G(b_2))$ and
$(F(a_2), G(b_2)) \preceq_3 (F(a_3), G(b_3))$. Then $F(a_1) \preceq_1 F(a_2)$, $G(b_1) \preceq_2 G(b_2)$ and $F(a_2) \preceq_1 F(a_3)$, $G(b_2) \preceq_2 G(b_3)$. These provide that $F(a_1) \preceq_1 F(a_3)$ and $G(b_1) \preceq_2 G(b_3)$. Hence $(F(a_1), G(b_1)) \preceq_3 (F(a_3), G(b_3))$.

Definition 7.3.7 *Let "$\preceq$" be a partial order relation on (F, A) and $F(a)$, $F(b)$ $\in (F, A)$. $F(a)$ and $F(b)$ are said to be comparable in the ordered relation "$\preceq$" if $F(a) \preceq F(b)$ or $F(b) \preceq F(a)$. $F(a)$ and $F(b)$ are said to be incomparable if they are not comparable.*

Definition 7.3.8 *If $((F, A), \preceq)$ is a po-soft set in which every two members are comparable then, it is called a soft chain.*

7.4 Lattice structure of soft sets

In this Section, we study on lattice structure of soft set by considering the following definitions.

Definition 7.4.1 *Let (G, B) be a non-empty subset of $((F, A), \preceq)$. Then,*

(i) An element $F(a) \in (F, A)$ is called an upper bound of (G, B) if $G(x) \preceq F(a)$, $\forall\ G(x) \in (G, B)$.

(ii) If $F(a)$ is an upper bound of (G, B) such that $F(a) \preceq F(b)$ for all upper bounds of (G, B) then $F(a)$ is called a least upper bound or supremum of (G, B). We denote sup(G, B) for supremum of (G, B).

(iii) An element $F(a) \in (F, A)$ is called a lower bound of (G, B) if $F(a) \preceq G(x)$, $\forall\ G(x) \in (G, B)$.

(iv) If $F(a)$ is a lower bound of (G, B) such that $F(b) \preceq F(a)$ for all lower bounds $F(b)$ of (G, B); then $F(a)$ is called a greatest lower bound or infimum of (G, B). We denote $inf(G, B)$ for infimum of (G, B).

Definition 7.4.2 *A po-soft set $((F, A), \preceq)$ is said to be a soft lattice if for every $F(a), F(b) \in (F, A)$, $a, b \in A$; sup$\{F(a), F(b)\}$ and $inf\{F(a), F(b)\}$ exist in (F, A).*
We write sup$\{F(a), F(b)\} = F(a) \bigtriangledown F(b)$ and $inf\{F(a), F(b)\} = F(a) \bigtriangleup F(b)$, where $\bigtriangledown$ and $\bigtriangleup$ denote the supremum and infimum between the soft sets respectively.

Example 7.4.1 *Let $(F, A) = \{F(a), F(b), F(c), F(d)\}$ is a soft set where $F(a) = \{u_1\}$, $F(b) = \{u_1, u_3, u_4\}$, $F(c) = \{u_1, u_2, u_4\}$, $F(d) = \{u_1, u_2, u_3, u_4\}$ and $A = \{a, b, c, d\}$. Then the soft set (F, A) is soft lattice. Here the ordered relation is set inclusion. Supremum and infimum are ordinary set union and intersection respectively. The Hasse diagram of this soft lattice is shown in Figure 7.4.1.*

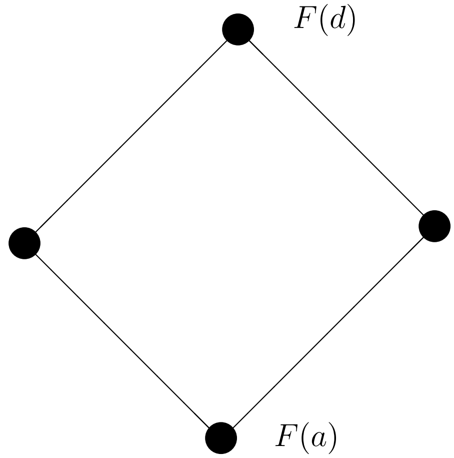

Figure 7.4.1: The Hasse diagram of Soft Lattice.

Definition 7.4.3 *A soft lattice* $((F, A), \preceq, \triangle, \bigtriangledown)$ *is said to be a distributive soft lattice if for* $F(a), F(b), F(c) \in (F, A)$*, then the following equality holds:*

$$F(a) \triangle (F(b) \bigtriangledown F(c)) = (F(a) \triangle F(b)) \bigtriangledown (F(a) \triangle F(c)).$$

Definition 7.4.4 *A soft lattice* $((F, A), \preceq, \bigtriangledown, \triangle)$ *is said to be a modular soft lattice if for* $F(a), F(b), F(c) \in (F, A)$ *with* $F(a) \succeq F(b)$*, then the following equality holds:*

$$F(a) \triangle (F(b) \bigtriangledown F(c)) = F(b) \bigtriangledown (F(a) \triangle F(c)).$$

7.5 Soft set relation

Here, we introduce a soft set relation determined by the parameters of soft set.

Definition 7.5.1 *Let* (F, A) *be a soft set. Let* ρ *be a relation on* A*. Soft set relation* R_ρ *on* (F, A) *induced by* ρ *which is defined as* $a\rho b \Rightarrow F(a)R_\rho F(b)$*,* $a, b \in A$*.*

Example 7.5.1 *Let* p_1*,* p_2*,* p_3 *and* p_4 *be the person suffer to the diseases such as 'blood sugar', 'high blood pressure', 'cholesterol' and 'uric acid' respectively.*

Let A be the age of persons i.e., $A = \{40, 45, 50, 55\}$. Let a binary relation ρ on A is defined as '$a\rho b$' iff a is less than b, $\forall\ a, b \in A$. Then the soft set relation on (F, A) is given by

$R_\rho = \{(F(40), F(45)),\ (F(40), F(50)),\ (F(40), F(55)),\ (F(45), F(50)),$
$(F(45), F(55)),\ (F(50), F(55))\}$.

Proposition 7.5.1 *Let (F, A) be a soft set. If ρ is an equivalence relation on A, then R_ρ is a soft equivalence relation on (F, A).*

Proof: *Reflexive:* ρ is an equivalence relation on A, therefore, $(a, a) \in \rho$, $\forall\ a \in A$. This implies that $(F(a), F(a)) \in R_\rho$, $\forall\ F(a) \in (F, A)$.
Symmetric: $(a, b) \in \rho$ and $(b, a) \in \rho$, so $(F(a), F(b)) \in (F, A)$ and $(F(b), F(a)) \in (F, A)$.
Transitive: Let $(a, b) \in \rho$ and $(b, c) \in \rho$, $a, b, c \in A$, then $(F(a), F(b)) \in R_\rho$ and $(F(b), F(c)) \in R_\rho$. Since ρ is an equivalence relation on A, therefore $(a, c) \in \rho$ and this implies that $(F(a), F(c)) \in R_\rho$.

