m-polar Fuzzy Graphs and Their Applications

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m-polar Fuzzy Graphs and Their Applications

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DEDICATED

TO

My Father
Mr. Sankar Ghorai
The reason of what I become today. Thanks for your great support and continuous care

My Mother
Late Malati Ghorai
For being my first teacher

My Wife
Mrs. Oindrila Mondal (Ghorai)
Whose affection, love, encouragement and prays of day and night make me able to get such success and honor

My Son
Ainesh Ghorai
Served as my inspiration and strength during stormy days

Declaration

I do hereby declare that the present Ph.D. thesis entitled "m-polar Fuzzy Graphs and Their Applications" embodies original research work solely carried out by me in the Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, West Bengal, India under the guidance of Professor Madhumangal Pal, Professor, Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, West Bengal, India and no part thereof has been submitted for any degree or diploma in any University/ Institution.

Date:

Place: Vidyasagar University, Midnapore (**Ganesh Ghorai**)

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CERTIFICATE

This is to certify that the thesis entitled ``*m-polar Fuzzy Graphs and Their Applications*" which is being submitted by **Ganesh Ghorai**, a research scholar in the **Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, India** in fulfillment of the degree of ***Doctor of Philosophy in Science*** is a record of bonafide research work carried out by him under my supervision and guidance. The thesis has fulfilled all the requirements as per the regulations of the Vidyasagar University, Midnapore. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

(Madhumangal Pal)

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Abstract

In this thesis, different types of m-polar fuzzy graphs have been considered. The major problems considered in the thesis are generalized m-polar fuzzy graphs and their properties, operations on m-polar fuzzy graphs, degree of vertices of m-polar fuzzy graphs, density of m-polar fuzzy graphs, m-polar fuzzy planar graphs, isomorphism and weak self complement m-polar fuzzy graphs, edge regular m-polar fuzzy graphs, the applications of m-polar fuzzy graphs, generalized regular bipolar fuzzy graphs and product bipolar fuzzy line graphs.

This thesis consists of ten chapters. In the first chapter, we provided the basic definitions of graph and different types of fuzzy graph which are needed in the subsequent chapters and further, a history of the problems.

In Chapter 2, we introduced generalized m-polar fuzzy graphs. Some operations have been defined to formulate these graphs. Some properties of strong m-polar fuzzy graphs, self complementary m-polar fuzzy graphs and self complementary strong m-polar fuzzy graphs are discussed.

In Chapter 3, we have defined three new operations on m-polar fuzzy graph such as direct product, semi-strong product and strong product. It is proved that any of the products of m-polar fuzzy graphs are again an m-polar fuzzy graph. Sufficient conditions are established for each one of them to be strong and also proved that strong product of two complete m-polar fuzzy graphs is complete. If any of the products of two m-polar fuzzy graphs G_1 and G_2 are strong, then it is shown that at least G_1 or G_2 must be strong. The degree of a vertex in m-polar fuzzy graphs which are obtained from two given m-polar fuzzy graphs G_1 and G_2 using the operations of Cartesian product, composition, direct product, semi-strong product and strong product. At the end of this chapter, 3-polar fuzzy influence graph is introduced as an application.

In Chapter 4, density of m-polar fuzzy graphs is defined and then introduced the notion of balanced m-polar fuzzy graphs. Some characterizations of balanced m-polar fuzzy graphs are given.

In Chaper 5, m-polar fuzzy planar graphs, m-polar fuzzy dual graphs are defined and some important properties are established. Here, the "degree of planarity" is used to measure the nature of planarity of an m-polar fuzzy planar graph. Also, we introduced some terms like m-polar fuzzy multiset, m-polar fuzzy multigraphs, m-polar fuzzy dual

graph. Some theorems have been proved on degree of planarity. Depending on the degree of planarity, the considerable edge has been introduced.

In Chapter 6, weak self complement m-polar fuzzy graphs is defined. A necessary condition is mentioned for an m-polar fuzzy graph to be weak self complement. Some properties of self complement and weak self complement m-polar fuzzy graphs are discussed. The order, size, busy vertices and free vertices of an m-polar fuzzy graphs are also defined and proved that isomorphic m-polar fuzzy graphs have same order, size and degree. Also, we have proved some results of busy vertices in isomorphic and weak isomorphic m-polar fuzzy graphs. A relative study of complement and operations on m-polar fuzzy graphs have been made. Some real life problems have been modeled using the concepts of m-polar fuzzy graphs.

In Chapter 7, the concept of edge regular, strongly regular and biregular m-polar fuzzy graph are introduced. Some properties of them are studied. Also, the concept of partially edge regular m-polar fuzzy graph and fully edge regular m-polar fuzzy graph are introduced with suitable illustrations. The notion of strongly edge irregular and strongly edge totally irregular m-polar fuzzy graphs. Some properties of them are also studied to characterize strongly edge irregular and strongly edge totally irregular m-polar fuzzy graphs.

In Chapter 8, we used m-polar fuzzy sets to introduce the notion of m-polar ψ-morphism on m-polar fuzzy graphs. The action of m-polar ψ-morphism on m-polar fuzzy graphs is studied and we established some results on weak and co-weak isomorphism. d_2-degree and total d_2-degree of a vertex in m-polar fuzzy graphs are defined and studied $(2, \overline{k})$-regularity and totally $(2, \overline{k})$-regularity. A real life situation of a company has been modeled in terms of 4-polar fuzzy graphs as an application.

In Chapter 9, we introduced generalized regular bipolar fuzzy graphs and investigated some its properties. Then, we define a product bipolar fuzzy intersection graph of a product bipolar fuzzy graph and the product bipolar fuzzy line graphs. Some characterizations of product bipolar fuzzy line graphs are also made.

Finally, Chapter 10 contains some concluding remarks and scopes of further research on the problems that have been studied in the thesis.

List of research papers

1. G. Ghorai and M. Pal, A note on "Regular bipolar fuzzy graphs" Neural Computing and Applications 21(1) 2012 197-205, *Neural Computing and Applications*, doi:10.1007/s00521-016-2771-0, (2016). [SCIE, Impact Factor 1.492]

2. G. Ghorai and M. Pal, Faces and dual of m-polar fuzzy planar graphs, *Journal of Intelligent and Fuzzy Systems*, **31**(3) 2043-2049 (2016). [SCIE, Impact Factor 1.004]

3. G. Ghorai and M. Pal, Some isomorphic properties of m-polar fuzzy graphs with applications, *SpringerPlus*, **5**(1) 1-21 (2016). [SCIE, Impact Factor 0.982]

4. G. Ghorai and M. Pal, A study on m-polar fuzzy planar graphs, *Int. J. of Computing Science and Mathematics*, **7**(3) 283-292 (2016). [Scopus Indexed Journal]

5. G. Ghorai and M. Pal, On some operations and density of m-polar fuzzy graphs, *Pacific Science Review A: Natural Science and Engineering*, **17**(1) 14-22 (2015). [Elsevier]

6. Some properties of m-polar fuzzy graphs, *Pacific Science Review A: Natural Science and Engineering*, **18**(1) 38-46 (2016). [Elsevier]

7. G. Ghorai and M. Pal, Novel concepts of strongly edge irregular m-polar fuzzy graphs, *International Journal of Applied and Computational Mathematics*, doi: 10.1007/s40819-016-0296-y, (2016). [Springer]

8. G. Ghorai and M. Pal, Ceratin types of product bipolar fuzzy graphs, *International Journal of Applied and Computational Mathematics*, doi: 10.1007/s40819-015-0112-0, (2015). [Springer]

9. G. Ghorai and M. Pal, On degrees of m-polar fuzzy graphs with application, communicated.

10. G. Ghorai and M. Pal, Morphism of m-polar fuzzy graphs with application, communicated.

Contents

Chapter 1

Introduction

Graphs can be used as a modeling tool for many problems of practical importance. For instance, a network of cities, which are represented by vertices, and connections among them make a graph. The well-known traveling salesman problem asks for the shortest possible tour, which visits all the cities exactly once. There are numerous applications like this. Graph theory was born in 1736 with Euler's paper in which he solved the Konigsberg bridge problem. This problem lead to the concept of Eulerian graph. In 1840, Mobious gave the idea of complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems. Graphs are very convenient tools for representing the relationships among objects, which are represented by vertices. In their turn, relationships among vertices are represented by connections. In general, any mathematical object involving points and connections among them can be called a graph or a hypergraph. For a great diversity of problems such pictorial representations may lead to a solution. Examples of such applications include databases, physical networks, organic molecules, map colorings, signal-flow graphs, web graphs, tracing mazes as well as less tangible interactions occurring in social networks, ecosystems and in a flow of a computer program. Thus, graphs can serve as a mathematical models to solve an appropriate graph-theoretic problem, and then interpret the solution in terms of the original problem. At present, graph theory is a dynamic field in both theory and applications.

There are several types of graphs which represent real world problems. These are discussed below.

1.1 Some preliminaries on graphs

Pictorial representation of a graph consists of vertices (representing objects) and edges (representing connections) between them. Formal definition is as follows:

Definition 1.1.1. (Graph) *A graph is an ordered pair $G = (V, E)$ of two sets V and E, where V is the set of vertices, nodes or points each representing the objects and E is the set of edges, arcs or lines which is a subset of $V \times V$, i.e. a relation defined on V.*

A multigraph [15] is a graph that may contain multiple edges between any two vertices, but it does not contain any self loops. A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding [15].

Graph has many variations such as directed graph, undirected graph, simple graph, finite graph, infinite graph, etc. In a directed graph the relation defined on V is not symmetric but in undirected graph the relation defined on V is symmetric. In a graph, loops may occur that is, a vertex has a relation to itself. Also, there may have more than one edges between two vertices, called parallel edges. Simple graphs have no multiple edges and loops at all. If in a graph, there are finite number of vertices and finite number of edges, the graph is called finite. Otherwise, it is infinite. Most commonly, unless stated otherwise, graph means undirected simple finite graph.

In a directed graph $\overrightarrow{G} = (V, \overrightarrow{E})$, a *walk* in $\overrightarrow{G}$ is an alternating sequence $W = v_1\overrightarrow{e_1}v_2\overrightarrow{e_2}\ldots v_{k-1}\overrightarrow{e_k}v_k$ of vertices v_i and arcs $\overrightarrow{e_i}$ of $\overrightarrow{G}$ such that tail of $\overrightarrow{e_i}$ is v_i and head is v_{i+1} for every $i = 1, 2, \ldots, k-1$. A walk is closed if $v_1 = v_k$. A trail is a walk in which all arcs are distinct. A *path* is a walk in which all vertices are distinct. A path $v_1, v_2, \ldots, v_k$ with $k \geq 3$ is a cycle if $v_1 = v_k$. The length of a path or a cycle is the number of its edges.

When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be incident with (on or to) each other. Two nonparallel edges are said to be adjacent if they are incident on a common vertex. Similarly, two vertices are said to be adjacent if they are the end vertices of the same edge. The number of edges incident on a vertex v_i with self-loops counted twice, is called the degree $d(v_i)$ of vertex v_i. The degree of a vertex is sometimes also referred to as its valency. Let us now consider a graph G with e edges and n vertices $v_1, v_2, \ldots, v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. That is, $\sum_{i=1}^{n} d(v_i) = 2e$.

Definition 1.1.2. *A graph in which all vertices are of equal degree is called a regular graph (or simply a regular).*

A vertex having no incident edge is called an isolated vertex. In other words, isolated vertices are vertices with zero degree. A regular graph of degree 0 has no lines at all. If G is regular of degree 1, then every component contains exactly one line; if it is regular of degree 2, every component is a cycle.

The minimum degree among the vertices of G is denoted $\delta(G)$ while $\triangle(G)$ is the largest such number. If $\delta(G) = \triangle(G) = r$, then all points have the same degree and G is called regular of degree r.

In geometry two figures are thought of as equivalent (and called congruent) if they have identical behavior in terms of geometric properties. Likewise, two graphs are thought of as equivalent (and called isomorphic) if they have identical behavior in terms of graph-theoretic properties. More precisely, two graphs G_1 and G_2 are said to be isomorphic (to each other) if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. In other words, suppose that edge e_1 is incident on vertices v_1 and v_2 in G_1 then the corresponding edge e_2 in G_2 must be incident on the vertices v_3 and v_4 that correspond to v_1 and v_2 respectively.

Definition 1.1.3. *A graph G is planar if it can be drawn in the plane with its edges only intersecting at vertices of G. So the graph is non-planar if it can not be drawn without crossing.*

In 1930, Kuratowski [64] invented some important results on planar graphs. A planar graph with cycles divides the plane into a set of regions, also called *faces*. The length of a face in a plane graph G is the total length of the closed walk(s) in G bounding the face. The portion of the plane lying outside a graph embedded in a plane is infinite region. In graph theory, the dual graph of a given planar graph G is a graph which has a vertex corresponding to each plane region of G, and the graph has an edge joining two neighboring regions for each edge in G, for a certain embedding of G. Whitney's planarity criterion [134] gives a characterization based on the existence of an algebraic dual. MacLane's planarity criterion [66] gives an algebraic characterization of finite planar graphs. Fraysseix Rosenstiehl's [44] planarity criterion gives a characterization based on the existence of a bipartition of the co-tree edges of a depth-first search tree. Schnyder's theorem [129] gives a characterization of planarity in terms of partial order dimension.

In mathematical area of graph theory, an intersection graph is a graph that represents the pattern of intersection of family of sets. An interval graph is the intersection of multiset of intervals on real line. Interval graphs are useful in resource allocation problem in operations research. Besides, interval graphs are used extensively in mathematical modeling, archaeology, developmental psychology, ecological modeling, mathematical sociology and organization theory. Tolerance graph [54] is another important graph. Tolerance graphs were introduced in order to generalize some well known applications of interval graphs. The main motivation was to model resource allocation and certain scheduling problems, in which resources, such as rooms and vehicles, can tolerate sharing among users. Tolerance graphs find in a natural way for applications in biology and bio informatics. The tolerance graphs find numerous other applications in constrained-based temporal reasoning, data transmission through networks to efficiently scheduling aircraft and crews, as well as contributing to genetic analysis and studies of the brain. The definition of tolerance graph is given below.

Definition 1.1.4. *[54] Tolerance graphs are generalization of interval graphs in which each vertex can be represented by an interval and a tolerance such that an edge occurs if and only if the overlap of corresponding intervals is at least as large as the tolerance associated with one of the vertices. Hence, a graph $G = (V, E)$ is a tolerance graph if there is a set $I = \{I_v : v \in V\}$ of closed real intervals and a set $\{T_v : v \in V\}$ of positive real numbers such that $(x, y) \in E$ if $|I_x \cap I_y| \geq min\{T_x, T_y\}$. The collection $\langle I, T\rangle$ of intervals and tolerances is called tolerance representation of the graph G.*

Bogart [26] et al. introduced proper and unit tolerance graphs. Brigham et al. [28] investigated different properties of tolerance competition graphs. Mertzios and Zaks [77] recognized of tolerance and bounded tolerance graphs.

Threshold graphs play an important role in graph theory as well as in several applied areas such as psychology, computer science, scheduling theory, etc. These graphs can be used to control the flow of information between processors, much like the traffic lights used in controlling the flow of the traffic. Acharya and Vartak [2] introduced open neighbourhood graphs. Chvatal and Hammer [39] solved set-packing problems and introduced threshold graphs. Andelic and Simic [12] discussed some notes on the threshold graphs. The definition of threshold graph is given below.

Definition 1.1.5. *[39] A graph $G = (V, E)$ is a threshold graph when there exists non-negative reals $w_v, v \in V$ and t such that $W(U) \leq t$ if and only if $U \subseteq V$ is stable set where $W(U) = \sum\limits_{v \in U} w_v$.*

So, $G = (V, E)$ is a threshold graph whenever one can assign vertex weights such that a set of vertices is stable if and only if its total weight does not exceed a certain threshold. The threshold dimension, $t(G)$ of a graph G is the minimum number k of threshold subgraphs $T_1, T_2, \ldots, T_k$ of G that cover the set of edges G. Threshold partition number, denoted by $tp(G)$, is the minimum number of edge disjoint threshold subgraphs needed to cover $E(G)$.

Formally, an edge cover of a graph G is a set of edges $C \subset E$ such that each vertex is incident with at least one edge in C. The set C is said to cover the vertices of G.

Definition 1.1.6. *[98] Ferrers digraph is a related digraph to threshold graph. A digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ is said to be a Ferrers digraph if it does not contain vertices x, y, z, w, not necessarily distinct, satisfying $\overrightarrow{(x, y)}, \overrightarrow{(z, w)} \in \overrightarrow{E}$ and $\overrightarrow{(x, w)}, \overrightarrow{(z, y)} \notin \overrightarrow{E}$. For a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, the underlying loop less graph $U(\overrightarrow{G}) = (V, E)$, where $E = \{(u, v) : u, v \in V, u \neq v, \overrightarrow{(u, v)} \in \overrightarrow{E}\}$.*

A split graph is a graph in which the vertices can be partitioned into a clique and an independent set.

Alternating 4-cycle of a graph $G = (V, E)$ is a configuration consisting of distinct vertices a, b, c, d such that $(a, b), (c, d) \in E$ and $(a, c), (b, d) \notin E$. By considering the presence or absence of edges $(a, d), (b, c)$, we see that the vertices of alternating 4-cycle induce a path P_4, a square C_4, or a matching $2K_2$.

For the graph $G = (V, E)$ with distinct positive vertex degrees $\delta_1 < \delta_2 < \ldots < \delta_m$ and $\delta_0 = 0$ (even no vertex of degree 0 exists), $\delta_{m+1} = |V| - 1$ degree partition is the sequence $D_i = \{v \in V : \deg(v) = \delta_i\}$ for $i = 0, 1, \ldots, m$.

Two vertices u and v are incomparable if they do not belong to the same tree or if there is no path from u to v and no path from v to u.

Directed graphs are similarly defined except they have directed edges. The formal definition is given below.

Definition 1.1.7. *A directed graph (digraph) $\overrightarrow{G}$ is a graph which consists of non-empty finite set $V(\overrightarrow{G})$ of elements called vertices and a finite set $\overrightarrow{E}(\overrightarrow{G})$ of ordered pairs of distinct vertices called arcs.*

The *out-neighborhood* [57] of a vertex v is the set $N^+(v) = \{u \in V - v : \overrightarrow{(v,u)} \in \overrightarrow{E}\}$. Similarly, the *in-neighborhood* [57] $N^-(v)$ of a vertex v is the set $\{w \in V - v : \overrightarrow{(w,v)} \in \overrightarrow{E}\}$. The *open neighborhood* of a vertex is the union of out-neighborhood and in-neighborhood of the vertex. A *walk* in $\overrightarrow{G}$ is an alternating sequence $W = x_1\overrightarrow{e_1}x_2\overrightarrow{e_2}\ldots x_{k-1}\overrightarrow{e_k}x_k$ of vertices x_i and arcs $\overrightarrow{e_i}$ of $\overrightarrow{G}$ such that tail of $\overrightarrow{e_i}$ is x_i and head is x_{i+1} for every $i = 1, 2, \ldots, k-1$. A walk is closed if $x_1 = x_k$. A trail is a walk in which all arcs are distinct. A *path* is a walk in which all vertices are distinct. A path $x_1, x_2, \ldots, x_k$ with $k \geq 3$ is a cycle if $x_1 = x_k$.

For an undirected graph, *open-neighborhood* [2] $N(x)$ of the vertex x is the set of all vertices adjacent to x in the graph. *Open neighborhood graph* [2] $N(G)$ of G is a graph whose vertex set is same as G and has an edge between two vertices x and y in $N(G)$ if and only if $N(x) \cap N(y) \neq \phi$ in G. *Closed neighborhood* $N[x]$ of x is the set $N(x) \cup \{x\}$. *Closed neighborhood graph* $N[G]$ of a graph G is similarly defined, except has an edge in $N[G]$ if and only if $N[x] \cap N[y] \neq \phi$ in G. *(p)-neighborhood graph* (read as open p-neighborhood graph) [27], $N_p(G)$ of a graph G is a graph whose vertex set is same as G and has an edge between two vertices x and y if and only if $|N(x) \cap N(y)| \geq p$ (note that $|X|$ is the number of elements in the crisp set X) in G. Similarly *[p]-neighborhood graph* (closed p-neighborhood graph) $N_p[G]$ [27] is defined except there is an edge if and only if $|N[x] \cap N[y]| \geq p$ in G.

In 1968, Cohen [41] introduced the notion of competition graphs in connection with a problem in ecology. Let $\overrightarrow{D} = (V, \overrightarrow{E})$ be a digraph, which corresponds to a food web. A vertex $x \in V(\overrightarrow{D})$ represents a species in the food web and an arc $\overrightarrow{(x,s)} \in \overrightarrow{E}(\overrightarrow{D})$ means that x preys on the species s. If two species x and y have a common prey s, they will compete for the prey s. Based on this analogy, Cohen defined a graph which represents the relations of competition among the species in the food web. The competition graph is also applicable in channel assignment, coding, modelling of complex economic and energy systems, etc. [101]. Cable et al. [38] introduced niche graphs to represent ecological problems. Lundgren and Maybee [71] introduced food webs with interval competition graph. The large research on competition graphs can be found in [59–61, 122]. Cho et al. [37] introduced m-step competition graph of a diagraph. Raychaudhuri and Roberts [101] introduced generalized competition graphs and their applications. Sano [119, 120] investigated several properties on the competition-common enemy graphs of digraphs.

Now, the competition graph is defined below.

Definition 1.1.8. *[41] The competition graph $C(\overrightarrow{G})$ of a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ is an undirected graph $G = (V, E)$ which has the same vertex set V and has an edge between distinct two vertices $x, y \in V$ if there exist a vertex $a \in V$ and arcs $\overrightarrow{(x, a)}, \overrightarrow{(y, a)} \in \overrightarrow{E}$ in $\overrightarrow{G}$. We say that a graph G is a competition graph if there exists a digraph $\overrightarrow{G}$ such that $C(\overrightarrow{G}) = G$.*

Another kind of competition graph is given below.

Definition 1.1.9. *[60] If p is a positive integer, the p-competition graph $C_p(\overrightarrow{G})$ corresponding to the digraph $\overrightarrow{G}$ is defined to have a vertex set V with an edge between x and y in V if and only if, for some distinct vertices $a_1, a_2, \ldots, a_p$ in V, $\overrightarrow{(x, a_1)}, \overrightarrow{(y, a_1)}, \overrightarrow{(x, a_2)}, \overrightarrow{(y, a_2)}, \ldots, \overrightarrow{(x, a_p)}, \overrightarrow{(y, a_p)}$ are arcs in $\overrightarrow{G}$.*

The m-step competition graph of a digraph is defined below.

Definition 1.1.10. *[37] Let $\overrightarrow{G}$ be a digraph. Suppose $\overrightarrow{G}$ represents a food web, where an arc from x to y implies that x is a predator of y. Let m be a positive integer. If there is a path from x to z of length m, then we say that x is an m-step predator of z, and z is its m-step prey. The m-step competition graph $C_m(\overrightarrow{G})$ is a graph with the same vertices as $\overrightarrow{G}$; the vertices x and y are joined by an edge in $C_m(\overrightarrow{G})$ if they share a common m-step prey in $\overrightarrow{G}$. We say that x and y are in m-step competition if they share an edge in $C_m(\overrightarrow{G})$.*

As is the case with most mathematical entities, it is convenient to consider a large graph as a combination of small ones and to derive its properties from those of the small ones. Since graphs are defined in terms of the sets of vertices and edges, it is natural to employ the set-theoretical terminology to define operations between graphs.

Definition 1.1.11. *[55] Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs.*

The Cartesian product $G^* = G_1^* \times G_2^* = (V, E)$ of graphs G_1^* and G_2^*. Then $V = V_1 \times V_2$ and $E = \{(x, x_2)(x, y_2) : x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) : z \in V_2, x_1y_1 \in E_1\}$.

The composition of the graph G_1^* with G_2^* is denoted by $G_1^*[G_2^*] = (V_1 \times V_2, E^0)$, where $E^0 = E \cup \{(x_1, x_2)(y_1, y_2) : x_1y_1 \in E_1, x_2 \neq y_2\}$ and E is defined in $G_1^* \times G_2^*$. Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$.

The union of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1^* and G_2^* is denoted by $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$.

The join of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup E'$, where E' is the set of all edges joining the nodes of V_1 and V_2 and assume that $V_1 \cap V_2 = \emptyset$. The join of G_1^* and G_2^* is denoted by $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$.