Proposition 7.5.2 *Let (F, A) be a soft set. If ρ is a soft partial order relation on A, then R_ρ is a partial order relation on (F, A).*

Proof: Proof is similar to the proof of Proposition 7.5.1.

7.6 Conclusion

Here, we have newly incorporated the Cartesian product between the two soft sets. In addition to the above, we have introduced a binary relation on soft set in different ways and have studied some of its important properties. Also, the soft set relation has been determined by the parameters of soft set which is a unique characteristic of this chapter. We have generated the lattice structure of soft set using soft set relation. Different examples are supplied to show the effectiveness and usefulness of the proposed study in this chapter.

Chapter 8

Soft Congruence Relation Over Lattice*

In this chapter, we first describe soft congruence relation over a lattice. We then define the concepts of complete soft congruence relation. Besides this, the concepts of upper and lower approximations of a subset in a lattice are depicted based on this soft congruence relation. We then give their related properties with examples to investigate their characterizations.

A congruence relation over lattice describes two operations. Many researchers have studied the algebraic structure of rough set such as semi-ring, group, ring and lattice and have been concentrated on congruence relation. Here we use soft congruence relation instead of equivalence relation to granulate the universe. Soft congruence relation may provide a new direction to study algebraic structure of rough set and soft set over lattice. In this chapter, we describe the soft binary relation as well as soft congruence relation over lattice. We obtain some important properties of soft binary relation considering lattice as a universal set. Moreover, based on the ideas of congruence relation, we define soft congruence relation on lattice. Beside this, we approximate the subset of a lattice under soft congruence approximation space, and investigate the characteristic of ideal of the lattice under this soft congruence approximation space.

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8.1 Preliminaries

In this section, we present some basic definitions and results of soft set theory which are the most useful in the sequel. Let U be an initial universe set and let E be a set of parameters which are usually initial attributes, characteristic, properties of objects in U. Also let $P(U)$ denote the power set of U and $A \subseteq E$.

Definition 8.1.1 *(60) Union of two soft sets (F, A) and (G, B) over a common universe U is a soft set (H, C) where $C = A \cup B$ and $\forall\ a \in C$, H is defined as follows:*

$$H(a) = \begin{cases} F(a), & \text{if } a \in A - B, \\ G(a), & \text{if } a \in B - A, \\ F(a) \cup G(a), & \text{if } a \in A \cap B. \end{cases}$$

We write $(H, C) = (F, A) \sqcup (G, B)$ where the symbol "$\sqcup$" stands for union between two soft sets.

Definition 8.1.2 *(2) Intersection of two soft sets (γ, A) and (δ, B) over a common universe U is a soft set (Y, D) where $D = A \cup B$, and $\forall\ a \in D$, and Y is described as follows:*

$$Y(a) = \begin{cases} \gamma(a), & \text{if } a \in A - B, \\ \delta(a), & \text{if } a \in B - A, \\ \gamma(a) \cap \delta(a), & \text{if } a \in A \cap B. \end{cases}$$

We write $(Y, D) = (\gamma, A) \sqcap (\delta, B)$ where the notation "$\sqcap$" stands for intersection between two soft sets.

Definition 8.1.3 *(16) Let (F, A) and (G, B) be two soft sets over U. Then $AND-$product of (F, A) and (G, B) is defined as follows:*

$$(H, C) = (F, A) \wedge (G, B)$$

where $C = A \times B$ and $H(a, b) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$.

8.2 Soft congruence relation on lattice

In this section, we introduce the concept of soft congruence relation over lattice and thereafter we present their related properties. Let us first, we discuss some basic notions of lattice theory. Suppose that $(L, \leq)$ is a partially order set. For all $a, b \in L$ if $a \vee b$ and $a \wedge b$ exist, then L is called a lattice, where $a \vee b$ and $a \wedge b$ are supremum and infimum of $\{a, b\}$ respectively. Let L be a lattice and $A \subseteq L$. Then A is called a sublattice of L if $a \in A$; $b \in A$ imply $a \vee b \in A$ and $a \wedge b \in A$. A is called an ideal if

(i) $a \in A$ and $b \in A$ imply $a \vee b \in A$,

(ii) $a \in L$, $b \in A$ imply $a \wedge b \in A$.

Definition 8.2.1 *An equivalence relation ρ on a lattice L is called a congruence relation if $a\rho b$ and $c\rho d$ hold and imply that $(a \wedge c)\ \rho\ (b \wedge d)$ and $(a \vee c)\ \rho\ (b \vee d)$ together hold. Since ρ is an equivalence relation on L, then ρ would partition L in equivalence classes where for any $a \in L$, equivalence class of a is given as*

$$[a]_\rho = \{x \in L : x\rho a\}.$$

Definition 8.2.2 *(33) Let (ρ, A) be a soft set over $L \times L$, where ρ is a set valued function defined by $\rho : A \to P(L \times L)$. Then (ρ, A) is called a soft binary relation over L.*

Definition 8.2.3 *Let (ρ, A) be soft binary relation over L. (ρ, A) is called a soft equivalence relation over L if each $\rho(e)[\neq \phi]$, $e \in A$ is an equivalence relation on L.*

Definition 8.2.4 *A soft equivalence relation (ρ, A) over L is called a soft congruence relation over L if each non null $\rho(e)$, $e \in A$ is a congruence relation on L.*

Example 8.2.1 *Let $L = \{a, b, c, d\}$ be a lattice. The partial order on L is defined as shown in Figure 8.2.1 and $A = \{\alpha, \beta\}$. Let us consider a set valued function $\rho : A \to P(L \times L)$ which is given by*

$$\rho(\alpha) = \{(a, a), (b, b), (c, c), (d, d), (a, c), (c, a), (b, d), (d, b)\}$$

and

$$\begin{aligned}\rho(\beta) \;=\; &\{(a, a), (b, b), (c, c), (d, d), (a, c), (c, a), (b, c), (c, b), (a, b), (b, a), (d, c),\\ &(c, d), (a, d), (d, a), (b, d), (d, b)\}\end{aligned}$$

Then (ρ, A) is a soft congruence relation on L.

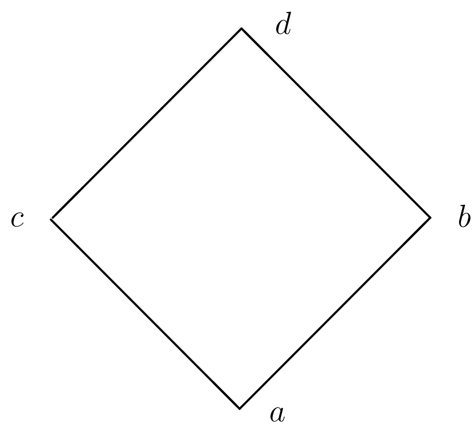

Figure 8.2.1 : *Lattice.*

Proposition 8.2.1 *Let (F, A) and (G, B) be soft congruence relations over L. Then $(H, C) = (F, A) \sqcup (G, B)$ is soft congruence relation over L if $F(e) \subseteq G(e)$ or $G(e) \subseteq F(e)$ for all $e \in A \cup B$.*

Proof: From Definition 8.1.1, we know that

$$H(e) = \begin{cases} F(e), & \text{if } \; e \in A - B, \\ G(e), & \text{if } \; e \in B - A, \\ F(e) \cup G(e), & \text{if } \; e \in A \cap B, \end{cases}$$

for all $e \in C$. Suppose that $e \in A - B$. Then $H(e) = F(e)$. Since $F(e)$ is a congruence relation on L, $H(e)$ is a congruence relation on L. Suppose that $e \in B - A$. Then $H(e) = G(e)$. Since $G(e)$ is a congruence relation on L, $H(e)$ is a congruence relation on L. Let $e \in A \cap B$. Since $F(e) \subseteq G(e)$ or $G(e) \subseteq F(e)$, $H(e) = F(e) \cup G(e) = F(e)$ or $H(e) = F(e) \cup G(e) = G(e)$. Since $F(e)$ and $G(e)$ are congruence relations, $H(e)$ is congruence relation. Hence the proof is completed.

Proposition 8.2.2 *Let (F, A) and (G, B) be two soft congruence relations over L. Then, $(H, C) = (F, A) \sqcap (G, B)$ is soft congruence relation over L.*

Proof: From Definition 8.1.2, we know that

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cap G(e), & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. Suppose that $e \in A - B$. Then $H(e) = F(e)$. Since $F(e)$ is a congruence relation on L, $F(e)$ is a congruence relation on L. Suppose that $e \in B - A$. Then $H(e) = G(e)$. Since $G(e)$ is a congruence relation on L, $G(e)$ is a congruence relation on L. Let $e \in A \cap B$, then $H(e) = F(e) \cap G(e)$. Since intersection of two congruence relations is a congruence relation, $H(e)$ is a congruence relation over L. Hence $(F, A) \sqcap (G, B)$ is congruence relation.

8.3 Approximations under soft congruence relation

Let (ρ, A) be a soft congruence relation on L. Then each $\rho(e)$, $e \in A$ is a congruence relation over L. Let $\psi = \bigcap_{\alpha \in A} \rho(e)$. Then ψ is a congruence relation on L. The equivalence class of $x \in L$ under this congruence relation is described as $[x]_\psi = \{y \in L : (x, y) \in \psi\}$.

We define the pair (L, ψ) as soft congruence approximation space.

Definition 8.3.1 *Let (L, ψ) be a soft congruence approximation space and X be a non-empty subset of L. Then the lower and upper approximations of X are defined as:*

(i) $\psi_\star(X) = \{y \in L : [y]_\psi \subseteq X\}$,

(ii) $\psi^\star(X) = \{y \in L : [y]_\psi \cap X \neq \phi\}$.

If $\psi_\star(X) = \psi^\star(X)$ *then* X *is called definable otherwise* X *is called rough.*

Example 8.3.1 *Considering Example 8.2.1, then*

$$\psi = \bigcap_{\alpha \in A} \rho(\alpha) = \{(a,a),(b,b),(c,c),(d,d),(a,c),(c,a),(b,d),(d,b)\}$$

is a congruence relation on L*. The congruence classes are given by* $\{a,c\}$, $\{b,d\}$*. Let* $X = \{a,c,d\}$*, then in the soft congruence approximation space* (L,ψ), $\psi_\star(X) = \{a,c\}$ *and* $\psi^\star(X) = \{a,b,c,d\}$*. Clearly* X *is rough.*

Definition 8.3.2 *Let* ψ *be a congruence relation on* L *and* S *be a non-empty subset of* L*.* S *is called an upper rough ideal (sublattice) of* L *if* $\psi^\star(S)$ *is an ideal (sublattice).* S *is called a lower rough ideal (sublattice) if* $\psi_\star(S)$ *is an ideal (sublattice).* S *is called a rough ideal of* L *if it is both an upper rough ideal and a lower rough ideal.*

Definition 8.3.3 *Let* ψ *be a congruence relation on* L*, then* ψ *is called a complete congruence relation if* $[a]_\psi \vee [b]_\psi = [a \vee b]_\psi$ *and* $[a]_\psi \wedge [b]_\psi = [a \wedge b]_\psi$ *for all* $a, b \in L$*.*
If ψ *is a complete soft congruence relation on* L*, we define the pair* (L,ψ) *as complete soft congruence approximation space over* L*.*

Proposition 8.3.1 *Let* ψ *be a soft congruence relation on a non-empty set* L*. If* A *and* B *are non-empty subsets of* L*, then*

(1) $\psi_\star(A) \subseteq A \subseteq \psi^\star(A)$,

(2) $\psi^\star(A \cup B) = \psi^\star(A) \cup \psi^\star(B)$,

(3) $\psi_\star(A \cap B) = \psi_\star(A) \cap \psi_\star(B)$,

(4) $A \subseteq B \Rightarrow \psi^\star(A) \subseteq \psi^\star(B)$ *and* $\psi_\star(A) \subseteq \psi_\star(B)$,

(5) $\psi_\star(A \cup B) \supseteq \psi_\star(A) \cup \psi_\star(B)$,

(6) $\psi^\star(A \cap B) \subseteq \psi^\star(A) \cap \psi^\star(B)$,

Proposition 8.3.2 *Let L be a lattice and (L, ψ) be a complete soft congruence approximation space over L. If A is a sublattice of L, then A is an upper rough sublattice.*

Proof: Let $a, b \in \psi^\star(A)$, then $[a]_\psi \cap A \neq \phi$ and $[b]_\psi \cap A \neq \phi$, so there exist x and y such that $x \in [a]_\psi \cap A$ and $y \in [b]_\psi \cap A$. So, $x, y \in A$ and since A is a sublattice of L, hence $x \vee y \in A$. Again $x, y \in A$ and ψ is a complete soft congruence relation on L, therefore $x \vee y \in [a]_\psi \vee [b]_\psi = [a \vee b]_\psi$. Hence $x \vee y \in [a \vee b]_\psi \cap A$, so $a \vee b \in \psi^\star(A)$. Similarly we can prove $a \wedge b \in \psi^\star(A)$. So, the proposition is obvious.