A (crisp) *hypergraph* on a set X is a pair $H^* = (X, E)$ where X is a finite set and E is a finite family of nonempty subsets of X which satisfy the condition: Every member of X is contained in some member of E. X is called the vertex set and E is the edge set of H^*. Multiple or repeated edges are allowed. A hypergraph $H^* = (X, E)$ is *simple* if E contains no repeated edges and whenever $E_1, E_2 \in E$ and $E_1 \subset E_2$, then $E_1 = E_2$. A hypergraph $H^* = (X; E_1, E_2, \ldots, E_k)$ where $X = \{x_1, x_2, \ldots, x_n\}$ can be mapped to a hypergraph $H^{**} = (E; x_1, x_2, \ldots, x_n)$ whose vertices are the points $e_1, e_2, \ldots, e_k$ (corresponding to $E_1, E_2, \ldots, E_k$), and whose edges are the sets $X_1, X_2, \ldots, X_n$ (corresponding to $x_1, x_2, \ldots, x_n$ respectively) where $X_j = \{x_j \in E_i, i \leq k\}, j = 1, 2, \ldots, n$. The hypergraph H^{**} is called the *dual hypergraph* of H.

Suppose $H_1^* = (X_1, E_1)$ and $H_2^* = (X_2, E_2)$ are crisp hypergraphs. Then H_1^* is *partial hypergraph* of H_2^* if $E_1 \subseteq E_2$, this relationship is denoted by $H_1^* \leq H_2^*$. A sequence of crisp hypergraphs $H_i^* = (X_i, E_i)$, $1 \leq i \leq n$ is said to be *ordered* if $H_1 < H_2 < \ldots < H_n$. The sequence $\{H_i^* | 1 \leq i \leq n\}$ is *simply ordered* if it is ordered and if whenever $E \in E_{i+1} \backslash E_i$, then $E \not\subseteq X_i$.

Almost all of our traditional tools for formal modeling, reasoning and computing are crisp, deterministic and explicit in character. Explicitness accepts that parameters of a system either belong to the system or not. In reality, if complexity of systems increases, the explicitness of systems reduces. Uncertainty has a pivotal role in any efforts to maximize the usefulness of parameters of systems or models. One of the meanings attributed to the term 'uncertainty' is "vagueness", i.e. the difficulty of making sharp or precise distinction. This applies even to many terms used in our day to day life, such as 'tall', 'nice', 'congested', etc. It is important to realize that this imprecision or vagueness that are characteristic of natural language does not necessarily imply a loss of accuracy or meaningfulness. A mathematical frame work to describe this phenomena was suggested by Zadeh [138–141] in his seminal paper entitled "Fuzzy Sets". Kosko [65], in his book, calls this as mismatch problem: The world is gray but science is black and white. Thus, the membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree. The research on fuzzy sets is increasing till the date due to its wide applications. Major applications of fuzzy set include image processing, optimization of networks, video traffic modelling and classification, shortest path problem, neural networks, etc.

1.2 Fuzzy Sets

In 1965, fuzzy set theory was introduced by Zadeh, Iranian-American Mathematician and Professor of computer science, as a generalization of Cantor's set theory. In the literature of fuzzy sets, the word fuzzy often stands for the word vague (formless, unclear).

Definition 1.2.1. (Crisp set) *A classical set is a collection of well defined objects with a crisp boundary. A crisp set A is characterized by a characteristic function which is denoted by χ_A and is defined by*

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$$

The idea of membership function of a fuzzy set is coming from the characteristic function of crisp set.

Definition 1.2.2. (Fuzzy set) *A fuzzy set A on a set X is characterized by a mapping $m : X \to [0,1]$, which is called the membership function. A fuzzy set is denoted by $A = (X, m_A)$.*

In the theory of fuzzy sets the membership degrees of elements range over the interval $[0,1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 denote that an element completely belongs to its corresponding fuzzy set and the membership degree 0 denote that an element does not belong to the fuzzy set. The membership degrees in the interval $(0, 1)$ denote the partial belongingness to the fuzzy set.

A (crisp) *multiset* over a non-empty set V is simply a mapping $d : V \to N$, where N is the set of natural numbers. Yager [135] first discussed fuzzy multisets, although he used the term "fuzzy bag". An element of nonempty set V may occur more than once with possibly the same or different membership values. A natural generalization of this interpretation of multiset leads to the notion of *fuzzy multiset*, or *fuzzy bag*, over a non-empty set V as a mapping $\tilde{C} : V \times [0, 1] \to N$. The membership values of $v \in V$ are denoted by $v_{\mu^j}, j = 1, 2, \ldots, p$ where $p = \max\{j : v_{\mu^j} \neq 0\}$. So the fuzzy multiset can be denoted as $M = \{(v, v_{\mu^j}), j = 1, 2, \ldots, p | v \in V\}$.

Operations on fuzzy sets

The fuzzy set theory is extended with definitions for set theoretic operations. Zadeh first defined basic operations. Over time, other authors have suggested additional and alternative operations. The following definitions provide an overview of a selection of fundamental operations on fuzzy set and characteristics in order to provide a general understanding of fuzzy set theory. Furthermore, different types of set operations that combine fuzzy sets are presented.

Definition 1.2.3. *[138] Let $A = (X, m_A)$ and $B = (X, m_B)$ be two fuzzy sets in X. Then,*

(i) $A \subseteq B$ if and only if $m_A(x) \leq m_B(x)$ (sometimes $A \subseteq B$ is denoted as $A \leq B$) for all $x \in X$,

(ii) $A = B$ if and only if $m_A(x) = m_B(x)$ for all $x \in X$,

(iii) the union of two fuzzy sets A and B is denoted by $A \cup B$ and is defined by the membership function $m_{A \cup B}(x) = \max\{m_A(x), m_B(x)\}$ for all $x \in X$,

(iv) the intersection of two fuzzy sets A and B is denoted by $A \cap B$ and is defined by the membership function $m_{A \cap B}(x) = \min\{m_A(x), m_B(x)\}$ for all $x \in X$.

Definition 1.2.4. (Cut level set) *[83] Let $A = (X, m_A)$ be a fuzzy set. The t-cut level set of A is the crisp set $A_t = \{x : m_A(x) > t\}$.*

The support of A is $\text{supp}(A) = \{x \in X \mid m_A(x) \neq 0\}$. The core of A is the crisp set of all members whose membership values are 1. A is non trivial if $\text{supp}(A)$ is nonempty. The height of A is $h(A) = max\{m_A(x) \mid x \in X\}$. A is normal if $h(A) = 1$. The family of all fuzzy subsets is denoted by $\mathcal{F}(x)$.

A fuzzy set $A = (X, m_A)$ is said to be convex if its membership value satisfies the following condition $m_A[\lambda x_1 + (1-\lambda)x_2] \geq min[m_A(x_1), m_A(x_2)]$ for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$. The families of sets most often considered in connection with intersection graphs are families of intervals of a linearly ordered set. Let X be a linearly ordered set. A fuzzy interval $\mathcal{I}$ on X is normal, convex fuzzy subset of X. A fuzzy number is a real fuzzy interval. *The cardinality of a fuzzy set* [140] $A = (X, m_A)$ is a positive real number $c(A)$ or $|A|$ is the sum of membership values of the elements of X.

1.3 Fuzzy graphs

Nowadays, graphs do not represent all the systems properly due to the uncertainty or haziness of the parameters of systems. For example, a social network may be represented as a graph where vertices represent an account (person, institution, etc.) and edges represent the relation between the accounts. If the relations among accounts are to be measured as good or bad according to the frequency of contacts among the accounts, fuzziness should be added to representation. This and many other problems lead to define fuzzy graphs. Rosenfeld [102] first introduced the fuzzy graphs considering fuzzy relations on fuzzy sets in 1975. Using this concept of fuzzy graph, Koczy [63] used fuzzy graphs in the evaluation and optimization of networks. After that fuzzy graph theory is a vast research area. Applications of fuzzy graph include data mining, image segmentation, clustering, image capturing, networking, communication, planning, scheduling.

The definition of a fuzzy graph is given below.

Definition 1.3.1. (Fuzzy graph) *[102] A fuzzy graph $\xi = (V, \sigma, \mu)$ is a non-empty set V together with a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that for all $x, y \in V$, $\mu(x, y) \leq \min\{\sigma(x), \sigma(y)\}$, where $\sigma(x)$ and $\mu(x, y)$ represent the membership values of the vertex x and of the edge (x, y) in ξ respectively.*

A loop at a vertex x in a fuzzy graph is represented by $\mu(x, x) \neq 0$. An edge is non-trivial if $\mu(x, y) \neq 0$.

An example of a fuzzy graph is shown in Figure 1.1.

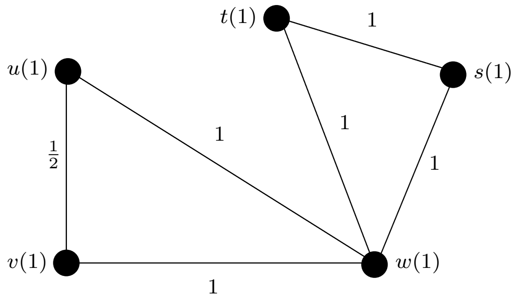

Figure 1.1: An example of fuzzy graph

Some terminologies of fuzzy graphs

Fuzzy subgraph of a fuzzy graph is a fuzzy graph whose vertex set is a subset of the vertex set of the given fuzzy graph. The formal definition is given below.

Definition 1.3.2. (Fuzzy subgraph) *[83] The fuzzy graph $\xi' = (V', \tau, \nu)$ is called a fuzzy subgraph of ξ if $\tau(x) \leq \sigma(x)$ for all $x \in V'$ and $\nu(x, y) \leq \mu(x, y)$ for all $x, y \in V'$ where $V' \subset V$.*

For the fuzzy graph $\xi = (V, \sigma, \mu)$, an edge (x, y) is called strong [42] if

$$\frac{1}{2}\{\sigma(x) \wedge \sigma(y)\} \leq \mu(x, y)$$

and it is called weak, otherwise. The strength of an edge (u, v) is denoted by

$$I_{(u,v)} = \frac{\mu(u, v)}{\sigma(u) \wedge \sigma(v)}.$$

An underlying crisp graph of a fuzzy graph $\xi = (V, \sigma, \mu)$ is a crisp graph $\xi^* = (V, \sigma^*, \mu^*)$ where $\sigma^* = \{u \in V(\xi) \,|\, \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \,|\, \mu(u, v) > 0\}$.

A path P in a fuzzy graph $\xi = (V, \sigma, \mu)$ is a sequence of distinct vertices $v_1, v_2, \ldots,$ $v_n (n \geq 2)$ such that $\mu(v_i, v_{i+1}) > 0$, $i = 1, 2, \ldots, (n-1)$. Here, $(n-1)$ is called the length of the path P. A path P is a cycle if $v_1 = v_n$ and $n \geq 4$. That is $P = (V, \sigma, \mu)$ is a cycle in ξ if and only if (V, σ^*, μ^*) is a cycle in ξ^*. A cycle $P = (V, \sigma, \mu)$ is a fuzzy cycle if it contains more than one weak edge (i.e., there is no unique $(x, y) \in \nu^*$ such that $\mu(x, y) = \wedge\{\mu(u, v) : (u, v) \in \nu^*\}$). Notice that, if a fuzzy graph is complete then the fuzzy graph is strong, but not vice versa. A fuzzy subgraph $\xi' = (V', \tau, \nu)$ of a fuzzy graph $\xi = (V, \sigma, \mu)$ is said to be a fuzzy clique if $(\xi')^*$ is a clique and every cycle in ξ' is a fuzzy cycle.

The union of two fuzzy graphs $\xi_1 = (V_1, \sigma_1, \mu_1)$ and $\xi_2 = (V, \sigma_2, \mu_2)$ is denoted by $\xi_1 \cup \xi_2 = (V_1 \cup V_2, \sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$, where for all $x \in V_1 \cup V_2$, $(\sigma_1 \cup \sigma_2)(x) = \sigma_1(x) \vee \sigma_2(x)$ with $\sigma_1(x) = 0$ whenever $x \notin \sigma_1^*$ and $\sigma_2(x) = 0$ whenever $x \notin \sigma_2^*$, for all $(x, y) \in V_1 \cup V_2 \times V_1 \cup V_2$, $(\mu_1 \cup \mu_2)(x, y) = \mu_1(x, y) \vee \mu_2(x, y)$ with $\mu_1(x, y) = 0$ whenever $(x, y) \notin \mu_1^*$ and $\mu_2(x, y) = 0$ whenever $(x, y) \notin \mu_2^*$.

The strength of connectedness between two vertices u and v is

$\mu^\infty(u, v) = \sup\{\mu^k(u, v) | k = 1, 2, \cdots\}$, where $\mu^k(u, v) = \sup\{\mu(u, u_1) \wedge \mu(u_1, u_2) \wedge \ldots \wedge \mu(u_{k-1}, v) | u_1, u_2, \ldots, u_{k-1} \in V\}$. In a fuzzy graph, an arc (u, v) is said to be strong arc [73] or strong edge, if $\mu(u, v) \geq \mu^\infty(u, v)$ and otherwise it is weak.

Definition 1.3.3. *A fuzzy graph $\xi = (V, \sigma, \mu)$ is said to be bipartite if the vertex set V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further, if $\mu(v_1, v_2) = \min\{\sigma(v_1), \sigma(v_2)\}$ for all $v_1 \in V_1$ and $v_2 \in V_2$, then ξ is called a complete bipartite fuzzy graph.*

If an edge (x, y) of a fuzzy graph satisfies the condition $\mu(x, y) = \min\{\sigma(x), \sigma(y)\}$, then this edge is called effective edge [86]. Two vertices are said to be effective adjacent if they are the end vertices of the same effective edge. Then the effective incident degree of a fuzzy graph is defined as number of effective incident edges on a vertex v. If all the edges of a fuzzy graph are effective, then the fuzzy graph becomes complete fuzzy graph. A pendent vertex in a fuzzy graph is defined as a vertex of an effective incident degree one. A fuzzy edge is called a fuzzy *pendant edge* [113], if one end vertex is fuzzy pendant vertex. The membership value of the pendant edge is the minimum among the membership values of the end vertices.

A fuzzy graph $\xi = (V, \sigma, \mu)$ is said to be *regular* [85] if $d(v) = k$, a positive real number, for all $v \in V$. If each vertex of ξ has same total degree k, then ξ is said to be a *totally regular* fuzzy graph. A fuzzy graph is said to be *irregular* [88], if there is a vertex which is adjacent to vertices with distinct degrees. A fuzzy graph is said to be *neighbourly irregular* [88], if every two adjacent vertices of the graph have different degrees. A fuzzy graph is said to be *totally irregular*, if there is a vertex which is adjacent to vertices with distinct total degrees. If every two adjacent vertices have distinct total degrees of a fuzzy graph then it is called *neighbourly total irregular* [88]. A fuzzy graph is called *highly irregular* [88] if every vertex of G is adjacent to vertices with distinct degrees.

Like complete graph, the definition of complete fuzzy graph is given below.

Definition 1.3.4. (Complete fuzzy graph) *[10] A fuzzy graph $\xi = (V, \sigma, \mu)$ is complete if $\mu(u, v) = \min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$, where (u, v) denotes the edge between the vertices u and v.*

The *complement* of fuzzy graph $\xi = (V, \sigma, \mu)$ [83] is the fuzzy graph $\xi' = (V, \sigma', \mu')$ where $\sigma'(u) = \sigma(u)$ for all $u \in V$ and

$$\mu'(u, v) = \begin{cases} 0, & \text{if } \mu(u, v) > 0 \\ \sigma(u) \wedge \sigma(v), & \text{otherwise.} \end{cases}$$

Operations on fuzzy graphs

Many operations are defined on fuzzy graphs. Some of them are introduced here.

Definition 1.3.5. *[10] The semi-strong product of two fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively, where it is assumed that $V_1 \cap V_2 = \emptyset$, is defined to be the fuzzy graph $G_1 \bullet G_2 = (\sigma_1 \bullet \sigma_2, \mu_1 \bullet \mu_2)$ of the graph $G^* = (V_1 \times V_2, E)$ such that $E = \{(u, v_1)(u, v_2) | u \in V_1, v_1v_2 \in E_2\} \cup \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\}$ and*

(i) $(\sigma_1 \bullet \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$ *for all* $(u, v) \in V_1 \times V_2$,

(ii) $(\mu_1 \bullet \mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1v_2)$,

(iii) $(\mu_1 \bullet \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$.

Definition 1.3.6. *[10] The strong product of two fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively, where it is assumed that $V_1 \cap V_2 = \emptyset$, is defined to be the fuzzy graph $G_1 \otimes G_2 = (\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2)$ of the graph $G^* = (V_1 \times V_2, E)$ such that $E = \{(u, v_1)(u, v_2) | u \in V_1, v_1v_2 \in E_2\} \cup \{(u_1, w)(u_2, w) | w \in V_2, u_1u_2 \in E_1\} \cup \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\}$ and*

(i) $(\sigma_1 \otimes \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$ *for all* $(u, v) \in V_1 \times V_2$,

(ii) $(\mu_1 \otimes \mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1v_2)$,

(iii) $(\mu_1 \otimes \mu_2)((u_1, w)(u_2, w)) = \sigma_2(w) \wedge \mu_1(u_1u_2)$,

(iv) $(\mu_1 \otimes \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$.

Definition 1.3.7. *[10] The direct product of two fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively such that*

$V_1 \cap V_2 = \emptyset$, is defined to be the fuzzy graph $G_1 \sqcap G_2 = (\sigma_1 \sqcap \sigma_2, \mu_1 \sqcap \mu_2)$ of the graph $G^* = (V_1 \times V_2, E)$ such that $E = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\}$ and

(i) $(\sigma_1 \sqcap \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$ for all $(u, v) \in V_1 \times V_2$,

(ii) $(\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$.

Directed fuzzy graphs (or simply fuzzy digraph) are the fuzzy graphs in which the fuzzy relations between edges are not necessarily symmetric. The definition of directed fuzzy graph is as follows:

Definition 1.3.8. (Directed fuzzy graph) *[82] Directed fuzzy graph $\overrightarrow{\xi} = (V, \sigma, \mu)$ is a non-empty set V together with a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that for all $x, y \in V$, $\mu(\overrightarrow{x, y}) \leq \sigma(x) \wedge \sigma(y)$.*

Since $\overrightarrow{\mu}$ is well defined, a fuzzy digraph has at most two directed edges (which must have opposite directions) between any two vertices. Here $\overrightarrow{\mu}(u, v)$ is denoted by the membership value of the edge $\overrightarrow{(u, v)}$. The loop at a vertex x is represented by $\overrightarrow{\mu}(x, x) \neq 0$. Here $\overrightarrow{\mu}$ need not be symmetric as $\overrightarrow{\mu}(x, y)$ and $\overrightarrow{\mu}(y, x)$ may have different values. The *underlying crisp graph of a directed fuzzy graph* is the graph similarly obtained except the directed arcs are replaced by undirected edges.

There are many variations in fuzzy graphs such as (i) Fuzzy intersection graph, (ii) Fuzzy hypergraph, (iii) Fuzzy threshold graph, (iv) Fuzzy tolerance graph, (v) Fuzzy planar graph, (vi) Interval-valued fuzzy graph, (vii) Intuitionistic fuzzy graph, (viii) Bipolar fuzzy graph, (ix) m-polar fuzzy graph, etc.

We now briefly describe these one by one as follows:

1.3.1 Fuzzy intersection graph

McAllister [76] first introduced the fuzzy intersection graph. The definition of fuzzy intersection graph is given below.

Definition 1.3.9. *Let $\mathcal{F} = \{A_1 = (X, m_1), A_2 = (X, m_2), \ldots, A_n = (X, m_n)\}$ be a finite family of fuzzy sets defined on a set X and consider $\mathcal{F}$ as crisp vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The fuzzy intersection graph of $\mathcal{F}$ is the fuzzy graph $Int(\mathcal{F}) = (V, \sigma, \mu)$ where $\sigma : V \to [0, 1]$ is defined by $\sigma(v_i) = h(A_i)$ and $\mu : V \times V \to [0, 1]$ is defined by*

$$\mu(v_i, v_j) = \begin{cases} h(A_i \cap A_j), & \text{if } i \neq j \\ 0, & \text{if } i = j. \end{cases}$$

1.3.2 Fuzzy hypergraphs

Goetschel [51] introduced fuzzy hypergraphs. The definition of fuzzy hypergraph is given below:

Definition 1.3.10. *Let X be a finite set and let $\mathcal{E}$ be a finite family of nontrivial fuzzy sets on X (or subsets of X) such that $X = \bigcup\{suppA|A \in \mathcal{E}\}$. Then the pair $\mathcal{H}= (X, \mathcal{E})$ is a fuzzy hypergraph on X.*

X and $\mathcal{E}$ are respectively vertex set and fuzzy edge set of $\mathcal{H}$. The height of $\mathcal{H}$, $h(\mathcal{H})$, is defined by $h(\mathcal{H}) = max\{h(A)|A \in \mathcal{E}\}$. A fuzzy hypergraph is *simple* if $\mathcal{E}$ has no repeated fuzzy edges and whenever $A, B \in \mathcal{E}$ and $A \subseteq B$, then $A = B$. A fuzzy hypergraph $\mathcal{H}= (X, \mathcal{E})$ is *support simple* if whenever $A, B \in \mathcal{E}$, $A \subseteq B$ and $supp(A) = supp(B)$, then $A = B$. Suppose $A = (X_1, \mu) \in \xi, X_1 \subseteq X$ and $c \in (0, 1]$. The $c-$cut of A is defined by $A^c = \{x \in X|\mu(x) \geq c\}$. If $\mathcal{E}^c = \{A^c| \in \mathcal{E}/\{\phi\}\}$ and $X^c = \bigcup\{A^c|A \in \mathcal{E}\}$. If $\mathcal{E}^c \neq \phi$, then the (crisp) hypergraph $H^c = (X^c, \mathcal{E}^c)$ is the $c-$ *level hypergraph* of $\mathcal{H}$.

Suppose $\mathcal{H}_1 = (X, \mathcal{E}_1)$ and $\mathcal{H}_2 = (X, \mathcal{E}_2)$ are fuzzy hypergraphs. Then $\mathcal{H}_1$ is partial hypergraph of $\mathcal{H}_2$ if $\mathcal{E}_1 \subseteq \mathcal{E}_2$. A fuzzy set $A = (X, \mu)$ with $\mu : X \to [0, 1]$ is an *elementary fuzzy set* if μ is constant function or μ has range $\{0, a\}, 0 \neq a$. An *elementary fuzzy hypergraph* is a fuzzy hypergraph in which all fuzzy edges are elementary.

A fuzzy hypergraph $\mathcal{H}= (X, \mathcal{E})$ is a m *tempered fuzzy hypergraph* of a crisp hypergraph $H^* = (X, E)$ if there exists a fuzzy set $A = (X, m)$ such that $m : X \to (0, 1]$ and $\mathcal{E}= \{\gamma_{E_i}|E_i \in E\}$ where

$$\gamma_{E_i}(x) = \begin{cases} min\{m(e)|e \in E_i\} & \text{if } x \in E_i \\ 0, & otherwise \end{cases}$$

A *fuzzy transversal* $\mathcal{T}= (X, \tau)$ of $\mathcal{H}$ is a fuzzy set defined on X with the property that $\tau_{h(A)} \cap \mu_{h(A)} \neq \phi$ for each $A \in \mathcal{E}$ (recall that $h(A)$ is the height of A). A *minimal fuzzy transversal* $\mathcal{T}$ for $\mathcal{H}$ is a transversal of $\mathcal{H}$ with the property that if $T_1 < T$, then T_1 is not a fuzzy transversal of $\mathcal{H}$.

1.3.3 Fuzzy threshold graph

Multi-processor scheduling, bin packing, and the knapsack problem are the different variations of set-packing problems and are being very well studied problem in combinatorial optimization. These problems have large impact on design and analysis of fuzzy threshold graph. All of these problems involve packing items of different sizes into bins of finite capacities. Consider a parallel system consists of a set of independent processing units each of which has a set of time-sharable resources such as CPU, one or more disks, network controllers, etc. Here all units have variable capacities as well as resources. Fuzzy threshold graph is defined as follows.