Proposition 8.3.3 *Let L be a lattice and (L, ψ) be a complete soft congruence approximation space over L. If A is a sublattice of L, then A is a lower rough sublattice of L if it is non-empty.*

Proof: Let $a, b \in \psi_\star(A)$, then $[a]_\psi \subseteq A$ and $[b]_\psi \subseteq A$. But ψ is a complete soft congruence relation on L, therefore $[a]_\psi \vee [b]_\psi = [a \vee b]_\psi$. Let k be any element of $[a \vee b]_\psi$. So there exist $x \in [a]_\psi$ and $y \in [b]_\psi$ such that $k = x \vee y$. Now $x \in [a]_\psi \subseteq A$ and $y \in [b]_\psi \subseteq A$, i.e., $x, y \in A$, and since A is a sublattice, $k = x \vee y \in A$, that is $k \in [a \vee b]_\psi$ imply $k \in A$. So $[a \vee b]_\psi \subseteq A$. Therefore $a \vee b \in \psi_\star(A)$. Similarly we can prove $a \wedge b \in \psi_\star(A)$. Hence the proof is completed.

Proposition 8.3.4 *Let ψ be a congruence relation on a lattice L. If A and B are ideals of L, then $\psi^\star(A \cap B) = \psi^\star(A) \cap \psi^\star(B)$.*

Proof: Let $x \in \psi^\star(A) \cap \psi^\star(B)$. Then $[x]_\psi \cap A \neq \phi$ and $[x]_\psi \cap B \neq \phi$. Then there exist $y \in A$ and $z \in B$ such that $x\psi y$ and $x\psi z$ hold. So we can write $(x \vee x)\psi(y \vee z)$ that is $x\psi(y \vee z)$. Since A and B are ideals of L, we have $y \wedge z \in (A \cap B)$ and hence $[x]_\psi \cap (A \cap B) \neq \phi$ which implies $x \in \psi^\star(A \cap B)$ that is $\psi^\star(A) \cap \psi^\star(B) \subseteq \psi^\star(A \cap B)$. Also by Proposition 8.3.1, $\psi^\star(A \cap B) \subseteq \psi^\star(A) \cap \psi^\star(B)$. Hence we have $\psi^\star(A \cap B) = \psi^\star(A) \cap \psi^\star(B)$.

Proposition 8.3.5 *Let ψ be a complete soft congruence relation on a lattice L. If A is an ideal of L, then A is an upper rough sublattice of L.*

Proof: If A is an ideal of L, then A is a sublattice of L and then this proposition follows from the Proposition 8.3.2.

Proposition 8.3.6 *Let (F, A) and (G, B) be soft congruence relations over L. Then $(H, C) = (F, A) \wedge (G, B)$ is soft congruence relation over L.*

Proof: Let (a_1, b_1) and $(a_2, b_2) \in H(\alpha, \beta)$ for all $(\alpha, \beta) \in A \times B$. Then (a_1, b_1) and $(a_2, b_2) \in F(\alpha)$, and (a_1, b_1) and $(a_2, b_2) \in G(\beta)$. Since (F, A) is congruence relation, $(a_1 \wedge a_2, b_1 \wedge b_2), (a_1 \vee a_2, b_1 \vee b_2) \in F(\alpha)$. Similarly, $(a_1 \wedge a_2, b_1 \wedge b_2), (a_1 \vee a_2, b_1 \vee b_2) \in G(\beta)$. Thus, (H, C) is congruence relation over L.

Proposition 8.3.7 *Let $\{(F, A_i) : i \in I\}$ be a non-empty family of soft congruence relation over L. Then,*

$$\bigwedge \{(F, A_i) : i \in I\}$$

is soft congruence relation over L.

8.4 Conclusion

Soft congruence relation is a new kind of soft set relation. In this chapter, we have established the soft congruence relation over lattice. Several properties of soft congruence relation have been studied. Approximations of subset of a lattice have been studied with respect to soft congruence relation. That is the roughness of a subset of lattices has been discussed using the soft set relation. We have also discussed the properties of lattice ideal with respect to the soft congruence relation. In addition to the above, we have concluded that the concept of the chapter has opened a new platform for algebraic study.

Chapter 9

Fuzzy Rough Soft Set and Its Application to Lattice*

A modified soft rough (MSR) approximation space is constructed on employing the soft set in the chapter. Approximation of a soft set is considered with respect to another soft set in an MSR approximation space. Roughness of a rough soft set is measured under an MSR approximation space and hereby the concept of fuzzy rough soft set is defined. By defining fuzzy rough soft set in an MSR approximation space, flavour of theories on soft sets and rough sets and fuzzy sets are retained altogether. Some properties of fuzzy rough soft set are derived. Moreover, lattice theory is studied on the fuzzy rough soft set.

9.1 Introduction

At present, uncertainty is an important and interesting topic to the researchers as it has considered in many situations like engineering, economics, social science, computer science, environmental science, medical science etc. Fuzzy set theory, probability theory, rough set theory and soft set theory are successfully applied to solve the problem with uncertainties in these areas. The concept of fuzzy set was introduced by Zadeh (133) applied to solve the problem. Fuzzy set allows that objects belong to a set or a relation to a given degree ranging between 0 and 1 i.e., a membership function is needed to define it.

*A part of this chapter has communicated to the International Journal

A connection between the soft set and the rough set has been discussed by Feng et al. (32). They introduced the notion of the soft rough set, where instead of equivalence classes parameterized subsets of a set is employed to find lower and upper approximations of a subset. But in their discussion, some cases may be occurred, where upper approximation of a nonempty set may be empty. Again upper approximation of a subset X may not be contained in the set X. These situations do not occur in (classical) rough set theory. To overcome this difficulty, Shabir et al. (107) redefined the soft rough set model and called Modified Soft Rough (MSR) set whose lower and upper approximations are different from the (classical) rough set theory and soft set theory. MSR-sets satisfy all the basic properties of rough sets. Roy and Bera (101) approximated the soft set in the MSR approximation space and defined the notion of rough soft set. But in this investigation, an attempt is taken to connect the rough soft set with the fuzzy set in an MSR approximation space. In this chapter, we calculate the measure of roughness of rough soft set in a modified soft rough approximation space and introduce the notion of fuzzy rough soft set. We also define here absolute fuzzy rough soft set and null fuzzy rough soft set. We study the properties like subset, union, intersection on fuzzy rough soft set and provide some examples to analyze the definitions. We also present some propositions on fuzzy rough soft set. An order relation on fuzzy rough soft set is also included on fuzzy rough soft set and its application is discussed through an example with the help of Hasse diagram.

9.2 Preliminaries

In rough set theory, indiscernibility relation, generated by information about objects of interest are of basic importance. When two objects have the same value over a certain group of attributes, we say they are indiscernible with respect to this group of attributes, or have the same description with respect to the indiscernibility relation. Indiscernibility relation is an equivalence relation. By this equivalence relation, we form equivalence class and all the equivalence classes form a partition of the universe, which are the basic building blocks of universal set called granules. Any subset of objects of the universe are approximated by two sets, called the lower and the upper approximations and can be viewed as the sets of elements which certainly and possibly belong to the set. Pair of these two approximations is called Rough Set.