Definition 1.3.11. (Fuzzy threshold graph) *[109] A fuzzy graph $G = (V, \sigma, \mu)$ is called a fuzzy threshold graph if there exists a non-negative real number t such that $\sum_{u \in U} \sigma(u) \le t$ if and only if $U \subseteq V$ is an independent set in G.*

1.3.4 Fuzzy tolerance graph

Fuzzy tolerance of a fuzzy interval is denoted by $\mathcal{T}$ and is defined by an arbitrary fuzzy interval whose core length is a positive real number. If the real number is taken as L and $|i_k - i_{k-1}| = L$ where $i_k, i_{k-1} \in R$, a set of real numbers, then the fuzzy tolerance is a fuzzy set of the interval $[i_{k-1}, i_k]$.

Fuzzy tolerance graph is defined by Samanta and Pal in [108]. They defined the fuzzy tolerance graph $\mathcal{G} = (V, \sigma, \mu)$ as the fuzzy intersection graph of finite family of fuzzy intervals $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\}$ on the real line along with tolerances $\mathcal{T} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n\}$ associated to each vertex of $v_i \in V$, where, $\sigma : V \to [0, 1]$ is defined by $\sigma(v_i) = h(\mathcal{I}_i) = 1$ for all $v_i \in V$ and $\mu : V \times V \to [0, 1]$ is defined by

$$\mu(v_i, v_j) = \begin{cases} 1, & \text{if } c(\mathcal{I}_i \cap \mathcal{I}_j) \ge \min\{c(\mathcal{T}_i), c(\mathcal{T}_j)\} \\ \frac{s(\mathcal{I}_i \cap \mathcal{I}_j) - \min\{s(\mathcal{T}_i), s(\mathcal{T}_j)\}}{s(\mathcal{I}_i \cap \mathcal{I}_j)} h(\mathcal{I}_i \cap \mathcal{I}_j), & \text{else if } s(\mathcal{I}_i \cap \mathcal{I}_j) \ge \\ & \min\{s(\mathcal{T}_i), s(\mathcal{T}_j)\} \\ 0, & \text{otherwise.} \end{cases}$$

1.3.5 Fuzzy planar graph

Day by day, the necessity of flyovers, subway tunnels, pipelines, metro lines increases due to demand in human kind. Number of crossing of routes increases the chance of accident. The cost of crossing of subways in underground is also high. But, the underground routes reduce the traffic jam. The system of routes without crossing is ideal for a city. But, lack of space and money often requires crossings of routes. It is true that, two congested crossing of routes is more safer than a congested and non-congested road crossing. The term "congested" has no specific meaning and measurement. To understand the exact load of a route we generally use the terms for routes like "congested", "very congested", "highly congested" routes, etc. These linguistic terms can be dealt in mathematics by giving some positive membership values and negative membership values in fuzzy sense. In mathematical sense, strong route means highly congested route and weak route means low congested route. Thus crossing between a strong route and a weak route is better than the crossing between two strong routes. That is, in city planning, crossing between strong routes and weak routes are allowed. The terms "strong route" and "weak route" lead strong edge and weak edge of a bipolar fuzzy graph respectively. And the approval of using the crossing between strong and weak edges lead to the concept of bipolar fuzzy planar graph. Abdul-jabbar et al. [1] and Nirmala and Dhanabal [94] introduced the concept of fuzzy planar graph. Recently, Samanta and Pal [114, 115] introduced fuzzy planar graph in a different way where crossing of edges are allowed and studied different properties of it.

1.3.6 Fuzzy competition graph

Fuzzy competition graph is a generalization of competition graph. This graph is related to fuzzy digraph. Fuzzy k-competition graph and m-step competition graph are the variations of fuzzy competition graph. Before defining fuzzy competition graph, we define some related terms.

Fuzzy *out-neighbourhood* [112] of a vertex $v \in V$ of a directed fuzzy graph $\overrightarrow{\mathcal{D}} = (V, \sigma, \nu)$ is the fuzzy set $\mathcal{N}^+(v) = (X_v^+, m_v^+)$, where $X_v^+ = \{u|\nu(\overrightarrow{v,u}) > 0\}$ and $m_v^+ : X_v^+ \to [0,1]$ is defined by $m_v^+ = \nu(\overrightarrow{v,u})$.

Fuzzy *in-neighbourhood* [112] of a vertex $v \in V$ of a directed fuzzy graph $\overrightarrow{\mathcal{D}} = (V, \sigma, \nu)$ is the fuzzy set $\mathcal{N}^-(v) = (X_v^-, m_v^-)$, where $X_v^- = \{u|\nu(\overrightarrow{u,v}) > 0\}$ and $m_v^- : X_v^- \to [0,1]$ is defined by $m_v^- = \nu(\overrightarrow{u,v})$.

Fuzzy *neighbourhood* [112] of a vertex $v \in V$ of a fuzzy graph $\mathcal{G} = (V, \sigma, \mu)$ is the fuzzy set $\mathcal{N}(v) = (X_v, m_v)$, where $X_v = \{u|\mu(u,v) > 0\}$ and $m_v : X_v \to [0,1]$ is defined by $m_v = \mu(u,v)$.

The *m-step fuzzy out-neighbourhood* [115] of a vertex $v \in V$ of a directed fuzzy graph $\overrightarrow{\mathcal{D}} = (V, \sigma, \nu)$ is the fuzzy set $\mathcal{N}_m^+(v) = (X_v^+, m_v^+)$, where $X_v^+ = \{u|\overrightarrow{\mu_m}(\overrightarrow{v,u}) = \min\{\nu(\overrightarrow{v,u_1}),\ \nu(\overrightarrow{u_1,u_2}),\ \ldots, \nu(\overrightarrow{u_m,u})\} > 0,\ vu_1u_2\ldots u_mu$ is a path from v to $u\}$ and $m_v^+ : X_v^+ \to [0,1]$ is defined by $m_v^+ = \overrightarrow{\mu_m}(\overrightarrow{v,u})$. If there is more than one fuzzy path of length m then we should take the path which has minimum membership value $\overrightarrow{\mu_m}(\overrightarrow{v,u})$.

The definition of fuzzy competition graph is as follows.

Definition 1.3.12. (Fuzzy competition graph) *[112] The fuzzy competition graph of a fuzzy digraph $\overrightarrow{\mathcal{D}} = (V, \sigma, \nu)$ is an undirected graph $\mathcal{C}(\overrightarrow{\mathcal{D}}) = (V, \sigma, \mu)$ which has the same fuzzy vertex set as in $\overrightarrow{\mathcal{D}}$ and has a fuzzy edge between two vertices $u, v \in V$ in $\mathcal{C}(\overrightarrow{\mathcal{D}})$ if and only if $\mathcal{N}^+(u) \cap \mathcal{N}^+(v)$ is non-empty fuzzy set in $\overrightarrow{\mathcal{D}}$. The membership value of the edge (u, v) in $\mathcal{C}(\overrightarrow{\mathcal{D}})$ is $\mu(u, v) = (\sigma(u) \wedge \sigma(v))h(\mathcal{N}^+(u) \cap \mathcal{N}^+(v))$.*

The *m-step fuzzy competition graph* [115] of a digraph $\overrightarrow{\mathcal{D}} = (V, \sigma, \nu)$ is denoted by $\mathcal{C}_m(\overrightarrow{\mathcal{D}})$ and is defined by $\mathcal{C}_m(\overrightarrow{\mathcal{D}}) = (V, \sigma, \mu)$ where $\mu(u, v) = (\sigma(u) \wedge \sigma(v))h(\mathcal{N}_m^+(u) \cap \mathcal{N}_m^+(v))$ for all $u, v \in V$.

There is another variation of fuzzy competition graph, called p-competition fuzzy graph which is defined below.

Definition 1.3.13. (p-competition fuzzy graph) *[112] Let p be a positive integer. The p-competition fuzzy graph $C^p(\overrightarrow{\xi}) = (V, \sigma, \nu)$ of a fuzzy digraph $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ is an undirected fuzzy graph which has the same fuzzy vertex set as in $\overrightarrow{\xi}$ and has a fuzzy edge between two vertices x and y of V in $C^p(\overrightarrow{\xi})$ if and only if $|\operatorname{supp}(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))| \geq p$. The edge membership value of the edge (x, y) in $C^p(\overrightarrow{\xi})$ is $\nu(x, y) = \dfrac{n - p + 1}{n}[\sigma(x) \wedge \sigma(y)]h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$ where $n = |\operatorname{supp}(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))|$.*

Therefore, fuzzy p-competition graphs are graphs with edges between the vertices, if the vertices have exactly p number of common neighbourhoods. On the other hand, there is another variation of competition graph known as fuzzy k-competition graph [112], where edges between two vertices exists if the minimum membership value of the common out-neighbourhoods of the vertices is more than positive real number k. Formal definition is given below.

Definition 1.3.14. (Fuzzy k-competition graph) *[112] Let k be a non-negative number. The fuzzy k-competition graph $C_k(\overrightarrow{G}) = (V, \sigma, \nu)$ of a fuzzy digraph $\overrightarrow{G} = (V, \sigma, \mu)$ is an undirected fuzzy graph which has the same vertex set as in $\overrightarrow{G}$ and has a fuzzy edge between two vertices $x, y \in V$ in $C_k(\overrightarrow{G})$ if and only if $|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)| > k$. The edge membership value between x and y in $C_k(\overrightarrow{G})$ is $\nu(x, y) = \dfrac{k' - k}{k'}[\sigma(x) \wedge \sigma(y)]h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$ where $k' = |\mathcal{N}^+(x) \cap \mathcal{N}^+(y)|$.*

1.3.7 Interval-valued fuzzy graph

An **interval number** [3] D is an interval $[a^-, a^+]$ with $0 \leq a^- \leq a^+ \leq 1$. For two interval numbers $D_1 = [a_1^-, a_1^+]$ and $D_2 = [a_2^-, a_2^+]$, we have the following:

(i) $D_1 + D_2 = [a_1^-, a_1^+] + [a_2^-, a_2^+] = [a_1^- + a_2^- - a_1^- \cdot a_2^-, a_1^+ + a_2^+ - a_1^+ \cdot a_2^+]$,

(ii) $\min\{D_1, D_2\} = [\min\{a_1^-, a_2^-\}, \min\{a_1^+, a_2^+\}]$,

(iii) $\max\{D_1, D_2\} = [\max\{a_1^-, a_2^-\}, \max\{a_1^+, a_2^+\}]$,

(iv) $D_1 \leq D_2 \Leftrightarrow a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$,

(v) $D_1 = D_2 \Leftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,

(vi) $D_1 < D_2 \Leftrightarrow D_1 \leq D_2$ but $D_1 \neq D_2$,

(vii) $kD_1 = [ka_1^-, ka_2^+]$, where $0 \leq k \leq 1$.

An interval-valued fuzzy set A on a set X is a mapping $\mu_A : X \to [0,1] \times [0,1]$, called the membership function, i.e. $\mu_A(x) = [\mu_A^-(x), \mu_A^+(x)]$. The support of A is $\operatorname{supp}(A) = \{x \in X | \mu_A^-(x) \neq 0\}$ and the core of A is $\operatorname{core}(A) = \{x \in X | \mu_A^-(x) = 1\}$. The support length is $s(A) = |\operatorname{supp}(A)|$ and the core length is $c(A) = |\operatorname{core}(A)|$. The height of A is $h(A) = \max\{\mu_A(x) | x \in X\} = [h^-(A), h^+(A)] = [\max\{\mu_A^-(x)\}, \max\{\mu_A^+(x)\}], \forall x \in X$.

Let $F = \{A_1, A_2, \cdots, A_n\}$ be a finite family of interval-valued fuzzy subsets on a set X. The intersection of two interval-valued fuzzy sets (IVFSs) A_1 and A_2 is an interval-valued fuzzy set defined by

$$A_1 \cap A_2 = \left\{ \left(x, \left[\min\{\mu_{A_1}^-(x), \mu_{A_2}^-(x)\}, \min\{\mu_{A_1}^+(x), \mu_{A_2}^+(x)\}\right]\right) : x \in X \right\}.$$

The union of two IVFSs A_1 and A_2 is a IVFS defined by

$$A_1 \cup A_2 = \left\{ \left(x, \left[\max\{\mu_{A_1}^-(x), \mu_{A_2}^-(x)\}, \max\{\mu_{A_1}^+(x), \mu_{A_2}^+(x)\}\right]\right) : x \in X \right\}.$$

An interval-valued fuzzy relation B on a set X is denoted as $\mu_B : X \times X \to [0,1] \times [0,1]$ such that

$$\mu_B^-(x,y) \leq \min\{\mu_A^-(x), \mu_A^-(y)\}$$
$$\mu_B^+(x,y) \leq \min\{\mu_A^+(x), \mu_A^+(y)\}$$

An *interval-valued fuzzy graph* [4] of a crisp graph $G^* = (V, E)$ is a graph $G = (V, A, B)$, where $A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set on V and $B = [\mu_B^-, \mu_B^+]$ is an interval-valued fuzzy relation on E. An edge (x, y), $x, y \in V$ in an interval-valued fuzzy graph is said to be independent strong if $\mu_B^-(x,y) \geq \frac{1}{2}\min\{\mu_A^-(x), \mu_A^-(y)\}$. An interval-valued fuzzy digraph $\overrightarrow{G} = (V, A, \overrightarrow{B})$ is an interval-valued fuzzy graph where the fuzzy relation $\overrightarrow{B}$ is antisymmetric.

An interval-valued fuzzy graph $G = (V, A, B)$ is said to be *complete interval-valued fuzzy graph* if $\mu^-(x,y) = \min\{\sigma^-(x), \sigma^-(y)\}$ and $\mu^+(x,y) = \min\{\sigma^+(x), \sigma^+(y)\}$, $\forall x, y \in V$.

1.3.8 Intuitionistic fuzzy graph

An *intuitionistic fuzzy set* [13, 14] A on the set X is characterized by a mapping $m : X \to [0,1]$, which is called as a membership function and $n : X \to [0,1]$, which is called as a non-membership function. An intuitionistic fuzzy set is denoted by $A = (X, m_A, n_A)$. The membership function of the intersection of two intuitionistic fuzzy sets $A = (X, m_A, n_A)$ and $B = (X, m_B, n_B)$ is defined as $m_{A\cap B} = min\{m_A, m_B\}$ and the non-membership function $n_{A\cap B} = max\{n_A, n_B\}$. We write $A = (X, m_A, n_A) \subseteq B = (X, m_B, n_B)$ if and only if $m_A(x) \leq m_B(x)$ and $n_A(x) \geq n_B(x)$ for all $x \in X$. Parvathi and Karunambigai [97] defined the intuitionistic fuzzy graph as below.

Definition 1.3.15. *[97] An intuitionistic fuzzy graph is of the form $G = (V, \sigma, \mu)$ where $\sigma = (\sigma_1, \sigma_2)$, $\mu = (\mu_1, \mu_2)$ and $V = \{v_0, v_1, \ldots, v_n\}$ such that*

(i) $\sigma_1 : V \to [0,1]$ *and* $\sigma_2 : V \to [0,1]$, *denote the degree of membership and non-membership functions of the vertex set* V *respectively and* $0 \leq \sigma_1(v_i) + \sigma_2(v_i) \leq 1$ *for every* $v_i \in V$ $(i = 1, 2, \ldots, n)$,

(ii) $\mu_1 : V \times V \to [0,1]$ *and* $\mu_2 : V \times V \to [0,1]$, *where* $\mu_1(v_i, v_j)$ *and* $\mu_2(v_i, v_j)$ *denote the the degree of membership and non-membership value of the edge* (v_i, v_j) *respectively such that* $\mu_1(v_i, v_j) \leq min\{\sigma_1(v_i), \sigma_1(v_j)\}$ *and* $\mu_2(v_i, v_j) \geq max\{\sigma_2(v_i), \sigma_2(v_j)\}$, $0 \leq \mu_1(v_i, v_j) + \mu_2(v_i, v_j) \leq 1$ *for every* $(v_i, v_j) \in V \times V$.

1.3.9 Bipolar fuzzy graph

In 1994, the concept of bipolar fuzzy set is introduced by Zhang [143] as a generalization of fuzzy set. A bipolar fuzzy set is a generalization of Zadeh's fuzzy set. The range of the membership value of a bipolar fuzzy set is $[-1, 1]$. In a bipolar fuzzy set, the membership value 0 of an element means that the element is not connected with the corresponding property, the membership value within $(0, 1]$ of an element implies that the element satisfies the property with certain negotiations (higher the value indicates that there is lower amount of negotiations), and the negative membership value within $[-1, 0)$ of an element means that the element satisfies the implicit counter-property to some extent.

Let X be a non-empty set. A *bipolar fuzzy set* [142, 143] B in X is characterized by $B = \{(x, \mu_B^P(x), \mu_B^N(x))|x \in X\}$, where $\mu_B^P : X \to [0, 1]$ and $\mu_B^N : X \to [-1, 0]$ are positive membership function and negative membership function respectively. The positive membership value $\mu_B^P(x)$ is used to denote the amount which the element x satisfies the property corresponding to a bipolar fuzzy set B, and the negative membership value $\mu_B^N(x)$ denotes the amount which the element x satisfies the implicit counter-property to some extent corresponding to a bipolar fuzzy set B.

For every two bipolar fuzzy sets $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ on X,

$(A \cap B)(x) = (min(\mu_A^P(x), \mu_B^P(x)), max(\mu_A^N(x), \mu_B^N(x)))$.

$(A \cup B)(x) = (max(\mu_A^P(x), \mu_B^P(x)), min(\mu_A^N(x), \mu_B^N(x)))$.

Akram [3, 5, 6] introduced bipolar fuzzy graphs, regular bipolar fuzzy graphs and investigated some properties of it. Later on, Yang et al. [136] modified their definition of bipolar fuzzy graphs and introduced generalized bipolar fuzzy graphs. The definition is given as follows.

Definition 1.3.16. (Generalized bipolar fuzzy graph) *[136] A bipolar fuzzy graph of a graph $G^* = (V, E)$ is a pair $G = (V, A, B)$ where $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set in V and $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy relation on $\widetilde{V^2}$ such that $\mu_B^P(xy) \leq min\{\mu_A^P(x), \mu_A^P(y)\}$, $\mu_B^N(xy) \geq max\{\mu_A^N(x), \mu_A^N(y)\}$ for all $xy \in \widetilde{V^2}$ and $\mu_B^P(xy) = \mu_B^N(xy) = 0$ for all $xy \in (\widetilde{V^2} - E)$.*

1.3.10 m-polar fuzzy graph

In 2014, Chen et al. [36] introduced the notion of m-polar fuzzy set as a generalization of bipolar fuzzy set and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions and that we can obtain concisely one from the corresponding one. The idea behind this is that "multipolar information" (not just bipolar information which correspond to two-valued logic) exists because data of real world problems are sometimes come from multiple agents. For example, the exact degree of telecommunication safety of mankind is a point in $[0, 1]^n$ ($n \approx 7 \times 10^9$) because different persons have been monitored different times. There are many other examples such as truth degrees of a logic formula which are based on n logic implication operators ($n \geq 2$), similarity degrees of two logic formulas which are based on n logic implication operators ($n \geq 2$), ordering results of a magazine, ordering results of a university, and inclusion degrees (accuracy measures, rough measures, approximation qualities, fuzziness measures, and decision preformation evaluations) of a rough set.

Here $[0,1]^m$ (m-power of [0,1]) is considered to be a poset with point-wise order $\leq$, where m is a natural number. $\leq$ is defined by $x \leq y \Leftrightarrow$ for each $i = 1, 2, \ldots, m$; $p_i(x) \leq p_i(y)$ where $x, y \in [0,1]^m$ and $p_i : [0,1]^m \to [0,1]$ is the i-th projection mapping.

Definition 1.3.17. (m-polar fuzzy set) *[36] An m-polar fuzzy set (or a $[0,1]^m$-set) on X is a mapping $A : X \to [0,1]^m$. The set of all m-polar fuzzy sets on X is denoted by $m(X)$.*

Definition 1.3.18. (Operations) *[45] Let A and B are two m-polar fuzzy sets in X. Then $A \cup B$ and $A \cap B$ are also m-polar fuzzy sets in X defined by: for $i = 1, 2, \ldots, m$ and $x \in X$,*

$p_i \circ (A \cup B)(x) = max\{p_i \circ A(x), p_i \circ B(x)\}$,

$p_i \circ (A \cap B)(x) = min\{p_i \circ A(x), p_i \circ B(x)\}$,

$A \subseteq B$ *if and only if* $p_i \circ A(x) \leq p_i \circ B(x)$ *and*

$A = B$ *if and only if* $p_i \circ A(x) = p_i \circ B(x)$.

Definition 1.3.19. (m-polar fuzzy relation) *[45] Let A be an m-polar fuzzy set on a set X. An m-polar fuzzy relation on A is an m-polar fuzzy set B of $X \times X$ such that $B(x,y) \leq min\{A(x), A(y)\}$ for all $x, y \in X$, i.e. $p_i \circ B(x,y) \leq min\{p_i \circ A(x), p_i \circ A(y)\}$ for all $x, y \in X$, $i = 1, 2, \ldots, m$. An m-polar fuzzy relation B on X is called symmetric if $B(x,y) = B(y,x)$ for all $x, y \in X$.*

Chen et al. [36] defined m-polar fuzzy graph in the following way:

Definition 1.3.20. (m-polar fuzzy graph) *[36] An m-polar fuzzy graph with an underlying pair (V, E) (where $E \subseteq V \times V$ is symmetric) is defined to be a pair $G = (A, B)$, where $A : V \to [0,1]^m$ and $B : E \to [0,1]^m$ satisfying $B(xy) \leq min\{A(x), A(y)\}$ for all $xy \in E$.*

1.4 Review of literature

After introduction of fuzzy graphs, several researches have been done. McAllister [76] characterised the fuzzy intersection graphs. After that Craine [40] characterized fuzzy interval graphs. Then, Goetschel [51] introduced fuzzy hypergraphs as an extension of crisp hypergraphs. He also described another important branch of fuzzy hypergraph theory in his paper "Fuzzy colorings of fuzzy hypergraphs" [52]. In another paper, Goetschel and Voxman introduced the intersection in fuzzy hypergraphs [53]. Somasundaram et al. [123] discussed domination in fuzzy graphs. Mordeson and Nair [81] defined successor and source of (fuzzy) finite state machines and (fuzzy) directed graphs. Mordeson and Nair [83] has given the details of fuzzy graphs and hypergraphs. After that, fuzzy line graphs, the operations on fuzzy graphs and cycles and cocyles of fuzzy graphs was introduced by Mordeson and Peng [78–80]. Nair et al. [89–91] introduced triangle and parallelogram laws on fuzzy graphs, cliques and fuzzy cliques in fuzzy graphs, perfect and precisely perfect fuzzy graphs. Bhutani and Battou [22] described M- strong fuzzy graphs. Bhutani and Rosenfeld [20] introduced strong arcs in fuzzy graphs. Mathew and Sunitha [73,75] defined different types of arcs in fuzzy graphs and studied Mengers theorem for fuzzy graphs. In another paper [74], they analyzed node connectivity and arc connectivity of a fuzzy graph. Bhutani et al. [24] presented some results on degrees of end nodes and cut nodes in fuzzy graphs. Eslahchi and Onaghe [42] introduced vertex strength of fuzzy graphs. They also defined strong fuzzy edges in fuzzy graphs. After that, Nagoorgani and Radha [85, 88] introduced regular and irregular fuzzy graphs. Nagoorgani et al. [86, 87] also studied fuzzy effective distance k-dominating sets and isomorphism properties of strong fuzzy graphs. Jabbar et al. [1] introduced fuzzy dual graph. Nirmala and Dhanabal [94] introduced special planar fuzzy graph configurations. To put an emphasis on real problem, Samanta and Pal [114, 115] studied fuzzy planar graph in a different way.

There are several variations of competition graphs in Cohen's literature [41]. After Cohen, some derivations of competition graphs have been found. Such as, Cho et al. [37] introduced the m-step competition graph of a digraph. The p-competition graph of a digraph has been defined by Kim et al. [60]. Brigham et al. [28] introduced the tolerance competition graphs. The competition hypergraphs have been found in Sonnatag et al. [124]. A recent work on fuzzy k-competition graphs and p-competition fuzzy graphs is available in [112].

Tolerance graphs [54] are generalization of interval graphs in which each vertex can be represented by an interval and a tolerance such that an edge occurs if and only if the overlap of corresponding intervals is at least as large as the tolerance associated with one of the vertices. The original motivation of the paper was to solve the scheduling problems. After that, ϕ-tolerance competition graph was introduced by [28] as the generalization of p-competition graph.

Nayeem and Pal [93] have worked on shortest path problem on a network with imprecise edge weight. Surveys of the large literature related to competition graph can be found in [38]. To find shortest path in a complex network is a very emerging work in this modern edge. There are various techniques to find shortest paths in a network. The bipolar fuzzy hypergraph is a hypergraph in which each vertex and edge are assigned bipolar fuzzy sets. Samanta and Pal [110] have introduced the bipolar fuzzy hypergraphs which has emerge importance in complex networking systems.

Rosenfeld [102] introduced the concept of μ-distance in fuzzy graphs. Concepts of eccentricity and centre in fuzzy graphs are introduced by Bhattacharya [18] using μ-distance. Sameena and Sunitha [117, 118] have further studied on the g-distance of fuzzy graphs. Automorphism, fuzzy end nodes, geodesics in fuzzy graphs are studied by Bhutani et al. [19, 21, 23]. The g-eccentric nodes, g-boundary nodes and g-interior nodes of a fuzzy graph are introduced by Linda and Sunitha [70].

Bershtein et al. [17] defined the cliques fuzzy set. Then, cliques and clique covers in fuzzy graphs is introduced by Sun et al. [125]. Another variation of clique cover is edge clique cover which is studied by Javadi and Hajebi [56]. In most research work of clique cover, the main task is to find the clique cover number.

Chvatal and Hammer [39] first introduced the threshold graph. In 1979, Manca [72] has derived an efficient matrix method for testing a given graph to see whether or not it is a threshold graph. There is a great introduction to threshold graphs and their applications in [98]. Due to the importance of fuzzy graphs, Samanta and Pal have introduced the fuzzy threshold graphs in [109].

The reader may found the works on various extensions of fuzzy graphs in [29–34, 103–106]. For further studies on fuzzy graphs and its variations the literatures [7–9, 11, 17, 25, 43, 50, 111, 113–116] may be very helpful.

1.5 Motivation of the work

In many real world problems, sometimes data come from n agents ($n \geq 2$), i.e. "multipolar information" exists. These information can not be represented well by means of fuzzy graphs or bipolar fuzzy graphs. Therefore m-polar fuzzy set is applied to graphs to describe the relationships among several individuals. In this direction Chen et al. [36] first defined m-polar fuzzy graph. We then introduced generalized m-polar fuzzy graphs in Chapter 2. Some operations have been defined to formulate these graphs. Some properties of strong m-polar fuzzy graphs, self complementary m-polar fuzzy graphs and self complementary strong m-polar fuzzy graphs are discussed.

In Chapter 3, we have defined three new operations on m-polar fuzzy graph such as direct product, semi-strong product and strong product. It is proved that any of the products of m-polar fuzzy graphs are again an m-polar fuzzy graph. Sufficient conditions are established for each one of them to be strong and also proved that strong product of two complete m-polar fuzzy graphs is complete. If any of the products of two m-polar fuzzy graphs G_1 and G_2 are strong, then it is shown that at least G_1 or G_2 must be strong. The degree of a vertex in m-polar fuzzy graphs which are obtained from two given m-polar fuzzy graphs G_1 and G_2 using the operations of Cartesian product, composition, direct product, semi-strong product and strong product. At the end of this chapter, 3-polar fuzzy influence graph is introduced as an application.

In Chapter 4, density of m-polar fuzzy graphs is defined and then introduced the notion of balanced m-polar fuzzy graphs. Some characterizations of balanced m-polar fuzzy graphs are given.

There are many real world applications like design problems for circuits, subways, utility lines with a graph structure in which crossing between edges is a nuisance. This is not a big problem for electrical wires but it creates extra expenses for some types of lines, i.e. burying one subway tunnel under another. These applications are designed using the concept of planar graphs. In a city planning, subway tunnels, pipelines, metro lines, etc. are all essential. There are chances of accident due to crossing. Routes without crossing are preferable, but due to the lack of space crossing of such lines are allowed. Crossing between congested and non-congested routes are more preferable than the crossing between two congested routes. The term "congested" has no definite meaning. We generally use "congested", "very congested", "highly congested" routes, etc. These terms are called linguistic terms and they have some membership values. A congested route may be termed as strong route and low congested route may be termed as weak route. Thus, crossing between strong and weak route may be allowed in a city planning with certain amount of safety. The terms "strong route" and "weak route" lead to strong edge and weak edge of an m-polar fuzzy graph respectively and the permission of crossing between strong and weak edges leads to the concept of m-polar fuzzy planar graphs. The m-fuzzy planar graph is introduced in Chapter 5. m-polar fuzzy dual graphs and some subclasses of m-polar fuzzy planar graph are also introduced here. Besides, some relations between these graphs are established.

Self complement m-polar fuzzy graphs have many important significance in the theory of m-polar fuzzy graphs. If an m-polar fuzzy graph is not self complement then also we can say that it is self complement in some weaker sense. Simultaneously, we can establish some results with this graph. This motivates us to define weak self complement m-polar fuzzy graphs in Chapter 6. A necessary condition is mentioned for an m-polar fuzzy graph to be weak self complement. Some properties of self complement and weak self complement m-polar fuzzy graphs are discussed. The order, size, busy vertices and free vertices of an m-polar fuzzy graphs are also defined and proved that isomorphic m-polar fuzzy graphs have same order, size and degree. Also, we have proved some results of busy vertices in isomorphic and weak isomorphic m-polar fuzzy graphs. A relative study of complement and operations on m-polar fuzzy graphs have been made. Some real life problems have been modeled using the concepts of m-polar fuzzy graphs.

In Chapter 7, the concept of edge regular, strongly regular and biregular m-polar fuzzy graph are introduced. Some properties of them are studied. Also, the concept of partially edge regular m-polar fuzzy graph and fully edge regular m-polar fuzzy graph are introduced with suitable illustrations. The notion of strongly edge irregular and strongly edge totally irregular m-polar fuzzy graphs. Some properties of them are also studied to characterize strongly edge irregular and strongly edge totally irregular m-polar fuzzy graphs.

An m-polar fuzzy model is useful for multi-polar information, multi-agent, multi-attribute and multi-object network models which gives more precision, flexibility, and comparability to the system as compared to the classical, fuzzy and bipolar fuzzy models. In Chapter 8, we used m-polar fuzzy sets to introduce the notion of m-polar ψ-morphism on m-polar fuzzy graphs. The action of m-polar ψ-morphism on m-polar fuzzy graphs is studied and we established some results on weak and co-weak isomorphism. d_2-degree and total d_2-degree of a vertex in m-polar fuzzy graphs are defined and studied $(2, \overline{k})$-regularity and totally $(2, \overline{k})$-regularity. A real life situation of a company has been modeled in terms of 4-polar fuzzy graphs as an application.

In Chapter 9, we introduced generalized regular bipolar fuzzy graphs and investigated some its properties. Then we define a product bipolar fuzzy intersection graph of a product bipolar fuzzy graph and the product bipolar fuzzy line graphs. Some characterizations of product bipolar fuzzy line graphs are also made.

Chapter 10 is devoted to the conclusion of the thesis followed by the bibliography.

1.6 Summary

This chapter introduces and discusses some preliminary notions used in the rest of the Thesis. Several types of graphs and fuzzy graphs are discussed. Some fuzzy set theoretic definitions and notations are also focussed. Motivation of the work and survey of related works of the thesis are discussed in this chapter.

Chapter 2

Fundamentals of m-polar fuzzy graphs*

2.1 Introduction

In 2014, Chen et al. [36] introduced the notion of m-polar fuzzy set as a generalization of bipolar fuzzy set. The idea behind this is that "multipolar information" (not just bipolar information which correspond to two-valued logic) exists because data of real world problems are sometimes come from multiple agents. For example, the exact degree of telecommunication safety of mankind is a point in $[0, 1]^n$ ($n \approx 7 \times 10^9$) because different persons have been monitored different times. There are many other examples such as truth degrees of a logic formula which are based on n logic implication operators ($n \geq 2$), similarity degrees of two logic formulas which are based on n logic implication operators ($n \geq 2$), ordering results of a magazine, ordering results of a university, and inclusion degrees (accuracy measures, rough measures, approximation qualities, fuzziness measures, and decision preformation evaluations) of a rough set. An m- polar fuzzy model is useful for multi-polar information, multi-agent, multi-attribute and multi-object network models which gives more precision, flexibility, and comparability to the system as compared to the classical, fuzzy and bipolar fuzzy models. Chen et al. [36] first defined m-polar fuzzy graphs. In this chapter, we modified their definition and introduced generalized m-polar fuzzy graph. Cartesian product, composition, union and join of two m-polar fuzzy graphs are defined. Some important

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properties of isomorphisms, strong m-polar fuzzy graphs, self complementary m-polar fuzzy graphs and self complementary strong m-polar fuzzy graphs are discussed.

2.2 Generalized m-polar fuzzy graphs

Chen et al. [36] defined the m-polar fuzzy graph in the following way:

An m-polar fuzzy graph with an underlying pair (V, E) (where $E \subseteq V \times V$ is symmetric) is defined to be a pair $G = (A, B)$ where $A : V \to [0, 1]^m$ and $B : E \to [0, 1]^m$ satisfying $B(xy) \leq min\{A(x), A(y)\}$ for all $xy \in E$.

According to the above definition, B is actually an m-polar fuzzy set in $E \subseteq V \times V$. However, when the definition is used, B is actually an m-polar fuzzy set defined in $\widetilde{V^2}$ satisfying $B(xy) = \mathbf{0} = (0, 0, \ldots, 0)$ for all $xy \in (\widetilde{V^2} - E)$. The above definition will cause problems in calculating the complement of an m-polar fuzzy graphs. Therefore, a generalized m-polar fuzzy graphs is defined below.

Before defining generalized m-polar fuzzy graph, we assume the following:

For a given set V, define an equivalence relation $\sim$ on $V \times V - \{(x, x) : x \in V\}$ as follows: $(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow$ either $(x_1, y_1) = (x_2, y_2)$ or $x_1 = y_2$ and $y_1 = x_2$. The quotient set obtained in this way is denoted by $\widetilde{V^2}$ and the equivalence class that contains the element (x, y) is denoted as xy or yx.

Throughout the chapter, $G^* = (V, E)$ represents a crisp graph and $G = (V, A, B)$ is an m-polar fuzzy graph of G^*.

Definition 2.2.1. *An m-polar fuzzy graph (or generalized m-polar fuzzy graph) of* $G^* = (V, E)$ *is a pair* $G = (V, A, B)$ *where* $A : V \to [0, 1]^m$ *is an m-polar fuzzy set in* V *and* $B : \widetilde{V^2} \to [0, 1]^m$ *is an m-polar fuzzy set in* $\widetilde{V^2}$ *such that* $p_i \circ B(xy) \leq min\{p_i \circ A(x), p_i \circ A(y)\}$ *for all* $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$ *and* $B(xy) = \mathbf{0}$ *for all* $xy \in (\widetilde{V^2} - E)$, $\big(\mathbf{0} = (0, 0, \ldots, 0)$ *is the smallest element in* $[0, 1]^m\big)$.

Here, $p_i \circ A(x)$ *denotes the ith degree of membership of the vertex* x *and* $p_i \circ B(xy)$ *denotes the ith degree of membership of the edge* xy*.* A *is called the m-polar fuzzy vertex set of* G *and* B *as the m-polar fuzzy edge set of* G*.*

Example 2.2.1. *Let* $X = \{F_1, F_2, F_3, F_4\}$ *and* $M = \{M_1, M_2, M_3\}$ *be the set of four friends and three movies respectively. Suppose they planed to watch movie. This situation can be represented as a 4-polar fuzzy graph* G *by considering the vertex set* *as M and the edge set as $M \times M$. Let A be a 4-polar fuzzy set of M. The membership value of M_i represents the preference degrees of the movie M_i corresponding to the friends. Suppose $A(M_1) =< 0.9, 0.4, 0.6, 0.1 >$, $A(M_2) =< 0.5, 0.3, 0.8, 0.1 >$, $A(M_3) =< 0.8, 0.9, 0.8, 0.2 >$. This means the preference degrees of M_1 corresponding to F_1, F_2, F_3 and F_4 are 0.9, 0.4, 0.6 and 0.1 respectively. Similarly, for the others. An edge between any two nodes represents the degrees of common features (i.e., love story, comedy, fighting, horror) of the nodes. Let $B(M_1M_2) =< 0.4, 0.2, 0.2, 0.1 >$, $B(M_2M_3) =< 0.4, 0.2, 0.2, 0.2 >$, $B(M_3M_1) =< 0.4, 0.2, 0.3, 0.1 >$. This means the degrees of common features (i.e., love story, comedy, fighting, horror) of the movies M_1 and M_2 are 0.4, 0.2, 0.2 and 0.1. In other words, both the movies M_1 and M_2 are* 40% *love story,* 20% *comedy,* 20% *fighting and* 10% *horror. Similarly for the others. It is easy to verify that G of Fig. 2.1 is a 4-polar fuzzy graph.*

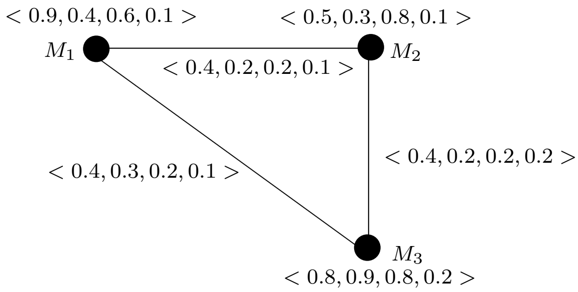

Figure 2.1: Example of 4-polar fuzzy graph G

Here after, we assume an m-polar fuzzy graph to be a generalized m-polar fuzzy graph.

2.3 Cartesian product, composition, union and join on m-polar fuzzy graphs

In this section, four types of operations such as Cartesian product, composition, union and join have been defined on m-polar fuzzy graphs to construct new types of m-polar fuzzy graphs.

Definition 2.3.1. *The Cartesian product $G_1 \times G_2$ of two m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively is defined as a triplet $(V_1 \times V_2, A_1 \times A_2, B_1 \times B_2)$ such that for $i = 1, 2, \ldots, m$*

(i) $p_i \circ (A_1 \times A_2)(x_1, x_2) = min\{p_i \circ A_1(x_1), p_i \circ A_2(x_2)\}$ *for all* $(x_1, x_2) \in V_1 \times V_2$.

(ii) $p_i \circ (B_1 \times B_2)((x, x_2)(x, y_2)) = min\{p_i \circ A_1(x), p_i \circ B_2(x_2y_2)\}$ *for all* $x \in V_1$, $x_2y_2 \in E_2$.

(iii) $p_i \circ (B_1 \times B_2)((x_1, z)(y_1, z)) = min\{p_i \circ B_1(x_1y_1), p_i \circ A_2(z)\}$ *for all* $z \in V_2$, $x_1y_1 \in E_1$.

(iv) $p_i \circ (B_1 \times B_2)((x_1, x_2)(y_1, y_2)) = 0$ *for all* $(x_1, x_2)(y_1, y_2) \in \widetilde{V_1 \times V_2}^2 - E$.

Example 2.3.1. *Let* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$ *be two graphs such that* $V_1 = \{a, b\}, V_2 = \{c, d\}, E_1 = \{ab\}$ *and* $E_2 = \{cd\}$. *Consider the 3-polar fuzzy graphs* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *of the graphs* G_1^* *and* G_2^* *respectively where* $A_1 = \{\frac{<0.3,0.4,0.6>}{a}, \frac{<0.3,0.5,0.7>}{b}\}$, $B_1 = \{\frac{<0.1,0.2,0.5>}{ab}\}$, $A_2 = \{\frac{<0.1,0.4,0.5>}{c}, \frac{<0.2,0.6,0.6>}{d}\}$, $B_2 = \{\frac{<0.1,0.3,0.4>}{cd}\}$. *Then it is easy to verify the following:*

$(B_1 \times B_2)((a, c)(a, d)) =< 0.1, 0.3, 0.4 >$, $(B_1 \times B_2)((a, c)(b, c)) =< 0.1, 0.2, 0.5 >$,
$(B_1 \times B_2)((b, c)(b, d)) =< 0.1, 0.3, 0.4 >$, $(B_1 \times B_2)((a, d)(b, d)) =< 0.1, 0.2, 0.5 >$,
$(B_1 \times B_2)((a, c)(b, d)) =< 0, 0, 0 >$, $(B_1 \times B_2)((b, c)(a, d)) =< 0, 0, 0 >$.

Hence, $G_1 \times G_2$ *is a 3-polar fuzzy graph of* $G_1^* \times G_2^*$ *(see Fig. 2.2).*

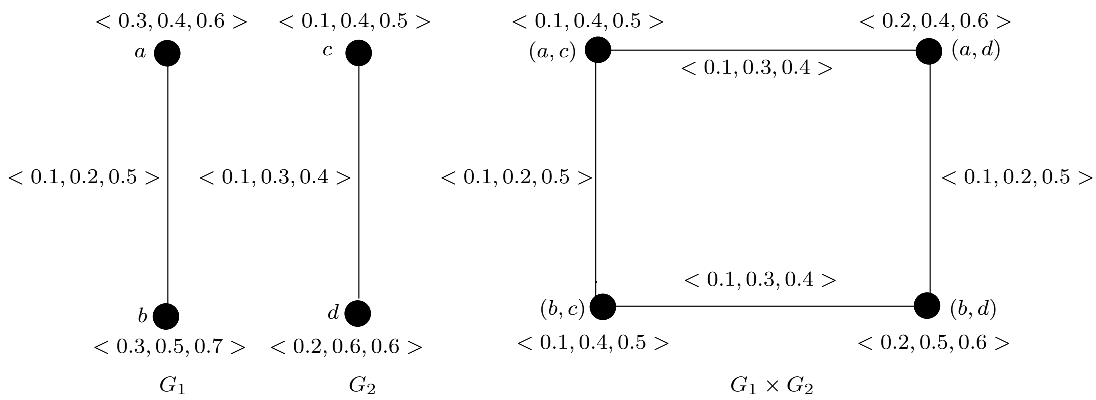

Figure 2.2: Cartesian product of two 3-polar fuzzy graphs G_1 and G_2

Proposition 2.3.1. *The Cartesian product* $G_1 \times G_2 = (V_1 \times V_2, A_1 \times A_2, B_1 \times B_2)$ *of two m-polar fuzzy graphs of the graphs* G_1^* *and* G_2^* *is an m-polar fuzzy graph of* $G_1^* \times G_2^*$.

Proof. Let $x \in V_1$, $x_2y_2 \in E_2$. Then for $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \times B_2)((x, x_2)(x, y_2))$
$= min\{p_i \circ A_1(x), p_i \circ B_2(x_2y_2)\}$
$\leq min\{p_i \circ A_1(x), min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2)\}\}$
$= min\{min\{p_i \circ A_1(x), p_i \circ A_2(x_2)\}, min\{p_i \circ A_1(x), p_i \circ A_2(y_2)\}\}$

$= min\{p_i \circ (A_1 \times A_2)(x, x_2), p_i \circ (A_1 \times A_2)(x, y_2)\}$.

Let $z \in V_2$, $x_1y_1 \in E_1$. Then for $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \times B_2)((x_1, z)(y_1, z))$

$= min\{p_i \circ B_1(x_1y_1), p_i \circ A_2(z)\}$

$\leq min\{min\{p_i \circ A_1(x_1), p_i \circ A_1(y_1)\}, p_i \circ A_2(z)\}\}$

$= min\{min\{p_i \circ A_1(x_1), p_i \circ A_2(z)\}, min\{p_i \circ A_1(y_1), p_i \circ A_2(z)\}\}$

$= min\{p_i \circ (A_1 \times A_2)(x_1, z), p_i \circ (A_1 \times A_2)(y_1, z)\}$.

Let $(x_1, x_2)(y_1, y_2) \in \widetilde{V_1 \times V_2}^2 - E$. Then for $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \times B_2)((x_1, x_2)(y_1, y_2)) = 0 \leq min\{p_i \circ (A_1 \times A_2)(x_1, x_2), p_i \circ (A_1 \times A_2)(y_1, y_2)\}$.

□

Definition 2.3.2. *The composition $G_1[G_2] = (V_1 \times V_2, A_1 \circ A_2, B_1 \circ B_2)$ of two m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively is defined as follows: for $i = 1, 2, \ldots, m$*

(i) $p_i \circ (A_1 \circ A_2)(x_1, x_2) = min\{p_i \circ A_1(x_1), p_i \circ A_2(x_2)\}$ *for all* $(x_1, x_2) \in V_1 \times V_2$.

(ii) $p_i \circ (B_1 \circ B_2)((x, x_2)(x, y_2)) = min\{p_i \circ A_1(x), p_i \circ B_2(x_2y_2)\}$ *for all* $x \in V_1$, $x_2y_2 \in E_2$.

(iii) $p_i \circ (B_1 \circ B_2)((x_1, z)(y_1, z)) = min\{p_i \circ B_1(x_1y_1), p_i \circ A_2(z)\}$ *for all* $z \in V_2$, $x_1y_1 \in E_1$.

(iv) $p_i \circ (B_1 \circ B_2)((x_1, x_2)(y_1, y_2)) = min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2), p_i \circ B_1(x_1y_1)\}$ *for all* $(x_1, x_2)(y_1, y_2) \in E^0 - E$, *where* $E = \{(x, x_2)(x, y_2) : x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) : z \in V_2, x_1y_1 \in E_1\}$ *and* $E^0 = E \cup \{(x_1, x_2)(y_1, y_2) : x_1y_1 \in E_1, x_2 \neq y_2\}$.

(v) $p_i \circ (B_1 \circ B_2)((x_1, x_2)(y_1, y_2)) = 0$ *for all* $(x_1, x_2)(y_1, y_2) \in \widetilde{V_1 \times V_2}^2 - E^0$.

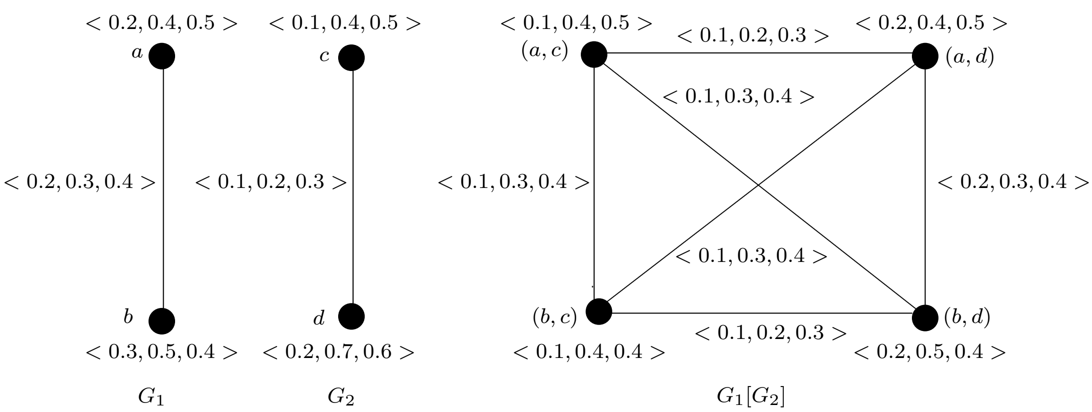

Figure 2.3: Composition of two 3-polar fuzzy graphs G_1 and G_2

Example 2.3.2. *Let G_1^* and G_2^* be same as in Example 2.3.1. Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two 3-polar fuzzy graphs of the graphs G_1^* and G_2^* respectively where $A_1 = \{\frac{<0.2,0.4,0.5>}{a}, \frac{<0.3,0.5,0.4>}{b}\}$, $B_1 = \{\frac{<0.2,0.3,0.4>}{ab}\}$, $A_2 = \{\frac{<0.1,0.4,0.5>}{c}, \frac{<0.2,0.7,0.6>}{d}\}$, $B_2 = \{\frac{<0.1,0.2,0.3>}{cd}\}$. Then we have,*

$(B_1 \circ B_2)((a,c)(a,d)) =< 0.1, 0.2, 0.3 >$, $(B_1 \circ B_2)((b,c)(b,d)) =< 0.1, 0.2, 0.3 >$,

$(B_1 \circ B_2)((a,c)(b,c)) =< 0.1, 0.3, 0.4 >$, $(B_1 \circ B_2)((a,d)(b,d)) =< 0.2, 0.3, 0.4 >$

$(B_1 \circ B_2)((a,c)(b,d)) =< 0.1, 0.3, 0.4 >$, $(B_1 \circ B_2)((b,c)(a,d)) =< 0.1, 0.3, 0.4 >$.