Definition 9.2.1 *A fuzzy set $\tilde{A}$ in X is characterized by a membership function $\mu_{\tilde{A}}(x)$ which associates with each points in X to a real number in the interval $[0,1]$ with the value of $\mu_{\tilde{A}}(x)$ at x representing the grade of membership $\mu_{\tilde{A}}(x)$ of x in A.*
A fuzzy set $\tilde{A}$ can be written an $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$. According to Zadeh (133), intersection, union, and complement of fuzzy set are defined componentwise as follows:
$(\mu_{\tilde{A}} \cap \mu_{\tilde{B}})(x) = \min\{\mu_{\tilde{A}},\ \mu_{\tilde{B}}\}$,
$(\mu_{\tilde{A}} \cup \mu_{\tilde{B}})(x) = \max\{\mu_{\tilde{A}},\ \mu_{\tilde{B}}\}$,
$(\mu_{\tilde{A}})^c(x) = 1 - \mu_{\tilde{A}}(x)$,
where $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ are membership functions of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ respectively in X and $x \in X$; and $(\mu_{\tilde{A}})^c(x)$ denotes the complement of the membership function $(\mu_{\tilde{A}})(x)$.

Definition 9.2.2 *(60) A pair $S = (F, A)$ is called a soft set over U, where $F : A \to P(U)$ denotes a set valued mapping and $P(U)$ is the power set of U.*

Definition 9.2.3 *(60) Let (G_1, B_1) and (G_2, B_2) be two soft sets over U. Then (G_1, B_1) AND $(G_2, B_2,)$ is denoted by $(G_1, B_1) \wedge (G_2, B_2,)$ and is defined by $(G_1, B_1) \wedge (G_2, B_2) = (H_1, A \times B)$, where $H_1(x, y) = G_1(x) \cap G_2(y)$, $\forall$ $(x, y) \in A \times B$.*

Definition 9.2.4 *(60) Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over U. Then (G_1, B_1) OR $(G_2, B_2,)$ is noted by $(G_1, B_1) \vee (G_2, B_2,)$ and is defined by $(G_1, B_1) \vee (G_2, B_2) = (H_2, A \times B)$, where $H_2(x, y) = G_1(x) \cup G_2(y)$, $\forall$ $(x, y) \in A \times B$.*

Definition 9.2.5 *(33) Let (F, A) be a soft set over U and $\psi : U \to P(A)$ be another mapping defined by $\psi(x) = \{a : x \in F(a)\}$. Then the pair (U, ψ)*

is called the MSR approximation space and for any $X \subseteq U$*, lower MSR-approximation,* $\underline{X}_\psi$ *and upper MSR-approximation,* $\overline{X}_\psi$ *respectively are defined as follows:*

$\underline{X}_\psi = \{x \in U : \psi(x) \neq \psi(y)\ \forall\ y \in X^c\}$*, where* X^c *is the complement of* X *i.e.,* $X^c = U - X$*,*

$\overline{X}_\psi = \{x \in U : \psi(x) = \psi(y) \text{ for some } y \in X\}$*.*

If $\underline{X}_\psi \neq \overline{X}_\psi$*, then* X *is said to be a modified soft rough set.*

Example 9.2.1 *Let* $U = \{p_1, p_2, p_3, p_4, p_5\}$ *be the universal set and a set of parameters* $A = \{e_1, e_2, e_3, e_4\}$*. Let the soft set* (F, A) *over* U *is stated in below:* $F(e_1) = \{p_1, p_2, p_4\}$, $F(e_2) = \{p_1, p_2, p_4, p_5\}$, $F(e_3) = \{p_5\}$, $F(e_4) = \{p_2, p_3, p_5\}$. *Then from the definition (i.e., Definition 6.2.2) of MSR set* $\psi : U \to P(A)$ *is constructed as follows:*

$\psi(p_1) = \{e_1, e_2\}$, $\psi(p_2) = \{e_1, e_2, e_4\}$, $\psi(p_3) = \{e_4\}$, $\psi(p_4) = \{e_1, e_2\}$, $\psi(p_5) = \{e_2, e_3, p_4\}$. *Let* $X = \{p_1, p_2, p_3\}$*. Then for the MSR approximation space* (U, ψ)*, we can write*

$\underline{X}_\psi = \{p_2, p_3\}$ *and* $\overline{X}_\psi = \{p_1, p_2, p_3, p_4\}$*. Clearly,* $\underline{X}_\psi \neq \overline{X}_\psi$ *and hence* X *is the modified soft rough set.*

Definition 9.2.6 *(9) Let* (F, A) *be a soft set over* U *and* (U, ψ) *be an MSR-approximation space with respect to* (F, A)*. Let* (G, B) *be another soft set over* U*. Then* (G, B) *is said to be rough soft set with respect to the parameter* $e \in B$ *if* $\underline{G(e)}_\psi \neq \overline{G(e)}_\psi$*.* (G, B) *is said to be a full rough soft set or a simply rough one if* $\underline{G(e)}_\psi \neq \overline{G(e)}_\psi$, $\forall\ e \in B$ *and it is denoted by* $RsG(e_B)$*. Therefore, rough soft set with respect to the parameter* e *is given by* $RsG(e) = (\underline{G(e)}_\psi, \overline{G(e)}_\psi)$*.*

Example 9.2.2 *Consider the universal set* U*, soft set* (F, A) *and set valued function* $\psi : U \to P(A)$ *as given in Example 9.2.1. Let* (G, B) *be another soft set over* U*, where* $B = \{e_1, e_2\}$*,* $G(e_1) = \{p_2, p_5\}$ *and* $G(e_2) = \{p_1, p_3, p_5\}$*. Then the lower MSR approximation and the upper MSR approximation set of* (G, B) *in* (U, ψ) *are* $\underline{G(e_1)}_\psi = \{p_2, p_5\}$*,* $\overline{G(e_1)}_\psi = \{p_2, p_5\}$*,* $\underline{G(e_2)}_\psi = \{p_3, p_5\}$*,* $\overline{G(e_2)}_\psi = \{p_1, p_3, p_4, p_5\}$*. Clearly,* (G, B) *is rough soft set with respect to the parameter* e_2*.*

Definition 9.2.7 *(9) Let (F, A) be a soft set over U and (U, ψ) be an MSR-approximation space. Let (G, B) be another soft set over U. Measure of roughness of (G, B) with respect to the parameter $e \in B$ is denoted by $R_{G(e)}$ and is defined as follows:*

$$R_{G(e)} = \frac{|\underline{G(e)}_{\psi}|}{|\overline{G(e)}_{\psi}|}.$$

where $|\underline{G(e)}_{\psi}|$ and $|\overline{G(e)}_{\psi}|$ denote the cardinality of the sets $\underline{G(e)}_{\psi}$ and $\overline{G(e)}_{\psi}$ respectively. Clearly, $0 \leq R_{G(e)} \leq 1$.