It can be easily checked that $G_1[G_2]$ is a 3-polar fuzzy graph of $G_1^*[G_2^*]$ (see Fig. 2.3).

Proposition 2.3.2. *The composition $G_1[G_2]$ of two m-polar fuzzy graphs G_1 and G_2 is an m-polar fuzzy graph.*

Proof. Let $x \in V_1$, $x_2y_2 \in E_2$. Then for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \times B_2)((x, x_2)(x, y_2))$

$= min\{p_i \circ A_1(x), p_i \circ B_2(x_2y_2)\}$

$\leq min\{p_i \circ A_1(x), min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2)\}\}$

$= min\{min\{p_i \circ A_1(x), p_i \circ A_2(x_2)\}, min\{p_i \circ A_1(x), p_i \circ A_2(y_2)\}\}$

$= min\{p_i \circ (A_1 \times A_2)(x, x_2), p_i \circ (A_1 \times A_2)(x, y_2)\}$.

Let $z \in V_2$, $x_1y_1 \in E_1$. The proof is similar to the above.

Let $(x_1, y_1)(x_2, y_2) \in E^0 - E$.

So $x_1y_1 \in E_1$ and $x_2 \neq y_2$.

Then we have for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \circ B_2)((x_1, x_2)(y_1, y_2))$

$= min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2), p_i \circ B_1(x_1y_1)\}$

$\leq min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2), min\{p_i \circ A_1(x_1), p_i \circ A_1(y_1)\}\}$

$= min\{min\{p_i \circ A_1(x_1), p_i \circ A_2(x_2)\}, min\{p_i \circ A_1(y_1), p_i \circ A_2(y_2)\}\}$

$= min\{p_i \circ (A_1 \times A_2)(x_1, x_2), p_i \circ (A_1 \times A_2)(y_1, y_2)\}$.

Hence $G_1[G_2]$ is an m-polar fuzzy graph. □

Definition 2.3.3. *The union $G_1 \cup G_2 = (V_1 \times V_2, A_1 \cup A_2, B_1 \cup B_2)$ of two m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively is defined as follows: for $i = 1, 2, \ldots, m$*

$$\text{(i) } p_i \circ (A_1 \cup A_2)(x_1, x_2) = \begin{cases} p_i \circ A_1(x) & if \quad x \in V_1 - V_2 \\ p_i \circ A_2(x) & if \quad x \in V_2 - V_1 \\ max\{p_i \circ A_1(x), p_i \circ A_2(x)\} & if \quad x \in V_1 \cap V_2. \end{cases}$$

$$\text{(ii) } p_i \circ (B_1 \cup B_2)(xy) = \begin{cases} p_i \circ B_1(xy) & if \quad xy \in E_1 - E_2 \\ p_i \circ B_2(xy) & if \quad xy \in E_2 - E_1 \\ max\{p_i \circ B_1(xy), p_i \circ B_2(xy)\} & if \quad xy \in E_1 \cap E_2. \end{cases}$$

$$\text{(iii) } p_i \circ (B_1 \cup B_2)(xy) = 0 \ if \ xy \in \widetilde{V_1 \times V_2}^2 - E_1 \cup E_2.$$

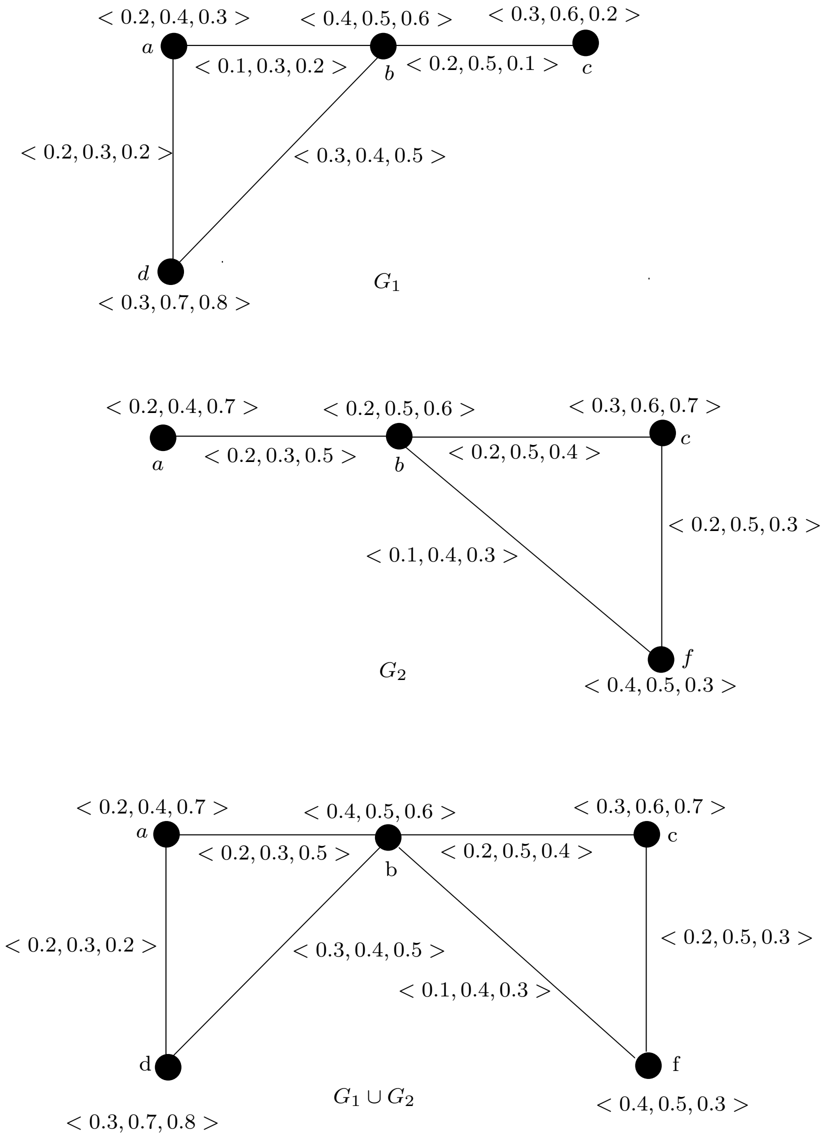

Figure 2.4: Union of two 3-polar fuzzy graphs G_1 and G_2

Example 2.3.3. *Let G_1^* and G_2^* be two graphs such that $V_1 = \{a, b, c, d\}, E_1 = \{ab, bc, ad, bd\}$, $V_2 = \{a, b, c, f\}$ and $\{ab, bc, bf, cf\}$. Consider the two 3-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ (see Fig. 2.4) where*

$A_1 = \{\frac{<0.2,0.4,0.3>}{a}, \frac{<0.4,0.5,0.6>}{b}, \frac{<0.3,0.6,0.2>}{c}, \frac{<0.3,0.7,0.8>}{d}\}$,
$B_1 = \{\frac{<0.1,0.3,0.2>}{ab}, \frac{<0.2,0.5,0.1>}{bc}, \frac{<0.2,0.3,0.2>}{ad}, \frac{<0.3,0.4,0.5>}{bd}, \frac{<0,0,0>}{cd}, \frac{<0,0,0>}{ac}\}$,
$A_2 = \{\frac{<0.2,0.4,0.7>}{a}, \frac{<0.2,0.5,0.6>}{b}, \frac{<0.3,0.6,0.7>}{c}, \frac{<0.4,0.5,0.3>}{f}\}$,
$B_2 = \{\frac{<0.2,0.3,0.5>}{ab}, \frac{<0.2,0.5,0.4>}{bc}, \frac{<0.2,0.5,0.3>}{cf}, \frac{<0.1,0.4,0.3>}{bf}, \frac{<0,0,0>}{af}, \frac{<0,0,0>}{ac}\}$.
Clearly, $G_1 \cup G_2$ is a 3-polar fuzzy graph.

Proposition 2.3.3. *The union $G_1 \cup G_2 = (V_1 \times V_2, A_1 \cup A_2, B_1 \cup B_2)$ of two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively is an m-polar fuzzy graph.*

Proof. Let $xy \in E_1 \cap E_2$. Then for $i = 1, 2, \ldots, m$
$p_i \circ (B_1 \cup B_2)(xy)$
$= max\{p_i \circ B_1(xy), p_i \circ B_2(xy)\}$
$\leq max\{min\{p_i \circ A_1(x), p_i \circ A_1(y)\}, min\{p_i \circ A_2(x), p_i \circ A_2(y)\}\}$
$= min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\}$.
Similarly, if $xy \in E_1 - E_2$, then
$p_i \circ (B_1 \cup B_2)(xy)$
$\leq min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\}$
and if $xy \in E_2 - E_1$, then
$p_i \circ (B_1 \cup B_2)(xy)$
$\leq min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\}$.
This completes the proof. □

Definition 2.3.4. *The join $G_1 + G_2 = (V_1 \cup V_2, A_1 + A_2, B_1 + B_2)$ of two m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively is defined as follows:*

(i) $p_i \circ (A_1 + A_2)(x) = p_i \circ (A_1 \cup A_2)(x)$ *if* $x \in V_1 \cup V_2$.
(ii) $p_i \circ (B_1 + B_2)(xy) = p_i \circ (B_1 \cup B_2)(xy)$ *if* $xy \in E_1 \cup E_2$.
(iii) $p_i \circ (B_1 + B_2)(xy) = min\{p_i \circ A_1(x), p_i \circ A_2(y)\}$ *if $xy \in E'$, where E' is the set of all edges joining the nodes of V_1 and V_2, assuming that $V_1 \cap V_2 = \emptyset$.*
(iv) $p_i \circ (B_1 + B_2)(xy) = 0$ *if* $xy \in \widetilde{V_1 \times V_2}^2 - E_1 \cup E_2 \cup E'$.

Proposition 2.3.4. *The join $G_1 + G_2 = (V_1 \cup V_2, A_1 + A_2, B_1 + B_2)$ of two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively is an m-polar fuzzy graph of $G_1^* + G_2^*$.*

Proof. Follows from the definition. □

Proposition 2.3.5. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two underlying graphs and $V_1 \cap V_2 = \emptyset$. Let A_1, A_2, B_1 and B_2 be m-polar fuzzy subsets of $V_1, V_2, \widetilde{V_1^2}$ and $\widetilde{V_2^2}$ respectively. Then $G_1 \cup G_2 = (V_1 \times V_2, A_1 \cup A_2, B_1 \cup B_2)$ is an m-polar fuzzy graph of $G_1^* \cup G_2^*$ if and only if $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ are m-polar fuzzy graphs of G_1^* and G_2^*, respectively.*

Proof. Suppose $G_1 \cup G_2$ is an m-polar fuzzy graph of $G_1^* \cup G_2^*$.

Let $xy \in E_1$. Then $xy \notin E_2$ and $x, y \in V_1 - V_2$ and for $i = 1, 2, \dots, m$

$p_i \circ B_1(xy)$

$= p_i \circ (B_1 \cup B_2)(xy)$

$\leq min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\}$

$= min\{p_i \circ A_1(x), p_i \circ A_1(y)\}$.

Let $xy \in (\widetilde{V_1^2} - E_1)$. Then for $i = 1, 2, \dots, m$

$p_i \circ B_1(xy) = p_i \circ B_1 \cup B_2(xy) = 0$.

This shows that $G_1 = (V_1, A_1, B_1)$ is an m-polar fuzzy graph of G_1^*.

Similarly, we can show that $G_2 = (V_2, A_2, B_2)$ is an m-polar fuzzy graph of G_2^*. The converse follows from Proposition 2.3.3. □

Proposition 2.3.6. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two underlying graphs and let $V_1 \cap V_2 = \emptyset$. Let A_1, A_2, B_1 and B_2 be m-polar fuzzy subsets of $V_1, V_2, \widetilde{V_1^2}$ and $\widetilde{V_2^2}$, respectively. Then $G_1 + G_2 = (V_1 \cup V_2, A_1 + A_2, B_1 + B_2)$ is an m-polar fuzzy graph of $G_1^* + G_2^*$ if and only if $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ are m-polar fuzzy graphs of G_1^* and G_2^*, respectively.*

Proof. Follows from Propositions 2.3.4 and 2.3.5. □

2.4 Isomorphisms of m-polar fuzzy graphs

In this section, different types of isomorphisms are defined on m-polar fuzzy graphs.

Definition 2.4.1. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. A homomorphism between G_1 and G_2 is a mapping $\phi : V_1 \to V_2$ such that for $i = 1, 2, \dots, m$*

(i) $p_i \circ A_1(x_1) \leq p_i \circ A_2(\phi(x_1))$ for all $x_1 \in V_1$,

(ii) $p_i \circ B_1(x_1y_1) \leq p_i \circ B_2(\phi(x_1)\phi(y_1))$ *for all* $x_1y_1 \in \widetilde{V_1^2}$.

Definition 2.4.2. *Let* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *be two m-polar fuzzy graphs of the graphs* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$ *respectively. An isomorphism between* G_1 *and* G_2 *is a bijective mapping* $\phi : V_1 \to V_2$ *such that for* $i = 1, 2, \ldots, m$

(i) $p_i \circ A_1(x_1) = p_i \circ A_2(\phi(x_1))$ *for all* $x_1 \in V_1$,

(ii) $p_i \circ B_1(x_1y_1) = p_i \circ B_2(\phi(x_1)\phi(y_1))$ *for all* $x_1y_1 \in \widetilde{V_1^2}$.

In this case, we write $G_1 \cong G_2$.

Remark 2.4.1. *If* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *are two m-polar fuzzy graphs. Then the canonical projection maps* $\pi_1 : V_1 \times V_2 \to V_1$ *and* $\pi_2 : V_1 \times V_2 \to V_2$ *are indeed homomorphisms from* $G_1 \times G_2$ *to* G_1 *and* $G_1 \times G_2$ *to* G_2 *respectively. This can be seen as follows:*

$p_i \circ (A_1 \times A_2)(x_1, x_2) = min\{p_i \circ A_1(x_1), p_i \circ A_2(x_2)\} \leq p_i \circ A_1(x_1) = p_i \circ A_1(\pi_1(x_1, x_2))$ *for all* $(x_1, x_2) \in V_1 \times V_2$ *and* $p_i \circ (B_1 \times B_2)((x_1, z)(y_1, z)) = min\{p_i \circ B_1(x_1y_1), p_i \circ A_2(z)\} \leq p_i \circ B_1(x_1y_1) = p_i \circ B_1(\pi_1(x_1, z)\pi_1(y_1, z))$ *for all* $z \in V_2$ *and* $x_1y_1 \in E_1$. *In a similar way we can check the other conditions also. This shows that the canonical projection maps* $\pi_1 : V_1 \times V_2 \to V_1$ *is a homomorphism from* $G_1 \times G_2$ *to* G_1.

Definition 2.4.3. *Let* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *be two m-polar fuzzy graphs of the graphs* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$ *respectively. A weak isomorphism between* G_1 *and* G_2 *is a bijective mapping* $\phi : V_1 \to V_2$ *which satisfies the following conditions:*

(i) ϕ *is a homomorphism, and*

(ii) $p_i \circ A_1(x_1) = p_i \circ A_2(\phi(x_1))$ *for all* $x_1 \in V_1$, $i = 1, 2, \ldots, m$.

In other words, a weak isomorphism preserves the weight of the nodes but not necessarily the weights of the arcs.

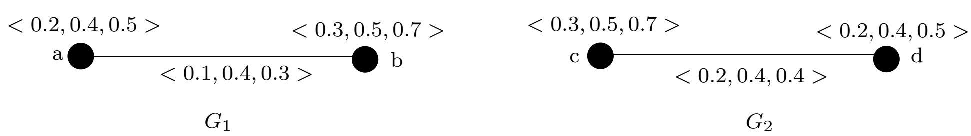

Figure 2.5: Weak isomorphism of G_1 and G_2

Example 2.4.1. *Consider two 3-polar fuzzy graphs G_1 and G_2 (see Fig. 2.5) of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively, where $V_1 = \{a, b\}, V_2 = \{c, d\}, E_1 = \{ab\}$ and $E_2 = \{cd\}$. Let us define a map $\phi : V_1 \to V_2$ such that $\phi(a) = d, \phi(b) = c$. Then we have*

$p_1 \circ A_1(a) = 0.2 = p_1 \circ A_2(d) = p_1 \circ A_2(\phi(a))$,

$p_2 \circ A_1(a) = 0.4 = p_2 \circ A_2(d) = p_2 \circ A_2(\phi(a))$,

$p_3 \circ A_1(a) = 0.5 = p_3 \circ A_2(d) = p_3 \circ A_2(\phi(a))$.

$p_1 \circ A_1(b) = 0.3 = p_1 \circ A_2(c) = p_1 \circ A_2(\phi(b))$,

$p_2 \circ A_1(b) = 0.5 = p_2 \circ A_2(c) = p_2 \circ A_2(\phi(b))$,

$p_3 \circ A_1(b) = 0.7 = p_3 \circ A_2(c) = p_3 \circ A_2(\phi(b))$.

$p_1 \circ B_1(ab) = 0.1 < 0.2 = p_1 \circ B_2(dc) = p_1 \circ B_2(\phi(a)\phi(b))$,

$p_2 \circ B_1(ab) = 0.4 = p_2 \circ B_2(dc) = p_2 \circ B_2(\phi(a)\phi(b))$,

$p_3 \circ B_1(ab) = 0.3 < 0.4 = p_3 \circ B_2(dc) = p_3 \circ B_2(\phi(a)\phi(b))$.

Hence, $B_1(ab) \neq B_2(\phi(a)\phi(b))$.

This shows that the map ϕ is a weak isomorphism but not an isomorphism.

Definition 2.4.4. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. A co-weak isomorphism between G_1 and G_2 is a bijective mapping $\phi : V_1 \to V_2$ which satisfies the following :*

(i) ϕ *is a homomorphism,*

(ii) $p_i \circ B_1(x_1y_1) = p_i \circ B_2(\phi(x_1y_1))$ *for all* $x_1y_1 \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$.

In other words, a co-weak isomorphism preserves the weight of the arcs but not necessarily the weights of the nodes.

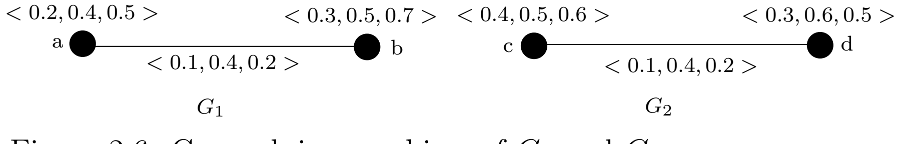

Figure 2.6: Co-weak isomorphism of G_1 and G_2

Example 2.4.2. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two crisp graphs defined in Example 2.4.1. Consider the 3-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of G_1^* and G_2^* (see Fig. 2.6). Consider the map $\phi : V_1 \to V_2$ defined by $\phi(a) = d, \phi(b) = c$. Then we have the following:*

$p_1 \circ A_1(a) = 0.2 < 0.3 = p_1 \circ A_2(d) = p_1 \circ A_2(\phi(a))$,

$p_2 \circ A_1(a) = 0.4 < 0.6 = p_2 \circ A_2(d) = p_2 \circ A_2(\phi(a))$,

$p_3 \circ A_1(a) = 0.5 = 0.5 = p_3 \circ A_2(d) = p_3 \circ A_2(\phi(a))$.

Therefore, $A_1(a) \neq A_2(d) = A_2(\phi(a))$.

Similarly, $A_1(b) \neq A_2(c) = A_2(\phi(b))$.

But, $p_1 \circ B_1(ab) = 0.1 = p_1 \circ B_2(dc) = p_1 \circ B_2(\phi(a)\phi(b))$,

$p_2 \circ B_1(ab) = 0.4 = p_2 \circ B_2(dc) = p_2 \circ B_2(\phi(a)\phi(b))$,

$p_3 \circ B_1(ab) = 0.2 = p_3 \circ B_2(dc) = p_3 \circ B_2(\phi(a)\phi(b))$.

Therefore, $B_1(ab) = B_2(dc) = B_2(\phi(a)\phi(b))$.

Hence, the map ϕ is a co-weak isomorphism but not an isomorphism.

2.5 Some properties of m-polar fuzzy graphs

The strong m-polar fuzzy graph is defined below.

Definition 2.5.1. *An m-polar fuzzy graph $G = (V, A, B)$ of the graph $G^* = (V, E)$ is called strong if $p_i \circ B(xy) = min\{p_i \circ A(x), p_i \circ A(y)\}$ for all $xy \in E$, $i = 1, 2, \ldots, m$.*

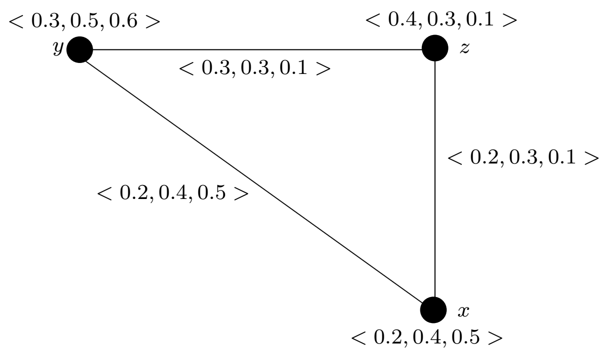

Figure 2.7: Strong 3-polar fuzzy graph G

Example 2.5.1. *Consider a graph $G^* = (V, E)$ such that $V = \{x, y, z\}, E = \{xy, yz, zx\}$. Let $G = (V, A, B)$ be the 3-polar fuzzy graph of G^*, where*

$A = \{\frac{<0.2,0.4,0.5>}{x}, \frac{<0.3,0.5,0.6>}{y}, \frac{<0.4,0.3,0.1>}{z}\}$,

$B = \{\frac{<0.2,0.4,0.5>}{xy}, \frac{<0.3,0.3,0.1>}{yz}, \frac{<0.2,0.3,0.1>}{zx}\}$.

Hence, G is a strong 3-polar fuzzy graph (see Fig. 2.7).

Proposition 2.5.1. *If G_1 and G_2 are the strong m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ $G_2^* = (V_2, E_2)$ respectively, then $G_1 \times G_2$, $G_1[G_2]$ and $G_1 + G_2$ are strong m-polar fuzzy graphs of the graphs $G_1^* \times G_2^*$, $G_1^*[G_2^*]$ and $G_1^* + G_2^*$.*

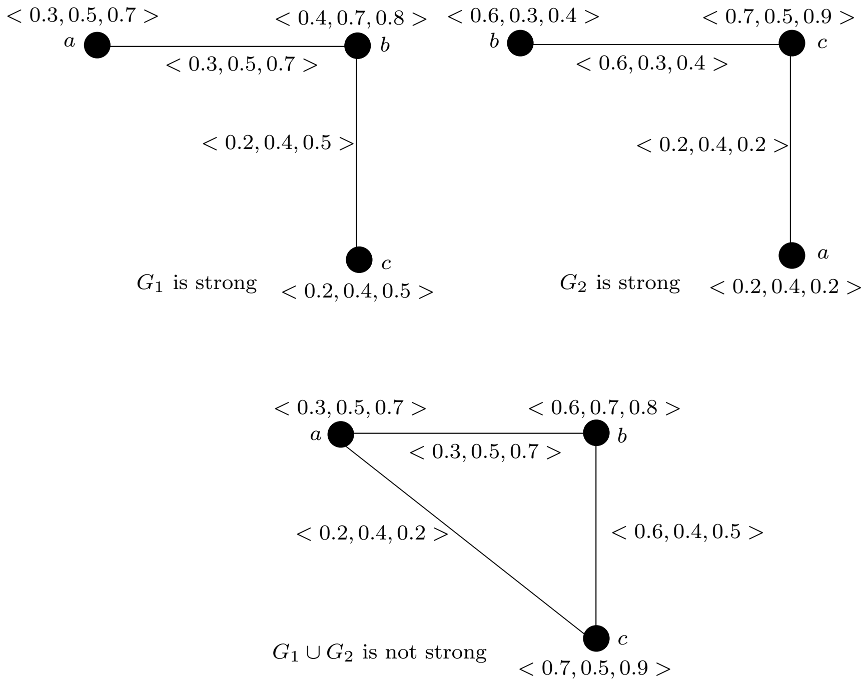

Figure 2.8: Union of two strong 3-polar graphs G_1 and G_2 is not strong

Proof. Follows from the Propositions 2.3.1, 2.3.2 and 2.3.4. □

Remark 2.5.1. *The union of two strong m-polar fuzzy graphs is not necessarily a strong m-polar fuzzy graph. For example, let us consider the 3-polar fuzzy graphs G_1 and G_2 as shown in Fig. 2.8.*

Proposition 2.5.2. *If $G_1 \times G_2$ is a strong m-polar fuzzy graph, then at least G_1 or G_2 must be strong.*

Proof. Suppose that both G_1 and G_2 are not strong m-polar fuzzy graphs. Then there exists at least one $x_1y_1 \in E_1$ and at least one $x_2y_2 \in E_2$ such that

(i) $B_1(x_1y_1) < min\{A_1(x_1), A_1(y_1)\}$ and $B_2(x_2y_2) < min\{A_2(x_2), A_2(y_2)\}$.

Without loss of generality, we assume that

(ii) $B_2(x_2y_2) \leq B_1(x_1y_1) < min\{A_1(x_1), A_1(y_1)\} \leq A_1(x_1)$.

Let $E = \{(x, x_2)(x, y_2) : x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) : z \in V_2, x_1y_1 \in E_1\}$. Consider $(x, x_2)(x, y_2) \in E$. Then, by definition of $G_1 \times G_2$ and inequality (i) we have

$(B_1 \times B_2)((x, x_2)(x, y_2))$

$= min\{A_1(x), B_2(x_2y_2)\} < min\{A_1(x), A_2(x_2), A_2(y_2)\}$

and $(A_1 \times A_2)(x, x_2) = min\{A_1(x), A_2(x_2)\}$,

$(A_1 \times A_2)(x, y_2) = min\{A_1(x), A_2(y_2)\}$.

Thus, $min\{(A_1 \times A_2)(x, x_2), (A_1 \times A_2)(x, y_2)\}$

$= min\{A_1(x), A_2(x_2), A_2(y_2)\}$.

Hence, $(B_1 \times B_2)((x, x_2)(x, y_2))$

$= min\{A_1(x), B_2(x_2y_2)\} < min\{(A_1 \times A_2)(x, x_2), (A_1 \times A_2)(x, y_2)\}$,

i.e. $G_1 \times G_2$ is not strong m-polar fuzzy graph, which is a contradiction. Hence if $G_1 \times G_2$ is strong m-polar fuzzy graph, then at least G_1 or G_2 must be strong m-polar fuzzy graph. □

Proposition 2.5.3. *If $G_1[G_2]$ is strong m-polar fuzzy graph, then at least G_1 or G_2 must be strong.*

Proof. Follows from the previous Proposition. □

Proposition 2.5.4. *Let $G = (V, A, B)$ be a strong m-polar fuzzy graph of a graph $G^* = (V, E)$. If $\overline{G} = (V, \overline{A}, \overline{B})$ satisfies $\overline{A} = A$ and $\overline{B}$ defined as follows:*

for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$

$$p_i \circ \overline{B}(xy) = \begin{cases} 0 & if \quad 0 < p_i \circ B(xy) \leq 1 \\ min\{p_i \circ A(x), p_i \circ A(y)\} & if \quad p_i \circ B(xy) = 0. \end{cases}$$

Then $\overline{G}$ is a strong m-polar fuzzy graph of $\overline{G^*} = (V, \widetilde{V^2} - E)$.

Proof. Obviously, the m-polar fuzzy sets $\overline{A}$ and $\overline{B}$ satisfy

$p_i \circ \overline{B}(xy) \leq min\{p_i \circ A(x), p_i \circ A(y)\}$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.

Now, let $xy \in \widetilde{V^2} - (\widetilde{V^2} - E) = E$.

As G is strong m-polar fuzzy graph, therefore $p_i \circ \overline{B}(xy) = min\{p_i \circ A(x), p_i \circ A(y)\}$ for $i = 1, 2, \ldots, m$.

If $B(xy) = 0$, then $p_i \circ B(xy) = 0$ for $i = 1, 2, \ldots, m$.

Therefore, $p_i \circ \overline{B}(xy)$

$= min\{p_i \circ A(x), p_i \circ A(y)\}$

$= p_i \circ B(xy) = 0$, $i = 1, 2, \ldots, m$.

Hence, $\overline{B}(xy) = \mathbf{0}$.

If for $i = 1, 2, \ldots, m$, $0 < p_i \circ B(xy) \leq 1$ then $p_i \circ \overline{B}(xy) = 0$, i.e. $\overline{B}(xy) = \mathbf{0}$.

Hence, for all $xy \in \widetilde{V^2} - (\widetilde{V^2} - E) = E$, $\overline{B}(xy) = \mathbf{0}$. Therefore, $\overline{G}$ is an m-polar fuzzy graph of $G^* = (V, \widetilde{V^2} - E)$.

On the other hand, for all $xy \in (\widetilde{V^2} - E)$, we have by Definition 2.2.1, $B(xy) = \mathbf{0}$, i.e. $p_i \circ B(xy) = 0$ for each $i = 1, 2, \dots, m$. Then $p_i \circ \overline{B}(xy) = min\{p_i \circ A(x), p_i \circ A(y)\}$, $i = 1, 2, \dots, m$. So, $\overline{G}$ is a strong m-polar fuzzy graph of $\overline{G^*} = (V, \widetilde{V^2} - E)$. □

Definition 2.5.2. *The strong m-polar fuzzy graph $\overline{G} = (V, \overline{A}, \overline{B})$ defined above is called the complement of the strong m-polar fuzzy graph $G = (V, A, B)$.*

Definition 2.5.3. *A strong m-polar fuzzy graph G is called self complementary if $G \cong \overline{G}$.*

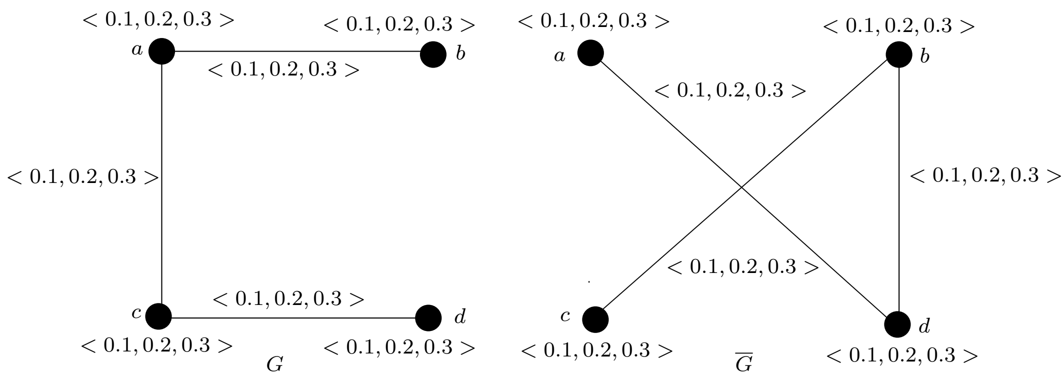

Figure 2.9: Self-complementary 3-polar fuzzy graphs

Example 2.5.2. *Let $G^* = (V, E)$ be a graph, where $V = \{a, b, c, d\}$ and $E = \{ab, ac, cd\}$ and $G = (V, A, B)$(see Fig. 2.9) be a strong 3-polar fuzzy graph of G^*, where*

$A = \{\frac{<0.1,0.2,0.3>}{a}, \frac{<0.1,0.2,0.3>}{b}, \frac{<<0.1,0.2,0.3>}{c}, \frac{<0.1,0.2,0.3>}{d}\}$,

$B = \{\frac{<0.1,0.2,0.3>}{ab}, \frac{<0.1,0.2,0.3>}{ac}, \frac{<0.1,0.2,0.3>}{cd}, \frac{<0,0,0>}{bd}, \frac{<0,0,0>}{ad}, \frac{<0,0,0>}{bc}\}$.

Then G is self complementary.

Let $\overline{G} = (V, \overline{A}, \overline{B})$ be the complement of G, where $\overline{A} = A$,

$\overline{B} = \{\frac{<0,0,0>}{ab}, \frac{<0,0,0>}{ac}, \frac{<0,0,0>}{cd}, \frac{<0.1,0.2,0.3>}{bd}, \frac{<0.1,0.2,0.3>}{ad}, \frac{<0.1,0.2,0.3>}{bc}\}$.

Let us now define a mapping $\phi : V \to V$ by $\phi(a) = b$, $\phi(b) = c$, $\phi(c) = d$, $\phi(d) = a$.

Then clearly, ϕ is a bijective mapping and

$A(a) = \overline{A}(\phi(a))$, $A(b) = \overline{A}(\phi(b))$, $A(c) = \overline{A}(\phi(c))$, $A(d) = \overline{A}(\phi(d))$.

$B(ab) =< 0.1, 0.2, 0.3 >= \overline{B}(\phi(a)\phi(b))$, $B(ac) =< 0.1, 0.2, 0.3 >= \overline{B}(\phi(a)\phi(c))$,

$B(cd) =< 0.1, 0.2, 0.3 >= \overline{B}(\phi(c)\phi(d))$, $B(bc) =< 0, 0, 0 >= \overline{B}(\phi(b)\phi(c))$,

$B(bd) =< 0, 0, 0 >= \overline{B}(\phi(b)\phi(d))$, $B(ad) =< 0, 0, 0 >= \overline{B}(\phi(a)\phi(d))$.

Hence, ϕ is an isomorphism from G onto $\overline{G}$, i.e. $G \cong \overline{G}$.

Proposition 2.5.5. *Let $G = (V, A, B)$ be a strong m-polar fuzzy graph of the graph $G^* = (V, E)$ and $\overline{G} = (V, \overline{A}, \overline{B})$ be the complement of G. Then,*

$p_i \circ \overline{B}(xy) = min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy)$ *for all* $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.

Proof. Let $xy \in \widetilde{V^2}$.

If $0 < p_i \circ B(xy) \leq 1$ for each $i = 1, 2, \ldots, m$ then by Definition 2.2.1, $xy \in E$.

As G is strong, $min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy) = 0 = p_i \circ \overline{B}(xy)$, $i = 1, 2, \ldots, m$.

If $p_i \circ B(xy) = 0$ for $i = 1, 2, \ldots, m$ then

$min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy)$

$= min\{p_i \circ A(x), p_i \circ A(y)\}$

$= p_i \circ \overline{B}(xy)$.

Hence the result. □

Proposition 2.5.6. *Let G be a self complementary strong m-polar fuzzy graph. Then $\sum_{x \neq y} p_i \circ B(xy) = \frac{1}{2} \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\}$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.*

Proof. Let $G = (V, A, B)$ be a self complementary strong m-polar fuzzy graph. Then $p_i \circ B(xy) = min\{p_i \circ A(x), p_i \circ A(y)\}\}$ for all $xy \in E$, $i = 1, 2, \ldots, m$ and there exists an isomorphism $\phi : G \to \overline{G}$ such that $p_i \circ A(x) = p_i \circ \overline{A}(\phi(x))$ for all $x \in V$ and $p_i \circ B(xy) = p_i \circ \overline{B}(\phi(x)\phi(y))$ for all $xy \in \widetilde{V^2}$.

Let $xy \in \widetilde{V^2}$. Then by Proposition 2.5.5, for $i = 1, 2, \ldots, m$

$p_i \circ \overline{B}(\phi(x)\phi(y)) = min\{p_i \circ A(\phi(x)), p_i \circ A(\phi(y))\} - p_i \circ B(\phi(x)\phi(y))$,

i.e. $p_i \circ B(xy) = min\{p_i \circ A(\phi(x)), p_i \circ A(\phi(y))\} - p_i \circ B(\phi(x)\phi(y))$.

Therefore,

$$\begin{aligned}\sum_{x \neq y} p_i \circ B(xy) + \sum_{x \neq y} p_i \circ B(\phi(x)\phi(y)) &= \sum_{x \neq y} min\{p_i \circ A(\phi(x)), p_i \circ A(\phi(y))\} \\ &= \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\},\end{aligned}$$

i.e. $2 \sum_{x \neq y} p_i \circ B(xy) = \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\}$,

i.e. $\sum_{x \neq y} p_i \circ B(xy) = \frac{1}{2} \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\}$. □

Proposition 2.5.7. *Let $G = (V, A, B)$ be a strong m-polar fuzzy graph of $G^* = (V, E)$. If $p_i \circ B(xy) = \frac{1}{2} min\{p_i \circ A(x), p_i \circ A(y)\}$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$ then G is self complementary.*

Proof. If $G = (V, A, B)$ is a strong m-polar fuzzy graph satisfying

$p_i \circ B(xy) = \frac{1}{2} min\{p_i \circ A(x), p_i \circ A(y)\}$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$ then the identity mapping $I : V \to V$ is an isomorphism from G to $\overline{G}$.

Clearly I satisfies the first condition for isomorphism, i.e. $A(x) = \overline{A}(I(x))$ for all $x \in V$ and by Proposition 2.5.5, we have for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$

$p_i \circ \overline{B}(I(x)I(y))$

$= p_i \circ \overline{B}(xy)$

$= min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy)$

$= min\{p_i \circ A(x), p_i \circ A(y)\} - \frac{1}{2}min\{p_i \circ A(x), p_i \circ A(y)\}$

$= \frac{1}{2}min\{p_i \circ A(x), p_i \circ A(y)\}$

$= p_i \circ B(xy)$.

i.e. $p_i \circ \overline{B}(I(x)I(y)) = p_i \circ B(xy)$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$,

i.e. I also satisfies the second condition for isomorphism.

Therefore, $G \cong \overline{G}$, i.e. G is self complementary. □

From Propositions 2.5.6 and 2.5.7, we have the following.

Corollary 2.5.1. *Let $G = (V, A, B)$ be a strong m-polar fuzzy graph of $G^* = (V, E)$. Then G is self complementary if and only if $p_i \circ B(xy) = \frac{1}{2}min\{p_i \circ A(x), p_i \circ A(y)\}$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.*

Proposition 2.5.8. *Let G_1 and G_2 be two strong m-polar fuzzy graphs. Then $G_1 \cong G_2$ if and only if $\overline{G_1} \cong \overline{G_2}$.*

Proof. Assume that, $G_1 \cong G_2$.

Then there exists a bijective mapping $\phi : V_1 \to V_2$ satisfying $A_1(x) = A_2(\phi(x))$ for all $x \in V_1$ and $p_i \circ B_1(xy) = p_i \circ B_2(\phi(x)\phi(y))$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.

Let $xy \in \widetilde{V_1^2}$.

If $p_i \circ B_1(xy) = 0$ for $i = 1, 2, \ldots, m$ then

$p_i \circ \overline{B_1}(xy)$

$= min\{p_i \circ A_1(x), p_i \circ A_1(y)\}$

$= min\{p_i \circ A_2(\phi(x), p_i \circ A_2(\phi(y)\}$

$= p_i \circ \overline{B_2}(\phi(x)\phi(y))$.

If $0 < p_i \circ B_1(xy) \leq 1$ for $i = 1, 2, \ldots, m$ then, $0 < p_i \circ B_2(\phi(x)\phi(y)) \leq 1$.

Therefore, $p_i \circ \overline{B_1}(xy) = 0 = p_i \circ \overline{B_2}(\phi(x)\phi(y))$.

So, $G_1 \cong G_2$.

Conversely, let $\overline{G_1} \cong \overline{G_2}$.

Then there exists a bijective mapping $\psi : V_1 \to V_2$ satisfying $\overline{A_1}(x) = \overline{A_2}(\psi(x))$ for all $x \in V_1$ and $p_i \circ \overline{B_1}(xy) = p_i \circ \overline{B_2}(\psi(x)\psi(y))$ for all $xy \in \widetilde{V_1^2}$.

Let $xy \in \widetilde{V_1^2}$.

If $p_i \circ B_1(xy) = 0$ for $i = 1, 2, \ldots, m$ then

$p_i \circ \overline{B_2}(\psi(x)\psi(y))$

$= p_i \circ \overline{B_1}(xy)$

$= min\{p_i \circ A_1(x), p_i \circ A_1(y)\}$

$= min\{p_i \circ \overline{A_1}(x), p_i \circ \overline{A_1}(y)\}$

$= min\{p_i \circ \overline{A_2}(\psi(x)), p_i \circ \overline{A_2}(\psi(y))\}$

$= min\{p_i \circ A_2(\psi(x)), p_i \circ A_2(\psi(y))\}$.

Again, $p_i \circ \overline{B_2}(\psi(x)\psi(y))$

$= min\{p_i \circ A_2(\psi(x)), p_i \circ A_2(\psi(y))\} - p_i \circ B_2(\psi(x)\psi(y))$

So, $p_i \circ B_2(\psi(x)\psi(y)) = 0 = p_i \circ B_1(xy))$, $i = 1, 2, \ldots, m$.

If $0 < p_i \circ B_1(xy) \leq 1$ for $i = 1, 2, \ldots, m$ then

$p_i \circ \overline{B_2}(\psi(x)\psi(y))$

$= p_i \circ \overline{B_1}(\psi(x)\psi(y)) = 0$.

Thus we have,

$p_i \circ B_2(\psi(x)\psi(y))$

$= min\{p_i \circ A_2(\psi(x)), p_i \circ A_2(\psi(y))\} - 0$

$= min\{p_i \circ \overline{A_2}(\psi(x)), p_i \circ \overline{A_2}(\psi(y))\}$

$= min\{p_i \circ \overline{A_1}(\psi(x)), p_i \circ \overline{A_1}(\psi(y))\}$

$= p_i \circ B_1(xy)$.

Hence, $G_1 \cong G_2$. $\square$

2.6 Applications

1-polar fuzzy graphs is nothing but the most familiar fuzzy graphs which has many applications in cluster analysis, solving fuzzy intersection equations, database theory, problem concerning group structure, etc. The further possible applications of m-polar fuzzy graphs in real world problems can be viewed in case of bipolar fuzzy graphs, i.e. 2-polar fuzzy graphs. Bipolar fuzzy graphs has many applications in social networks, engineering, computer science, database theory, expert systems, neural networks, artificial intelligence, signal processing, pattern recognition, robotics, computer networks, medical diagnosis, etc. Also, m-polar fuzzy graphs ($m > 2$) is very useful in many decision making situations. This happens when a group of friends decides which movie to watch, when a company decides which product design to manufacture, and when a democratic country elects its leader. For instance, consider the case of a company. In a company, a group of people decides which product design to manufacture. In such case, different product design can be taken as nodes. An edge is drawn between two nodes if there is some m-polar fuzzy relationship between them. We assume that the membership value of each node represents the degrees of preference of the product design corresponding to the group of people of the company. The degrees of preference (within $[0, 1]$) represent the individual preference of the peoples. Thus, a node has multi-preference degrees corresponding to a product design. Similarly, the degree of relationship between the nodes measures the edge relationship value. Between two product designs, one design may have better looks, may be very demandable, may be cheap, etc. So, there are multipolar information between two product designs. This type of network is an ideal example of m-polar fuzzy graphs. It is very important for a company to decide which product design to manufacture so that they can make profit as much as possible. A very good product design is gladly acceptable to the peoples if it is also cheap in price. The determination of which product design to manufacture is called the decision making problem. By taking the very good decision (very good product design), one company can spread their product all over the world keeping in mind that the product design is very good, demandable, cheap, easily accessible, etc. Moreover, the results of m-polar fuzzy graphs can be applicable in various areas of engineering, computer science, artificial intelligence, neural networks, social networks, etc.

2.7 Summary

Graph theory is an extremely useful tool in solving the combinatorial problems in different areas including algebra, number theory, geometry, topology, operations research, optimization and computer science. Since researches or modelings on real world problems often involve multi-agent, multi-attribute, multi-object, multi-index, multi-polar information, uncertainty, or and limits process, therefore m-polar fuzzy graphs is very useful. The m-polar fuzzy models gives more precision, flexibility and comparability to the system as compared to the classical, fuzzy and bipolar fuzzy models. Therefore, the concept of generalized m-polar fuzzy graph is introduced and studied several important results of it.

Chapter 3

Operations and degrees of m-polar fuzzy graphs*

3.1 Introduction

Graph operations are very important topic in graph theory. Also, they are conveniently used in many combinatorial applications, computer science, geometry, algebra, number theory and operation research. In various situations they present a suitable construction means. For examples, in partition theory, we deal with complex objects. A typical such object is a fuzzy graph and fuzzy hypergraph with large chromatic number that we do not know how to compute exactly the chromatic number of these graphs. In such cases, these operations have the main role in solving problems. Hence, in this chapter, three new operations are defined on m-polar fuzzy graph such as direct product, semi-strong product and strong product. It is proved that any of the products of m-polar fuzzy graphs are again an m-polar fuzzy graph. Sufficient conditions are established for each one of them to be strong and also proved that strong product of two complete m-polar fuzzy graphs is complete. If any of the products of two m-polar fuzzy graphs G_1 and G_2 are strong, then it is shown that at least G_1 or G_2 must be strong. We study about the degree of a vertex in m-polar fuzzy graphs which are obtained from two given m-polar fuzzy graphs G_1 and G_2 using the operations of Cartesian product, composition, direct product, semi-strong product and strong product. Finally, the concept of product m-polar fuzzy graph is introduced

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and proved that every product m-polar fuzzy graph is an m-polar fuzzy graph. Some operations like union, direct product, ring sum are defined to construct new product m-polar fuzzy graphs.

3.2 Products on m-polar fuzzy graphs

Here direct product of two m-polar fuzzy graphs is defined.

Definition 3.2.1. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of the underlying graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively such that $V_1 \cap V_2 = \emptyset$. The direct product of G_1 and G_2 is defined to be the m-polar fuzzy graph $G_1 \sqcap G_2 = (A_1 \sqcap A_2, B_1 \sqcap B_2)$ of the graph $G^* = (V_1 \times V_2, E)$ where, $E = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\} \subseteq \widetilde{V_1 \times V_2}^2$ and for each $i = 1, 2, \ldots, m$*

(i) $p_i \circ (A_1 \sqcap A_2)(u, v) = p_i \circ A_1(u) \wedge p_i \circ A_2(v)$ for all $(u, v) \in V_1 \times V_2$,

(ii) $p_i \circ (B_1 \sqcap B_2)((u_1, v_1)(u_2, v_2)) = p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$ for all $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$,

(iii) $p_i \circ (B_1 \sqcap B_2)((w, x)(y, z)) = 0$ for all $(w, x)(y, z) \in (\widetilde{V_1 \times V_2}^2 - E)$.

Below, the direct product of m-polar fuzzy graphs is explained by an example.

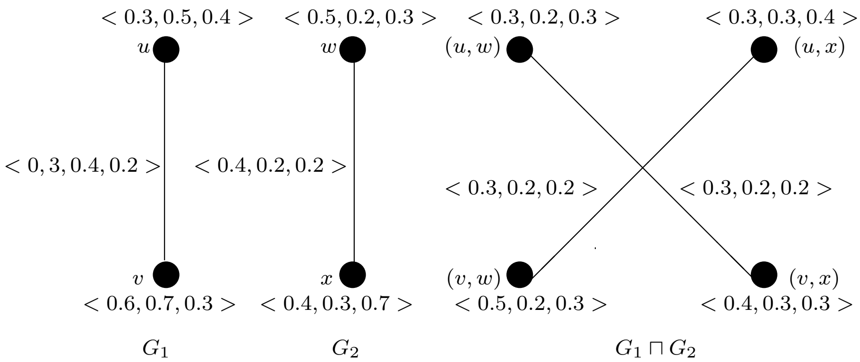

Figure 3.1: Direct product of G_1 and G_2

Example 3.2.1. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two crisp graphs such that $V_1 = \{u, v\}, V_2 = \{w, x\}, E_1 = \{uv\}$ and $E_2 = \{wx\}$. Consider two 3-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. Using the definition of direct product, $G_1 \sqcap G_2$ is constructed (see Fig. 3.1). It is easy to see that $G_1 \sqcap G_2$ is a 3-polar fuzzy graph.*

Theorem 3.2.1. *The direct product $G_1 \sqcap G_2$ of two m-polar fuzzy graphs G_1 and G_2 is an m-polar fuzzy graph.*

Proof. Let $(u_1, v_1)(u_2, v_2) \in E$. Then $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

Hence for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \sqcap B_2)((u_1, v_1)(u_2, v_2))$

$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$

$\leq p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \sqcap A_2)(u_1, v_1) \wedge p_i \circ (A_1 \sqcap A_2)(u_2, v_2)$

Also, for all $(w, x)(y, z) \in (\widetilde{V_1 \times V_2}^2 - E)$, $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \sqcap B_2)((w, x)(y, z)) = 0 \leq p_i \circ (A_1 \sqcap A_2)(w, x) \wedge p_i \circ (A_1 \sqcap A_2)(y, z)$.