9.3 Fuzzy rough soft set

Here, we discuss for every soft set (G, B) over U there is an associated fuzzy set. It is known that each soft set (G, B) over U, roughness of (G, B) with respect to the parameter $e \in B$, in MSR-approximation space is a number from the interval [0, 1]. Hence, we can define a fuzzy set for every soft set. We denote the notation (U, ψ) as an MSR approximation space with respect to A for the soft set (F, A) over U in this chapter.

Definition 9.3.1 *Let (G, B) be a soft set over U. Then fuzzy rough soft set of (G, B) over (U, ψ) is defined as: $\{(G(e), R_{G(e)}) : G(e) \in (G, B)\}$, where $R_{G(e)}$ is the roughness of (G, B) with respect to the parameter $e \in B$.*

Example 9.3.1 *Consider the Example 9.2.2, roughness of (G, B) is given by $R_{G(e_1)} = 1$ and $R_{G(e_2)} = \frac{1}{2}$. Therefore, the fuzzy rough soft set of (G, B) is given by $\{(G(e_1), 1), (G(e_2), \frac{1}{2})\}$.*

Definition 9.3.2 *A fuzzy rough soft set (G, B) over (U, ψ) is said to be null fuzzy rough soft set if $R_G(e) = 0$, $\forall\ e \in B$ and we use the symbol (G_ϕ, B).*

Definition 9.3.3 *A fuzzy rough soft set (G, B) over (U, ψ) is said to be absolute fuzzy rough soft set if $R_G(e) = 1$, $\forall\ e \in B$ and we denote it by (G_U, B).*

Example 9.3.2 *In Example 9.2.1, let (G, C) be another soft set over U which is defined as follows:*
$G(e_1) = \{p_1\}$, $G(e_2) = \{p_4\}$.
Then in the MSR approximation space (U, ψ), lower and upper MSR approximations of (G, C) are given by $\underline{G(e_1)}_\psi = \phi$, $\overline{G(e_1)}_\psi = \{p_1, p_4\}$, $\underline{G(e_2)}_\psi = \phi$, $\overline{G(e_2)}_\psi = \{p_1, p_4\}$. Also roughness of (G, C) is given by $R_{G(e_1)} = 0$ and $R_{G(e_2)} = 0$. So (G, C) is a null fuzzy rough soft set.
If we consider the soft set (G, D) over U defined as $G(e_1) = \{p_2, p_5\}$ and $G(e_2) = \{p_2, p_3, p_5\}$. Then in the MSR approximation space (U, ψ), $\underline{G(e_1)}_\psi = \{p_2, p_5\}$, $\overline{G(e_1)}_\psi = \{p_2, p_5\}$, $\underline{G(e_2)}_\psi = \{p_2, p_3, p_5\}$, $\overline{G(e_2)}_\psi = \{p_2, p_3, p_5\}$. Now $R_{G(e_1)} = 1$ and $R_{G(e_2)} = 1$. Hence (G, D) is an absolute fuzzy rough soft set.

Definition 9.3.4 *Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$ respectively. (G_1, B_1) is said to be fuzzy rough soft subset of (G_2, B_2) if*
(i) $B_1 \subseteq B_2$, and
(ii) $R_{G_1(e)} = R_{G_2(e)}\ \forall\ e \in B_1$.
We write $(G_1, B_1) \sqsubseteq_F (G_2, B_2)$, where the symbol '$\sqsubseteq_F$' denotes fuzzy rough soft subset.

Definition 9.3.5 *Two fuzzy rough soft sets, (G_1, B_1) and (G_2, B_2) over (U, ψ) is said to be equal if $(G_1, B_1) \sqsubseteq_F (G_2, B_2)$ and $(G_2, B_2) \sqsubseteq_F (G_1, B_1)$.*

Proposition 9.3.1 *If (G_1, B_1) is a soft subset of (G_2, B_2) then (G_1, B_1) is a fuzzy rough soft subset of (G_2, B_2).*

Proof: Let (G_1, B_1) be soft subset of (G_2, B_2), then
(i) $B_1 \subseteq B_2$, and
(ii) $G_1(e) = G_2(e)\ \forall\ e \in B_1$.
Therefore $\underline{G_1(e)}_\psi = \underline{G_2(e)}_\psi$ and $\overline{G_1(e)}_\psi = \overline{G_2(e)}_\psi$, $\forall\ e \in B_1$. This gives $R_{G_1(e)} = R_{G_2(e)}\ \forall\ e \in B_1$. This completes the proof of the proposition.

Proposition 9.3.2 *Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G(e_{B_1})}$ and $R_{G(e_{B_2})}$ respectively. Then*

$H(e) = (G_1, B_1) \sqcup (G_2, B_2),\ B_1 \cap B_2 = \phi$ is a fuzzy rough soft set. The membership function of fuzzy rough soft set is denoted by $R_{H(e)}$ and is given as follows:

$$R_{H(e)} = \begin{cases} R_{G(e_{B_1})}, & \text{if } e \in B_1 - B_2, \\ R_{G(e_{B_2})}, & \text{if } e \in B_2 - B_1. \end{cases}$$

Definition 9.3.6 *Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$ respectively. Then the union of (G_1, B_1) and (G_2, B_2) is defined as $(G_1, B_1) \sqcup_F (G_2, B_2) = (H, C)$, where $C = B_1 \cup B_2$; the symbol '$\sqcup_F$' denotes fuzzy rough soft union, and the membership function is described as follows:*

$$R_{H(e)} = \begin{cases} R_{G_1(e_{B_1})}, & \text{if } e \in B_1 - B_2, \\ R_{G_2(e_{B_2})}, & \text{if } e \in B_2 - B_1, \\ \max\{R_{G_1(e)}, R_{G_2(e)}\}, & \text{if } e \in B_1 \cap B_2. \end{cases}$$

Definition 9.3.7 *Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$ respectively. The intersection of (G_1, B_1) and (G_2, B_2) is defined as $(G_1, B_1) \sqcap_F (G_2, B_2) = (H, C)$, where $C = B_1 \cap B_2$; the symbol '$\sqcap_F$' means fuzzy rough soft intersection, and the membership function is given by $R_{H(e)} = \min\{R_{G_1(e)}, R_{G_2(e}\}$, $e \in C$.*

Definition 9.3.8 *Complement of a fuzzy rough soft set (G, B) with membership function $R_{G(e_B)}$ is denoted by (G^c, B) and the rough membership function is given by $R_{G^c(e_B)} = 1 - R_{G(e_B)}$.*

Definition 9.3.9 *Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$ respectively. Then (G_1, B_1) AND (G_2, B_2), denoted by $(G_1, B_1) \wedge_F (G_2, B_2)$, defined by $(G_1, B_1) \wedge_F (G_2, B_2) = (H_1, A \times B)$, where $H_1(x, y) = G_1(x) \cap G_2(y)$ and the membership function is given by*
$R_{H_1(x,y)} = \min\{R_{G_1(x)}, R_{G_2(y)}\}$, $\forall\ (x, y) \in A \times B$.