This shows that $G_1 \sqcap G_2$ is an m-polar fuzzy graph. □

Theorem 3.2.2. *If $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ are two strong m-polar fuzzy graphs, then $G_1 \sqcap G_2$ is also strong.*

Proof. Let $(u_1, v_1)(u_2, v_2) \in E$.

Since G_1 and G_2 are strong, therefore for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \sqcap B_2)((u_1, v_1)(u_2, v_2))$

$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$

$= p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \sqcap A_2)(u_1, v_1) \wedge p_i \circ (A_1 \sqcap A_2)(u_2, v_2)$.

Hence, $G_1 \sqcap G_2$ is strong m-polar fuzzy graph. □

Now, semi-strong product is defined between two m-polar fuzzy graphs to construct a new m-polar fuzzy graph.

Definition 3.2.2. *The semi-strong product of two m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ of $G_1^* = (V_1, E_1)$ and $G_2 = (V_2, A_2, B_2)$ of $G_2^* = (V_2, E_2)$, where it is assumed that $V_1 \cap V_2 = \emptyset$, is defined to be the m-polar fuzzy graph $G_1 \bullet G_2 = (A_1 \bullet A_2, B_1 \bullet B_2)$ of $G^* = (V_1 \times V_2, E)$, where $E = \{(u, v_1)(u, v_2) | u \in V_1, v_1v_2 \in E_2\} \cup \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\} \subseteq \widetilde{V_1 \times V_2}^2$ satisfying the following: for each $i = 1, 2, \ldots, m$*

(i) $p_i \circ (A_1 \bullet A_2)(u, v) = p_i \circ A_1(u) \wedge p_i \circ A_2(v)$ *for all* $(u, v) \in V_1 \times V_2$,

(ii) $p_i \circ (B_1 \bullet B_2)((u, v_1)(u, v_2)) = p_i \circ A_1(u) \wedge p_i \circ B_2(v_1v_2)$ *for all* $u \in V_1$ *and* $v_1v_2 \in E_2$,

(iii) $p_i \circ (B_1 \bullet B_2)((u_1, v_1)(u_2, v_2)) = p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$ *for all* $u_1u_2 \in E_1$, $v_1v_2 \in E_2$ *and*

(iv) $p_i \circ (B_1 \bullet B_2)((w, x)(y, z)) = 0$ *for all* $(w, x)(y, z) \in (\widetilde{V_1 \times V_2}^2 - E)$.

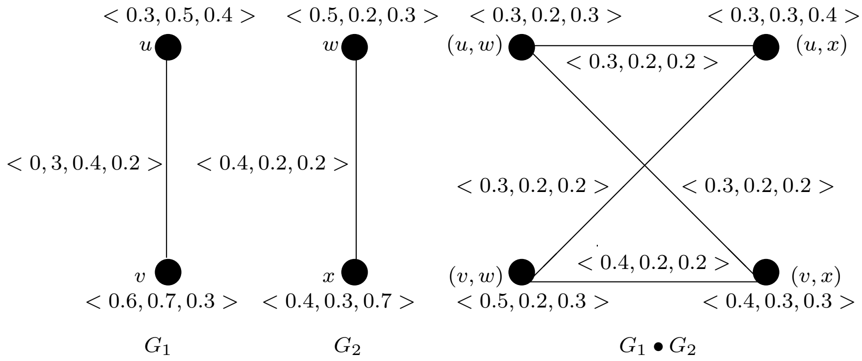

Figure 3.2: Semi-strong product of G_1 and G_2

We demonstrate this product in the following example.

Example 3.2.2. *Consider the 3-polar fuzzy graphs* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *as in Example 3.2.1. Then* $G_1 \bullet G_2$ *is calculated using the above definition. It is easy to see that* $G_1 \bullet G_2$ *is a 3-polar fuzzy graph (see Fig. 3.2).*

Theorem 3.2.3. *If* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *are* m*-polar fuzzy graphs, then* $G_1 \bullet G_2$ *is an* m*-polar fuzzy graph.*

Proof. Let $(u, v_1)(u, v_2) \in E$. Then $u \in V_1$ and $v_1 v_2 \in E_2$. Since G_2 is an m-polar fuzzy graph, we have for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \bullet B_2)((u, v_1)(u, v_2))$

$= p_i \circ A_1(u) \wedge p_i \circ B_2(v_1 v_2)$

$\leq p_i \circ A_1(u) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \bullet A_2)(u, v_1) \wedge p_i \circ (A_1 \bullet A_2)(u, v_2)$.

Let $(u_1, v_1)(u_2, v_2) \in E$.

Then $u_1 u_2 \in E_1$ and $v_1 v_2 \in E_2$.

G_1 and G_2 being m-polar fuzzy graphs, we have for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \bullet B_2)((u_1, v_1)(u_2, v_2))$

$= p_i \circ B_1(u_1 u_2) \wedge p_i \circ B_2(v_1 v_2)$

$\leq p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \bullet A_2)(u_1, v_1) \wedge p_i \circ (A_1 \bullet A_2)(u_2, v_2)$.

Finally, for all $(w, x)(y, z) \in (\widetilde{V_1 \times V_2}^2 - E)$, $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \bullet B_2)((w, x)(y, z)) = 0 \leq p_i \circ (A_1 \bullet A_2)(w, x) \wedge p_i \circ (A_1 \bullet A_2)(y, z)$. □

Theorem 3.2.4. *If* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *are strong* m*-polar fuzzy graphs, then* $G_1 \bullet G_2$ *is a strong* m*-polar fuzzy graph.*

Proof. Let $(u, v_1)(u, v_2) \in E$. Then $u \in V_1$ and $v_1v_2 \in E_2$.

G_2 being strong, we have for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \bullet B_2)((u, v_1)(u, v_2))$

$= p_i \circ A_1(u) \wedge p_i \circ B_2(v_1v_2)$

$= p_i \circ A_1(u) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \bullet A_2)(u, v_1) \wedge p_i \circ (A_1 \bullet A_2)(u, v_2)$.

If $(u_1, v_1)(u_2, v_2) \in E$. Then $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

Now, G_1 and G_2 being strong, we have for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \bullet B_2)((u_1, v_1)(u_2, v_2))$

$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$

$= p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \bullet A_2)(u_1, v_1) \wedge p_i \circ (A_1 \bullet A_2)(u_2, v_2)$.

Hence, $G_1 \bullet G_2$ is strong m-polar fuzzy graph. □

The strong product between m-polar fuzzy graphs is an important operation of m-polar fuzzy graph which is defined below.

Definition 3.2.3. *The strong product of two m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ of $G_1^* = (V_1, E_1)$ and $G_2 = (V_2, A_2, B_2)$ of $G_2^* = (V_2, E_2)$ such that $V_1 \cap V_2 = \emptyset$, is defined to be the m-polar fuzzy graph $G_1 \otimes G_2 = (A_1 \otimes A_2, B_1 \otimes B_2)$ of $G^* = (V_1 \times V_2, E)$, where $E = \{(u, v_1)(u, v_2) | u \in V_1, v_1v_2 \in E_2\} \cup \{(u_1, w)(u_2, w) | w \in V_2, u_1u_2 \in E_1\} \cup \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\} \subseteq \widetilde{V_1 \times V_2}^2$ such that the following condition holds: for each $i = 1, 2, \ldots, m$*

(i) $p_i \circ (A_1 \otimes A_2)(u, v) = p_i \circ A_1(u) \wedge p_i \circ A_2(v)$ *for all* $(u, v) \in V_1 \times V_2$,

(ii) $p_i \circ (B_1 \otimes B_2)((u, v_1)(u, v_2)) = p_i \circ A_1(u) \wedge p_i \circ B_2(v_1v_2)$ *for all* $u \in V_1$ *and* $v_1v_2 \in E_2$,

(iii) $p_i \circ (B_1 \otimes B_2)((u_1, w)(u_2, w)) = p_i \circ B_1(u_1u_2) \wedge p_i \circ A_2(w)$ *for all* $w \in V_2$ *and* $u_1u_2 \in E_1$,

(iv) $p_i \circ (B_1 \otimes B_2)((u_1, v_1)(u_2, v_2)) = p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$ *for all* $u_1u_2 \in E_1$ *and* $v_1v_2 \in E_2$ *and*

(v) $p_i \circ (B_1 \otimes B_2)((w, x)(y, z)) = 0$ *for all* $(w, x)(y, z) \in (\widetilde{V_1 \times V_2}^2 - E)$.

We now give an example which illustrates that the strong product of m-polar fuzzy graphs is again an m-polar fuzzy graph.

Example 3.2.3. *Consider the 3-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ as in Example 3.2.1. Also consider the strong product $G_1 \otimes G_2$ which is shown in Fig. 3.3.*

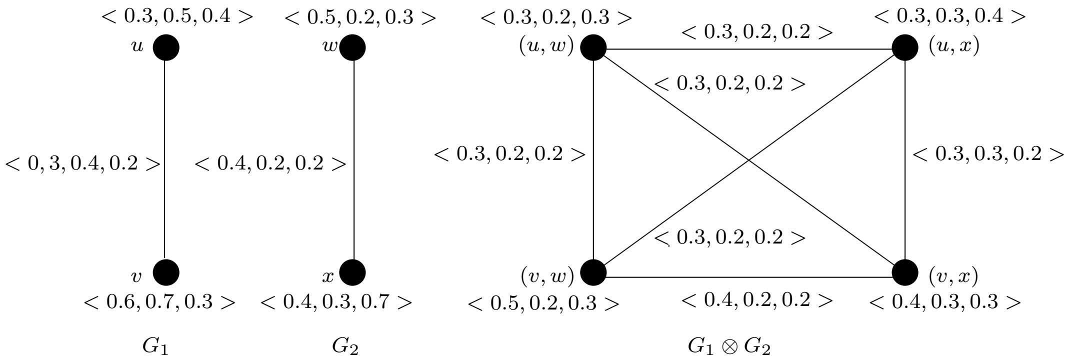

Figure 3.3: Strong product of G_1 and G_2

It is easily checked that, $G_1 \otimes G_2$ is a 3-polar fuzzy graph.

Theorem 3.2.5. *The strong product $G_1 \otimes G_2$ of two m-polar fuzzy graphs is an m-polar fuzzy graph.*

Proof. Follows from the definition of strong product. □

Theorem 3.2.6. *If $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ are complete m-polar fuzzy graphs, then $G_1 \otimes G_2$ is complete.*

Proof. By Theorem 3.2.5, we know that the strong product of m-polar fuzzy graphs is an m-polar fuzzy graph. Since G_1 and G_2 are complete, therefore every pair of vertices are adjacent in the graph $G_1 \otimes G_2$ and hence $E = \widetilde{V_1 \times V_2}^2$.

Let $(u, v_1)(u, v_2) \in E$. Since G_2 is complete, we have for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \otimes B_2)((u, v_1)(u, v_2))$

$= p_i \circ A_1(u) \wedge p_i \circ B_2(v_1 v_2)$

$= p_i \circ A_1(u) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \otimes A_2)(u, v_1) \wedge p_i \circ (A_1 \otimes A_2)(u, v_2)$.

Let $(u_1, w)(u_2, w) \in E$.

Since G_1 is complete, we have for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \otimes B_2)((u_1, w)(u_2, w))$

$= p_i \circ B_1(u_1 u_2) \wedge p_i \circ A_2(w)$

$= p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(w)$

$= p_i \circ (A_1 \otimes A_2)(u_1, w) \wedge p_i \circ (A_1 \otimes A_2)(u_2, w)$.

Finally, let $(u_1, v_1)(u_2, v_2) \in E$.

Then since G_1 and G_2 are complete, we have for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \otimes B_2)((u_1, v_1)(u_2, v_2))$

$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$

$= p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \otimes A_2)(u_1, v_1) \wedge p_i \circ (A_1 \otimes A_2)(u_2, v_2)$.

Hence, $G_1 \otimes G_2$ is complete. □

Theorem 3.2.7. *If $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ are m-polar fuzzy graphs such that $G_1 \sqcap G_2$ is strong, then at least one of G_1 and G_2 must be strong.*

Proof. Let us assume that both G_1 and G_2 are not strong m-polar fuzzy graphs. Then there exist at least one $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$ such that for each $i = 1, 2, \ldots, m$

$p_i \circ B_1(u_1v_1) < p_i \circ A_1(u_1) \wedge p_i \circ A_1(v_1)$ and

$p_i \circ B_2(u_2v_2) < p_i \circ A_2(u_2) \wedge p_i \circ A_2(v_2)$.

Now, for each $i = 1, 2, \ldots, m$ we have

$p_i \circ (B_1 \sqcap B_2)((u_1, v_1)(u_2, v_2))$

$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$

$< p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$ (from the above assumption)

$= p_i \circ (A_1 \sqcap A_2)(u_1, v_1) \wedge p_i \circ (A_1 \sqcap A_2)(u_2, v_2)$.

This shows that, $G_1 \sqcap G_2$ is not strong, which is a contradiction. So our assumption is wrong. This means at least one of G_1 and G_2 is strong. □

The following result follows from the preceding theorem.

Theorem 3.2.8. *If $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ are two m-polar fuzzy graphs such that $G_1 \bullet G_2$ or $G_1 \otimes G_2$ is strong, then at least one of G_1 and G_2 must be strong.*

3.3 Product m-polar fuzzy graphs

In this section, a new type of m-polar fuzzy graphs, known as product m-polar fuzzy graphs are defined.

Definition 3.3.1. *A product m-polar fuzzy graph of a graph $G^* = (V, E)$ is a pair $G = (V, A, B)$ where $A : V \to [0, 1]^m$ is an m-polar fuzzy set in V and $B : \widetilde{V^2} \to [0, 1]^m$*

is an m-polar fuzzy set in $\widetilde{V^2}$ *such that* $p_i \circ B(xy) \le p_i \circ A(x) \times p_i \circ A(y)$ *for all* $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.

Remark 3.3.1. *Since* $p_i \circ A(x)$ *and* $p_i \circ A(y)$ *are less than or equal to 1 for each* $i = 1, 2, \ldots, m$, *it follows that* $p_i \circ B(xy) \le p_i \circ A(x) \times p_i \circ A(y) \le p_i \circ A(x) \wedge p_i \circ A(y)$ *for all* $xy \in \widetilde{V^2}$. *Hence, every product m-polar fuzzy graph is an m-polar fuzzy graph.*

Definition 3.3.2. *A product m-polar fuzzy graph* $G = (V, A, B)$ *is said to be complete if* $p_i \circ B(xy) = p_i \circ A(x) \times p_i \circ A(y)$ *for each* $i = 1, 2, \ldots, m$ *and* $x, y \in V$.

Definition 3.3.3. *The complement of the product m-polar fuzzy graph* $G = (V, A, B)$ *is an m-polar fuzzy graph* $\overline{G} = (V, \overline{A}, \overline{B})$ *where* $\overline{A} = A$ *and* $\overline{B}$ *is defined by*

$p_i \circ \overline{B}(xy) = p_i \circ A(x) \times p_i \circ A(y) - p_i \circ B(xy)$ *for each* $i = 1, 2, \ldots, m$ *and* $xy \in \widetilde{V}^2$.

Remark 3.3.2. *Since for all* $xy \in \widetilde{V}^2$, $i = 1, 2, \ldots, m$

$p_i \circ \overline{B}(xy) = p_i \circ A(x) \times p_i \circ A(y) - p_i \circ B(xy) \le p_i \circ A(x) \times p_i \circ A(y)$,

therefore $\overline{G}$ *is a product m-polar fuzzy graph.*

Definition 3.3.4. *The union* $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ *of two product m-polar fuzzy graphs* $G_1 = (V_1, A_1, B_1)$ *of* $G_1^* = (V_1, E_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *of* $G_2^* = (V_2, E_2)$ *is defined as follows: for each* $i = 1, 2, \ldots, m$

$$(i)\ p_i \circ (A_1 \cup A_2)(x) = \begin{cases} p_i \circ A_1(x) & if \quad x \in V_1 - V_2 \\ p_i \circ A_2(x) & if \quad x \in V_2 - V_1 \\ p_i \circ A_1(x) \vee p_i \circ A_2(x) & if \quad x \in V_1 \cap V_2. \end{cases}$$

$$(ii)\ p_i \circ (B_1 \cup B_2)(xy) = \begin{cases} p_i \circ B_1(xy) & if \quad xy \in E_1 - E_2 \\ p_i \circ B_2(xy) & if \quad xy \in E_2 - E_1 \\ p_i \circ B_1(xy) \vee p_i \circ B_2(xy) & if \quad xy \in E_1 \cap E_2. \end{cases}$$

Proposition 3.3.1. *The direct product* $G_1 \sqcap G_2$ *of two product m-polar fuzzy graphs* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *is a product m-polar fuzzy graph.*

Proof. Let $(u_1, v_1)(u_2, v_2) \in E$. Then $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

Now for each $i = 1, 2, \ldots, m$ we have

$p_i \circ (B_1 \sqcap B_2)((u_1, v_1)(u_2, v_2))$

$= min\{p_i \circ B_1(u_1u_2), p_i \circ B_2(v_1v_2)\}$

$\le min\{p_i \circ A_1(u_1) \times p_i \circ A_1(u_2), p_i \circ A_2(v_1) \times p_i \circ A_2(v_2)\}$

$= min\{p_i \circ A_1(u_1), p_i \circ A_2(v_1)\} \times \{p_i \circ A_1(u_2), p_i \circ A_2(v_2)\}$

$= p_i \circ (A_1 \sqcap A_2)(u_1, v_1) \times p_i \circ (A_1 \sqcap A_2)(u_2, v_2)$.

Hence the result. □

Remark 3.3.3. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two complete product m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. Then $G_1 \sqcap G_2$ may not be complete. For example, let us take product 3-polar fuzzy graphs G_1 and G_2 which are complete but $G_1 \sqcap G_2$ is not complete (see Fig. 3.4).*

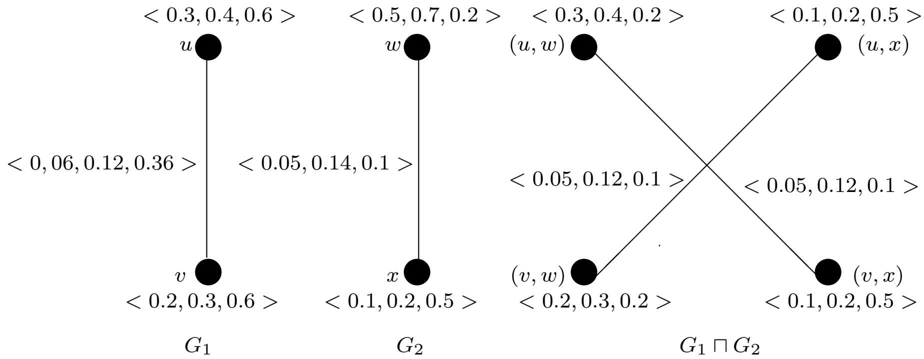

Figure 3.4: G_1 and G_2 are complete product 3-polar fuzzy graphs but $G_1 \sqcap G_2$ is not complete

Definition 3.3.5. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be the product m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. Then, the ring sum of G_1 and G_2 is denoted by $G = G_1 \oplus G_2 = (A_1 \oplus A_2, B_1 \oplus B_2)$ and defined as follows: for each $i = 1, 2, \ldots, m$*

(i) $p_i \circ (A_1 \oplus A_2)(u) = p_i \circ (A_1 \cup A_2)(u)$ *for all* $u \in V_1 \cup V_2$ *and*

(ii) $p_i \circ (B_1 \oplus B_2)(uv) = \begin{cases} p_i \circ B_1(uv) & if \quad uv \in E_1 - E_2 \\ p_i \circ B_2(uv) & if \quad uv \in E_2 - E_1 \\ 0 & otherwise. \end{cases}$

Proposition 3.3.2. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be the product m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. Then the ring sum $G = G_1 \oplus G_2 = (A_1 \oplus A_2, B_1 \oplus B_2)$ is a product m-polar fuzzy graph.*

Proof. We will show the following: for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \oplus B_2)(uv) \leq p_i \circ (A_1 \oplus A_2)(u) \times p_i \circ (A_1 \oplus A_2)(v)$ for all $uv \in E_1 \cup E_2$.

Case (i): Let $uv \in E_1 - E_2$ and $u, v \in V_1 - V_2$. Then for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \oplus B_2)(uv)$

$= p_i \circ B_1(uv)$

$\leq p_i \circ A_1(u) \times p_i \circ A_1(v)$

$= p_i \circ (A_1 \oplus A_2)(u) \times p_i \circ (A_1 \oplus A_2)(v).$

Case (ii): Let $uv \in E_1 - E_2$ and $u \in V_1 - V_2$, $v \in V_1 \cap V_2$.

Then for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \oplus B_2)(uv)$

$= p_i \circ B_1(uv)$

$\leq p_i \circ A_1(u) \times max\{p_i \circ A_1(v), p_i \circ A_2(v)\}$

$\leq p_i \circ (A_1 \cup A_2)(u) \times p_i \circ (A_1 \cup A_2)(v)$

$= p_i \circ (A_1 \oplus A_2)(u) \times p_i \circ (A_1 \oplus A_2)(v).$

Case (iii): Let $uv \in E_1 - E_2$ and $u, v \in V_1 \cap V_2$.

Then for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \oplus B_2)(uv)$

$= p_i \circ B_1(uv)$

$\leq max\{p_i \circ A_1(u), p_i \circ A_2(u)\} \times max\{p_i \circ A_1(v), p_i \circ A_2(v)\}$

$\leq p_i \circ (A_1 \cup A_2)(u) \times p_i \circ (A_1 \cup A_2)(v)$

$= p_i \circ (A_1 \oplus A_2)(u) \times p_i \circ (A_1 \oplus A_2)(v).$

Similarly, we can show that if $uv \in E_2 - E_1$, then also for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \oplus B_2)(uv) \leq p_i \circ (A_1 \oplus A_2)(u) \times p_i \circ (A_1 \oplus A_2)(v).$

Hence the result. □

3.4 Degrees of vertices in m-polar fuzzy graphs

In this section, we study about the degree of a vertex in m-polar fuzzy graphs which are obtained from two given m-polar fuzzy graphs G_1 and G_2 using the operations of Cartesian product, composition, direct product, semi-strong product and strong product.