Definition 9.3.10 *Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$ respectively. Then*

(G_1, B_1) *OR* (G_2, B_2) *is denoted by* $(G_1, B_1) \vee_F (G_2, B_2)$ *and is defined as* $(G_1, B_1) \vee_F (G_2, B_2) = (H_2, A \times B)$, *where* $H_2(x, y) = G_1(x) \cup G_2(y)$ *and the membership function is given by* $R_{H_2(x,y)} = \max\{R_{G_1(x)}, R_{G_2(y)}\}$, $\forall\ (x, y) \in A \times B$.

Proposition 9.3.3 *Let* (G_1, B_1) *and* (G_2, B_2) *be two fuzzy rough soft sets over* (U, ψ) *with the membership functions* $R_{G_1(e_{B_1})}$ *and* $R_{G_2(e_{B_2})}$ *respectively. Then*
(i) $(G_1, B_1) \vee_F (G_2, B_2) = (G_2, B_2) \vee_F (G_1, B_1)$
(ii) $(G_1, B_1) \wedge_F (G_2, B_2) = (G_2, B_2) \wedge_F (G_1, B_1)$.

Proposition 9.3.4 *Let* (G_1, B_1) *and* (G_2, B_2) *be two fuzzy rough soft sets over* (U, ψ) *with the membership functions* $R_{G_1(e_{B_1})}$ *and* $R_{G_2(e_{B_2})}$ *respectively. Then the following results hold:*
(i) $((G_1, B_1) \sqcup_F (G_2, B_2))^c = (G_1, B_1)^c \sqcap_F (G_2, B_2)^c$
(ii) $((G_1, B_1) \sqcap_F (G_2, B_2))^c = (G_1, B_1)^c \sqcup_F (G_2, B_2)^c$.

Proof:
(i) Case 1: Let $e \in B_1 - B_2$ and $R_{G_1(e)} = p$. Then $R_{G_2(e)} = 0$. Therefore, $R_{(G_1 \sqcup_F G_2)^c(e)} = 1 - p$. Also

$$\begin{aligned} R_{({G_1}^c\ \sqcap_F\ {G_2}^c)(e)} &= \min\{R_{{G_1}^c(e)}, R_{{G_2}^c(e)}\} \\ &= \min\{1 - p, 1\} \\ &= 1 - p. \end{aligned}$$

Case 2: If $e \in B_2 - B_1$, then the proof can be established in similar way as in Case 1.
Case 3: Suppose $e \in B_1 \cap B_2$, then

$$\begin{aligned} R_{({G_1}^c\ \sqcap_F\ {G_2}^c)(e)} &= \min\{R_{{G_1}^c(e)}, R_{{G_2}^c(e)}\} \\ &= 1 - \max\{R_{G_1(e)}, R_{G_2(e)}\} \\ &= R_{(G_1 \sqcup_F G_2)^c(e).} \end{aligned}$$

(ii) Proof is similar to that of proof (i).

Proposition 9.3.5 *For any two fuzzy rough soft sets (G_1, B_1) and (G_2, B_2) over (U, ψ), the following conditions establish.*
(i) $(G_1, B_1) \sqcup_F (G_2, B_2) = (G_2, B_2) \sqcup_F (G_1, B_1)$ *and*
$(G_1, B_1) \sqcap_F (G_2, B_2) = (G_2, B_2) \sqcap_F (G_1, B_1)$.
(ii) $(G_1, B_1) \sqcup_F (\phi, B) = (G_1, B_1)$ *and* $(G_1, B_1) \sqcap_F (\phi, B) = (\phi, B_2)$,
where (ϕ, B) denotes the null fuzzy rough soft set.
(iii) $(\phi, B) \sqcup_F (\phi, B) = (\phi, B)$, $(\phi, B) \sqcap_F (\phi, B) = (\phi, B)$.

Now, we define a binary relation '$\asymp$' on fuzzy rough soft set (G, B) over (U, ψ) as $G(e_1) \asymp G(e_2)$ if and only if $R_{G(e_1)} = R_{G(e_2)}$ for $e_1, e_2 \in B$.
Clearly, '$\asymp$' is an equivalence relation on (G, B). We denote equivalence class of $(G(e_1), R_{G(e_1)})$ by the relation '$\asymp$' as $[G(e_1)]_{\asymp}$.

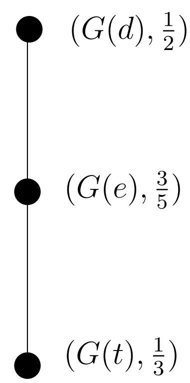

Figure 9.3.1: Chain of fuzzy rough soft set.

Proposition 9.3.6 *Every fuzzy rough soft set forms a chain by the order relation '$\asymp$'.*

Proof: Straightforward.

Example 9.3.3 *Let $U = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ be the set of people in a social gathering. Let the parameter set A described the shirts of three colors namely red, white and blue i.e., $A = \{r, w, b\}$, where r, w and b stand for red, white and blue respectively. Let us consider the soft set (F, A) with $F(r) = \{p_1, p_3, p_4\}$, $F(w) = \{p_1, p_2, p_4, p_5, p_6\}$ and $F(b) = \{p_1, p_2, p_3, p_6\}$.*

Then from definition of MSR set $\psi : U \to P(A)$ *is constructed as:* $\psi(p_1) = \{r, w, b\}$, $\psi(p_2) = \{w, b\}$, $\psi(p_3) = \{r, w\}$, $\psi(p_4) = \{r, w\}$, $\psi(p_5) = \{w\}$, $\psi(p_6) = \{w, b\}$. *Let* $B = \{doctor, teacher, engineer\} = \{d, t, e\}$, *where* d, t *and* e *denote doctor, teacher and engineer respectively. Let* (G, B) *be another soft set over* U *is defined as:* $G(d) = \{p_1, p_2, p_4\}$, $G(t) = \{p_1, p_2\}$, $G(e) = \{p_1, p_3, p_5, p_6\}$. *Then for the MSR approximation space* (U, ψ), *we can write* $\underline{G(d)}_\psi = \{p_1, p_4\}$, $\overline{G(d)}_\psi = \{p_1, p_2, p_4, p_6\}$, $\underline{G(t)}_\psi = \{p_1\}$, $\overline{G(t)}_\psi = \{p_1, p_2, p_6\}$, $\underline{G(e)}_\psi = \{p_1, p_3, p_5\}$, $\overline{G(e)}_\psi = \{p_1, p_2, p_3, p_5, p_6\}$. *Therefore, the fuzzy rough soft set of* (G, B) *is given by* $\{(G(d), \frac{1}{2}), (G(d), \frac{1}{3}), (G(d), \frac{3}{5})\}$. *Then* $[(G(d), \frac{1}{2})]_\asymp = \{(G(d), \frac{1}{2})\}$, $[(G(t), \frac{1}{3})]_\asymp = \{(G(t), \frac{1}{3})\}$, $[(G(e), \frac{3}{5})]_\asymp = \{(G(e), \frac{3}{5})\}$. *Clearly, fuzzy rough soft set of* (G, B) *forms a chain by the order relation '*$\asymp$*'. The Hasse diagram for this chain is depicted in Figure 9.3.1.*

9.4 Conclusion

Soft set theory, rough set theory and fuzzy set theory have been all treated as mathematical tools to deal with uncertainty for variety of problems. A possible hybridization of these theories is an interesting topic to the researchers. In this study, we have proposed the concept of fuzzy rough soft set in MSR approximation space which can be viewed as a pair of soft set and its roughness. We have defined the union and the intersection of fuzzy rough soft set with several examples. Also we have established important properties of fuzzy rough soft set.

Chapter 10

Conclusion and Scope of Future Works

10.1 Conclusion

Lattice and order set have wide fields of applications in computer science, engineering, discrete mathematics, data mining, number theory, group theory etc. In addition to the above, many applications utilize lattices and ordered set in fundamental ways. These include such areas as knowledge representation, text categorization and data mining, where order plays an fundamental organizing principle. Also, for the application of lattice and ordered set to inductive logic programming, ordered set form basic models. On the other hand in our complex world, there are many situations occur, where we cannot use traditional methods to solve problems in economics, engineering, environment, social science, medical science etc. because of various types of uncertainties present in these problems. Probability theory, fuzzy set theory, rough set theory, soft set theory are novel mathematical tools to solve real world uncertain problems approximately. That is why, the study on lattice theory under uncertain environments with the help of rough set, soft set and their hybridizations i.e., rough soft sets are initiated. We present a general frame work for the study of approximation in lattice. We have studied the properties of lattice in an approximation space based on Pawlak's notion of indiscernibility relation among the objects in a set. Rough modular lattice and rough distributive lattice are defined in Pawlak's approximation space. In order to study an algebraic connection between soft set and algebraic system like lattice theory in a soft approximation space. Notions of soft rough lattice are introduced. In this thesis, rough-ideal and rough-homomorphism are studied in rough set environments. We have initiated to study rough set and soft set in different types of approximation spaces. Rough soft set is defined in a modified soft rough approximation space. We have presented soft set relation in a new way; and based on this relation we have introduced lattice theory on soft sets. In this thesis we have presented a hybridization structure between fuzzy set and rough-soft set and, as a result fuzzy rough soft set is introduced. To enrich the theoretical development of lattice theory under soft set environment, the notion of soft congruence relation is introduced. In this study, we have tried to made a fusion between fuzzy set and rough soft set. Here we have measured the roughness of rough soft set and introduced the concept of fuzzy rough soft set in MSR-approximation space. Moreover, lattice theory is studied on the fuzzy rough soft set. In the whole thesis we have tried to incorporate lattice theory in uncertain environments.

In this thesis we have studied uncertainty in algebra with the help of rough set, soft set and soft rough set. We have constructed different approximation spaces based on Pawlak's approximation space and then have approximated a subset of the universal set. In this study, hybrid model combining rough sets with soft sets, rough soft sets are exploited to extend many practical applications based on rough sets or soft sets. As a consequence, the output results in each chapter certainly have been arrested the attention of researchers who are highly hopeful that this thesis will widely help in and contribute to growth and development of interest among interested researchers who are involving in such areas.

10.2 Scope of Future Works

There are many directions of future works emerging from this thesis. A few of them are appended below:

• The concept of rough modular and rough distributive lattice can be extended in knowledge representation problems.

• Soft rough ideal and soft rough homomorphism in connection with other algebra may be designed.

• Researchers may put their attention on soft rough approaches to ring ideal.

• Researchers can apply the hybrid structure (Rough-Soft) in decision making problem.

• Researchers can formulate soft rough ideal to some applied fields such as knowledge representation theorems, information system etc.

• The concept of soft rough ideal may be extended to more results on lattices under soft rough environment.

• Researchers can implement rough congruence relation and soft rough congruence relation over lattice.

• Soft congruence relation of soft lattice and fuzzy soft congruence relation can be designed based on our defining soft congruence relation.

• The researchers can define soft rough lattice newly based on our proposed soft set relation.

• Researchers may defined fuzzy rough soft relation, congruence relation and lattice ideal under fuzzy rough soft set, which may be the extended work of this thesis.

• In this thesis, we have studied lattice theory under uncertain environment and the obtained results may be apply in that situation where the information about the data are imprecise.

• Finally, one can applied lattice theory for practical problems on engineering, computer science, data mining, decision making problem, knowledge representation system and other real-life problems under uncertain environment.

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2. Roy, S. K. and Bera, S. (2014) Distributive lattice: a rough set approach, *Malaya Journal of Matematik (University Press, Singapore)*, Vol. **2**, No. 3, pp. 273-276.
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4. Roy, S. K. and Bera, S. (2015) Approximation of rough soft set and its application to lattice, *Fuzzy Information and Engineering (Elsevier)*, Vol. **7**, No. 3, pp. 379-387.
5. Bera, S. and Roy, S. K. (2016) Soft rough approach to lattice-ideal, *The Journal of Fuzzy Mathematics (International Fuzzy Mathematics Institute, USA,)* Vol. **24**, No. 1, pp. 49-56.
6. Bera, S., Roy, S. K. and Karaaslan, F. (2017) Soft congruence relation and its application to lattice, *Hacettepe Journal of Mathematics and Statistics (Hacettepe University, Turkey),* SCIE, IF: 0.277, DOI: 10.15672/HJMS.2017.436.

List of Communicated Papers

1. Bera, S. and Roy, S. K. Fuzzy rough soft set and its application to lattice.
2. Bera, S., Roy, S. K. and Çağman, N. An another approach for cartesian product on soft set relation and its application to lattice.
3. Bera, S. and Roy, S. K. Rough ideal and homomorphism and their applications to lattice.

List of Conferences/Seminar Attended

1. International Conference on **Frontier of Mathematical Sciences with Applications (ICFMSA),** Calcutta Mathematical Society (CMS), December 07-09, 2012.

2. National Seminar on **Emerging Trends in Mathematics,** Vidyasagar University, W.B. In collaboration with Calcutta Mathematical Society (CMS), December 19-20, 2012.

3. 1^{st} International Conference on **Recent Trends in Mathematics and Its Applications(ICRTMA),** Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, W.B., March 20-21, 2013.

4. International Conference on **Facets of Uncertainties and Applications (ICFUA),** ORSI, Calcutta Chapter and Department of Applied Mathematics, University of Calcutta, December 5-7, 2013.

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