3.4.1 Degree of a vertex in Cartesian product

Now, we compute the degree of a vertex in the Cartesian product. By the definition of Cartesian product, for any vertex $(x_1, x_2) \in V_1 \times V_2$, the degree of it is denoted by $d_{G_1 \times G_2}(x_1, x_2) = (p_1 \circ d_{G_1 \times G_2}(x_1, x_2), p_2 \circ d_{G_1 \times G_2}(x_1, x_2), \ldots, p_m \circ d_{G_1 \times G_2}(x_1, x_2))$ and is defined by

$$\begin{aligned}
p_i \circ d_{G_1 \times G_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E} p_i \circ (B_1 \times B_2)((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1=y_1, x_2y_2\in E_2} p_i \circ A_1(x_1) \wedge p_i \circ B_2(x_2y_2) \\
&+ \sum_{x_2=y_2, x_1y_1\in E_1} p_i \circ A_2(x_2) \wedge p_i \circ B_1(x_1y_1) \text{ for } i = 1, 2, \ldots, m.
\end{aligned}$$

Theorem 3.4.1. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs. If $B_2 \subseteq A_1$ and $B_1 \subseteq A_2$, then $d_{G_1\times G_2}(x_1, x_2) = d_{G_1}(x_1) + d_{G_2}(x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$.*

Proof. For each $i = 1, 2, \ldots, m$ we have,

$$\begin{aligned}
p_i \circ d_{G_1 \times G_2}(x_1, x_2) &= \sum_{x_1=y_1, x_2y_2\in E_2} p_i \circ A_1(x_1) \wedge p_i \circ B_2(x_2y_2) \\
&+ \sum_{x_2=y_2, x_1y_1\in E_1} p_i \circ A_2(x_2) \wedge p_i \circ B_1(x_1y_1) \\
&= \sum_{x_2y_2\in E_2} p_i \circ B_2(x_2y_2) + \sum_{x_1y_1\in E_1} p_i \circ B_1(x_1y_1) \\
&= p_i \circ d_{G_1}(x_1) + p_i \circ d_{G_2}(x_2).
\end{aligned}$$

Hence, $d_{G_1\times G_2}(x_1, x_2) = d_{G_1}(x_1) + d_{G_2}(x_2)$. □

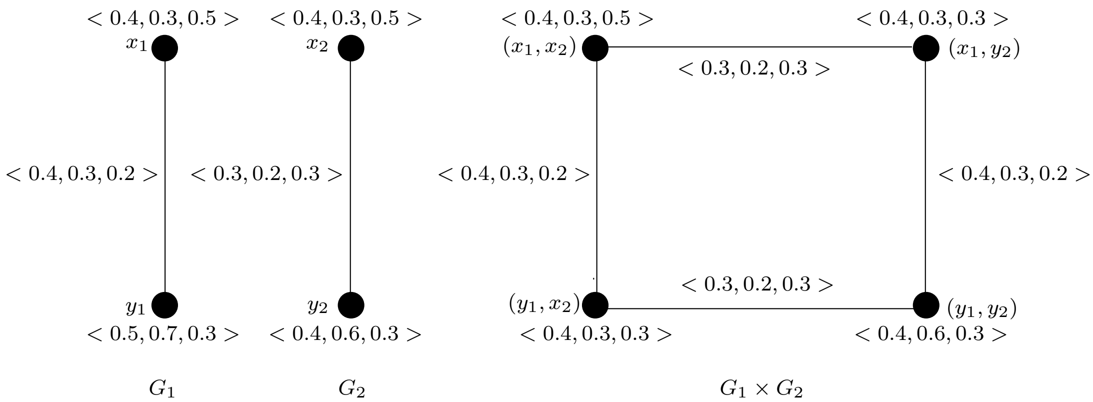

Figure 3.5: Cartesian product of G_1 and G_2

Example 3.4.1. *Let us consider the 3-polar fuzzy graphs G_1, G_2 and their Cartesian product $G_1 \times G_2$ (see Fig. 3.5). For this graph, $B_2 \subseteq A_1$ and $B_1 \subseteq A_2$. So, by Theorem 3.4.1,*

$$p_1 \circ d_{G_1\times G_2}(x_1, x_2) = p_1 \circ d_{G_1}(x_1) + p_1 \circ d_{G_2}(x_2)$$

$$= 0.4 + 0.3 = 0.7,$$

$$p_2 \circ d_{G_1 \times G_2}(x_1, x_2) = p_2 \circ d_{G_1}(x_1) + p_2 \circ d_{G_2}(x_2)$$
$$= 0.3 + 0.2 = 0.5,$$

$$p_3 \circ d_{G_1 \times G_2}(x_1, x_2) = p_3 \circ d_{G_1}(x_1) + p_3 \circ d_{G_2}(x_2)$$
$$= 0.2 + 0.3 = 0.5.$$

So, $d_{G_1 \times G_2}(x_1, x_2) = (0.7, 0.5, 0.5)$.

Also, from Fig. 3.5, $d_{G_1 \times G_2}(x_1, x_2)$

$$= (0.3 + 0.4, 0.2 + 0.3, 0.3 + 0.2)$$
$$= (0.7, 0.5, 0.5).$$

Hence, $d_{G_1 \times G_2}(x_1, x_2) = (0.7, 0.5, 0.5)$.

Similarly, we can find the degrees of all other vertices in $G_1 \times G_2$*. This can be verified from the Fig. 3.5 also.*

Theorem 3.4.2. *Let* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *be two m-polar fuzzy graphs such that* $A_1 \subseteq B_2$*, then* $B_1 \subseteq A_2$ *and conversely.*

Proof. By definition of m-polar fuzzy graphs, we have

$p_i \circ B_j(xy) \leq min\{p_i \circ A_j(x), p_i \circ A_j(y)\}$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$ and $j = 1, 2$.

Therefore, $p_i \circ B_j \leq max\{p_i \circ A_j\}$ and

$min\{p_i \circ B_j\} \leq p_i \circ A_j$ for $i = 1, 2, \ldots, m$ and $j = 1, 2$.

Also, since $A_1 \subseteq B_2$, $max\{p_i \circ A_1\} \leq min\{p_i \circ B_2\}$ for $i = 1, 2, \ldots, m$.

Hence, $p_i \circ B_1 \leq max\{p_i \circ A_1\}$

$\leq min\{p_i \circ B_2\}$

$\leq p_i \circ A_2$ for $i = 1, 2, \ldots, m$,

i.e. $B_1 \subseteq A_2$.

The converse part can be proved in a similar way. □

Theorem 3.4.3. *Let* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *be two m-polar fuzzy graphs.*

(i) If $A_1 \subseteq B_2$ *and* A_1 *is constant function with* $A_1(x) = (c_1, c_2, \ldots, c_m) = c$ *for all* $x \in V_1$*, then* $d_{G_1 \times G_2}(x_1, x_2) = d_{G_1}(x_1) + c d_{G_2^*}(x_2)$.

(ii) If $A_2 \subseteq B_1$ *and* A_2 *is constant function with* $A_2(x) = (k_1, k_2, \ldots, k_m) = k$ *for all* $x \in V_2$*, then* $d_{G_1 \times G_2}(x_1, x_2) = d_{G_2}(x_2) + k d_{G_1^*}(x_1)$.

Proof. (i) Because $A_1 \subseteq B_2$, by Theorem 3.4.2, $B_1 \subseteq A_2$.

Then for $i = 1, 2, \ldots, m$,

$$\begin{aligned}
p_i \circ d_{G_1 \times G_2}(x_1, x_2) &= \sum_{x_1 = y_1, x_2y_2 \in E_2} p_i \circ A_1(x_1) \wedge p_i \circ B_2(x_2y_2) \\
&+ \sum_{x_2 = y_2, x_1y_1 \in E_1} p_i \circ A_2(x_2) \wedge p_i \circ B_1(x_1y_1) \\
&= \sum_{x_2y_2 \in E_2} p_i \circ A_1(x_1) + \sum_{x_1y_1 \in E_1} p_i \circ B_1(x_1y_1) \\
&= \sum_{x_2y_2 \in E_2} c_i + p_i \circ d_{G_1}(x_1) \\
&= c_i d_{G_2^*}(x_2) + p_i \circ d_{G_1}(x_1).
\end{aligned}$$

Hence, $d_{G_1 \times G_2}(x_1, x_2) = d_{G_1}(x_1) + c d_{G_2^*}(x_2)$.

(ii) Proof is similar to the above case. □

3.4.2 Degree of a vertex in composition

Now, we calculate the degree of a vertex in the composition of two m-polar fuzzy graphs. By the definition of composition, for any vertex $(x_1, x_2) \in V_1 \times V_2$, the degree of it is denoted by $d_{G_1[G_2]}(x_1, x_2) = (p_1 \circ d_{G_1[G_2]}(x_1, x_2), p_2 \circ d_{G_1[G_2]}(x_1, x_2), \ldots, p_m \circ d_{G_1[G_2]}(x_1, x_2))$ and is defined by

$$\begin{aligned}
p_i \circ d_{G_1[G_2]}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E} p_i \circ (B_1 \circ B_2)((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1 = y_1, x_2y_2 \in E_2} p_i \circ A_1(x_1) \wedge p_i \circ B_2(x_2y_2) \\
&+ \sum_{x_2 = y_2, x_1y_1 \in E_1} p_i \circ A_2(x_2) \wedge p_i \circ B_1(x_1y_1) \\
&+ \sum_{x_2 \neq y_2, x_1y_1 \in E_1} p_i \circ A_2(x_2) \wedge p_i \circ A_2(y_2) \wedge p_i \circ B_1(x_1y_1)
\end{aligned}$$

for $i = 1, 2, \ldots, m$.

Theorem 3.4.4. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs. If $B_2 \subseteq A_1$ and $B_1 \subseteq A_2$, then $d_{G_1[G_2]}(x_1, x_2) = |V_2| d_{G_1}(x_1) + d_{G_2}(x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$.*

Proof. For each $i = 1, 2, \ldots, m$ we have,

$$\begin{aligned}
p_i \circ d_{G_1[G_2]}(x_1, x_2) &= \sum_{x_1=y_1, x_2y_2 \in E_2} p_i \circ A_1(x_1) \wedge p_i \circ B_2(x_2y_2) \\
&+ \sum_{x_2=y_2, x_1y_1 \in E_1} p_i \circ A_2(x_2) \wedge p_i \circ B_1(x_1y_1) \\
&+ \sum_{x_2 \neq y_2, x_1y_1 \in E_1} p_i \circ A_2(x_2) \wedge p_i \circ A_2(y_2) \wedge p_i \circ B_1(x_1y_1) \\
&= \sum_{x_2y_2 \in E_2} p_i \circ B_2(x_2y_2) + \sum_{x_2=y_2, x_1y_1 \in E_1} p_i \circ B_1(x_1y_1) \\
&+ \sum_{x_2 \neq y_2, x_1y_1 \in E_1} p_i \circ B_1(x_1y_1) \\
&\text{(Since } p_i \circ A_1 \geq p_i \circ B_2 \text{ and } p_i \circ A_2 \geq p_i \circ B_1) \\
&= p_i \circ d_{G_2}(x_2) + |V_2| p_i \circ d_{G_1}(x_1).
\end{aligned}$$

Hence, $d_{G_1[G_2]}(x_1, x_2) = |V_2| d_{G_1}(x_1) + d_{G_2}(x_2)$. □

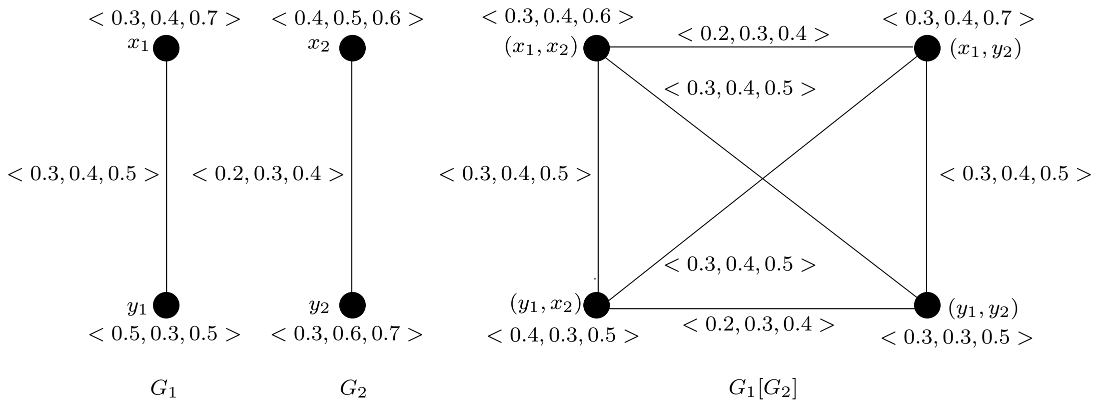

Figure 3.6: Composition of G_1 and G_2

Example 3.4.2. *Consider the 3-polar fuzzy graphs G_1, G_2 and their composition $G_1[G_2]$ (see Fig. 3.6). Here, $B_2 \subseteq A_1$ and $B_1 \subseteq A_2$. Therefore, by Theorem 3.4.4 we have,*

$p_1 \circ d_{G_1[G_2]}(x_1, x_2) = p_1 \circ d_{G_1}(x_1)|V_2| + p_1 \circ d_{G_2}(x_2) = 0.3 \times 2 + 0.2 = 0.8,$

$p_2 \circ d_{G_1[G_2]}(x_1, x_2) = p_2 \circ d_{G_1}(x_1)|V_2| + p_2 \circ d_{G_2}(x_2) = 0.4 \times 2 + 0.3 = 1.1,$

$p_3 \circ d_{G_1[G_2]}(x_1, x_2) = p_3 \circ d_{G_1}(x_1)|V_2| + p_3 \circ d_{G_2}(x_2) = 0.5 \times 2 + 0.4 = 1.4.$

Therefore, $d_{G_1[G_2]}(x_1, x_2) = (0.8, 1.1, 1.4)$.

Again from the Fig. 3.6,

$d_{G_1[G_2]}(x_1, x_2) = (p_1 \circ d_{G_1[G_2]}(x_1, x_2), p_2 \circ d_{G_1[G_2]}(x_1, x_2), p_3 \circ d_{G_1[G_2]}(x_1, x_2))$

$= (0.3 + 0.2 + 0.3, 0.4 + 0.3 + 0.4, 0.5 + 0.4 + 0.5) = (0.8, 1.1, 1.4).$

In the same way, we can find the degree of all vertices in $G_1[G_2]$. This can be verified from the Fig. 3.6.

Theorem 3.4.5. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs.*

(i) If $A_1 \subseteq B_2$ and A_1 is constant function with $A_1(x) = (c_1, c_2, \ldots, c_m) = c$ for all $x \in V_1$, then $d_{G_1[G_2]}(x_1, x_2) = |V_2|d_{G_1}(x_1) + cd_{G_2^*}(x_2)$.

(ii) If $A_2 \subseteq B_1$ and A_2 is constant function with $A_2(x) = (k_1, k_2, \ldots, k_m) = k$ for all $x \in V_2$, then $d_{G_1[G_2]}(x_1, x_2) = d_{G_2}(x_2) + k|V_2|d_{G_1^*}(x_1)$.

Proof. (i) Because $A_1 \subseteq B_2$, by Theorem 3.4.2, $B_1 \subseteq A_2$. Now for $i = 1, 2, \ldots, m$ we have,

$$\begin{aligned}
p_i \circ d_{G_1[G_2]}(x_1, x_2) &= \sum_{x_1=y_1, x_2y_2 \in E_2} p_i \circ A_1(x_1) \wedge p_i \circ B_2(x_2y_2) \\
&+ \sum_{x_2=y_2, x_1y_1 \in E_1} p_i \circ A_2(x_2) \wedge p_i \circ B_1(x_1y_1) \\
&+ \sum_{x_2 \neq y_2, x_1y_1 \in E_1} p_i \circ A_2(x_2) \wedge p_i \circ A_2(y_2) \wedge p_i \circ B_1(x_1y_1) \\
&= \sum_{x_2y_2 \in E_2} p_i \circ A_1(x_1) + \sum_{x_2=y_2, x_1y_1 \in E_1} p_i \circ B_1(x_1y_1) \\
&+ \sum_{x_2 \neq y_2, x_1y_1 \in E_1} p_i \circ B_1(x_1y_1) \\
&= \sum_{x_2y_2 \in E_2} c_i + |V_2| \sum_{x_1y_1 \in E_1} p_i \circ B_1(x_1y_1) \\
&= c_i d_{G_2^*}(x_2) + |V_2| p_i \circ d_{G_1}(x_1).
\end{aligned}$$

Hence, $d_{G_1[G_2]}(x_1, x_2) = |V_2|d_{G_1}(x_1) + cd_{G_2^*}(x_2)$.

(ii) Similarly to the above case. □

3.4.3 Degree of a vertex in direct product

Degree of a vertex in the direct product is as follows. By definition of direct product for any vertex $(x_1, x_2) \in V_1 \times V_2$, the degree of (x_1, x_2) is denoted by $d_{G_1 \sqcap G_2}(x_1, x_2) = (p_1 \circ d_{G_1 \sqcap G_2}(x_1, x_2), p_2 \circ d_{G_1 \sqcap G_2}(x_1, x_2), \ldots, p_m \circ d_{G_1 \sqcap G_2}(x_1, x_2))$ and is defined by

$$\begin{aligned}
p_i \circ d_{G_1 \sqcap G_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E} p_i \circ (B_1 \sqcap B_2)((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} p_i \circ B_1(x_1y_1) \wedge p_i \circ B_2(x_2y_2) \text{ for } i = 1, 2, \ldots, m.
\end{aligned}$$

Theorem 3.4.6. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs. If $B_1 \subseteq B_2$, then $d_{G_1 \sqcap G_2}(x_1, x_2) = d_{G_1}(x_1)$. Also, if $B_2 \subseteq B_1$, then $d_{G_1 \sqcap G_2}(x_1, x_2) = d_{G_2}(x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$.*

Proof. Let $B_1 \subseteq B_2$ i.e., $p_i \circ B_2 \geq p_i \circ B_1$ for each $i = 1, 2, \ldots, m$. Then we have,

$$\begin{aligned} p_i \circ d_{G_1 \sqcap G_2}(x_1, x_2) &= \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} p_i \circ B_1(x_1y_1) \wedge p_i \circ B_2(x_2y_2) \\ &= \sum_{x_1y_1 \in E_1} p_i \circ B_1(x_1y_1) = p_i \circ d_{G_1}(x_1) \quad \text{for } i = 1, 2, \ldots, m. \end{aligned}$$

Hence, $d_{G_1 \sqcap G_2}(x_1, x_2) = d_{G_1}(x_1)$.

Similarly, if $B_2 \subseteq B_1$ then $d_{G_1 \sqcap G_2}(x_1, x_2) = d_{G_2}(x_2)$. □

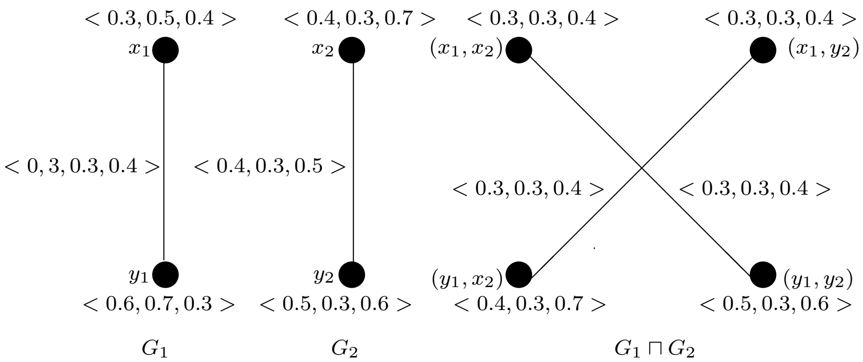

Figure 3.7: The direct product of G_1 and G_2

Example 3.4.3. *In this example we consider the direct product of two 3-polar fuzzy graphs and calculate the degree of vertices in the direct product. Let us now consider the 3-polar fuzzy graphs G_1, G_2 and their direct product $G_1 \sqcap G_2$ (see Fig. 3.7). Here, we see that $p_i \circ B_2 \geq p_i \circ B_1$ for $i = 1, 2, 3$, i.e. $B_1 \subseteq B_2$. Hence by Theorem 3.4.6, $p_1 \circ d_{G_1 \sqcap G_2}(x_1, x_2) = p_1 \circ d_{G_1}(x_1) = 0.3$, $p_2 \circ d_{G_1 \sqcap G_2}(x_1, x_2) = p_2 \circ d_{G_1}(x_1) = 0.3$, $p_3 \circ d_{G_1 \sqcap G_2}(x_1, x_2) = p_3 \circ d_{G_1}(x_1) = 0.4$. So, $d_{G_1 \sqcap G_2}(x_1, x_2) = (0.3, 0.3, 0.4)$. Similarly, we can find the degree of all other vertices in $G_1 \sqcap G_2$. This can also be verified from Fig. 3.7.*

3.4.4 Degree of a vertex in semi-strong product

Next, we consider the semi-strong product of two m-polar fuzzy graphs and calculate the degree of vertices of it. For any vertex vertex $(x_1, x_2) \in V_1 \times V_2$ in the semi-strong product $G_1 \bullet G_2$, the degree of (x_1, x_2) is denoted by $d_{G_1 \bullet G_2}(x_1, x_2) = (p_1 \circ$

$d_{G_1 \bullet G_2}(x_1, x_2), p_2 \circ d_{G_1 \bullet G_2}(x_1, x_2), \ldots, p_m \circ d_{G_1 \bullet G_2}(x_1, x_2))$ and is defined by

$$\begin{aligned} p_i \circ d_{G_1 \bullet G_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E} p_i \circ (B_1 \bullet B_2)((x_1, x_2)(y_1, y_2)) \\ &= \sum_{x_1 = y_1, x_2y_2 \in E_2} p_i \circ A_1(x_1) \wedge p_i \circ B_2(x_2y_2) \\ &+ \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} p_i \circ B_1(x_1y_1) \wedge p_i \circ B_2(x_2y_2) \text{ for } i = 1, 2, \ldots, m. \end{aligned}$$

Theorem 3.4.7. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs. If $B_1 \subseteq B_2 \subseteq A_1$, then $d_{G_1 \bullet G_2}(x_1, x_2) = d_{G_1}(x_1) + d_{G_2}(x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$.*

Proof. Let $B_1 \subseteq B_2 \subseteq A_1$, i.e. $p_i \circ A_1 \geq p_i \circ B_2 \geq p_i \circ B_1$ for each $i = 1, 2, \ldots, m$. Then, for $i = 1, 2, \ldots, m$ and $(x_1, x_2) \in V_1 \times V_2$,

$$\begin{aligned} p_i \circ d_{G_1 \bullet G_2}(x_1, x_2) &= \sum_{x_1 = y_1, x_2y_2 \in E_2} p_i \circ A_1(x_1) \wedge p_i \bullet B_2(x_2y_2) \\ &+ \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} p_i \circ B_1(x_1y_1) \wedge p_i \bullet B_2(x_2y_2) \\ &= \sum_{x_2y_2 \in E_2} p_i \circ B_2(x_2y_2) + \sum_{x_1y_1 \in E_1} p_i \circ B_1(x_1y_1) \\ &= p_i \circ d_{G_2}(x_2) + p_i \circ d_{G_1}(x_1). \end{aligned}$$

Hence, $d_{G_1 \bullet G_2}(x_1, x_2) = d_{G_1}(x_1) + d_{G_2}(x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$. □

Example 3.4.4. *Consider the 3-polar fuzzy graphs G_1, G_2 and their semi-strong product $G_1 \bullet G_2$ (see Fig. 3.8). Here, we see that $p_i \circ A_1 \geq p_i \circ B_2 \geq p_i \circ B_1$ for $i = 1, 2, 3$, i.e. $B_1 \subseteq B_2 \subseteq A_1$. Hence by Theorem 3.4.7, we have*

$p_1 \circ d_{G_1 \bullet G_2}(x_1, x_2) = p_1 \circ d_{G_1}(x_1) + p_1 \circ d_{G_2}(x_2) = 0.2 + 0.2 = 0.4,$

$p_2 \circ d_{G_1 \bullet G_2}(x_1, x_2) = p_2 \circ d_{G_1}(x_1) + p_2 \circ d_{G_2}(x_2) = 0.2 + 0.3 = 0.5,$

$p_3 \circ d_{G_1 \bullet G_2}(x_1, x_2) = p_3 \circ d_{G_1}(x_1) + p_3 \circ d_{G_2}(x_2) = 0.3 + 0.4 = 0.7.$

So, $d_{G_1 \sqcap G_2}(x_1, x_2) = (0.4, 0.5, 0.7)$.

Again, from the Fig. 3.8, we have

$d_{G_1 \sqcap G_2}(x_1, x_2) = (0.2 + 0.2, 0.2 + 0.3, 0.3 + 0.4) = (0.4, 0.5, 0.7).$

Similarly, we can find the degrees of all vertices in $G_1 \bullet G_2$ which can be verified from the figure also.

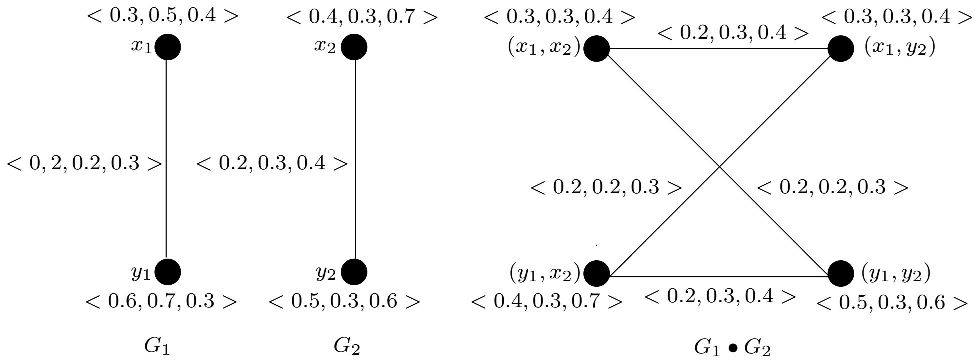

Figure 3.8: Semi-strong product of G_1 and G_2

3.4.5 Degree of a vertex in strong product

Finally, we compute the degree of a vertex in strong product of m-polar fuzzy graphs. By definition of strong product, for any vertex $(x_1, x_2) \in V_1 \times V_2$ in $G_1 \otimes G_2$, the degree of (x_1, x_2) is denoted by $d_{G_1 \otimes G_2}(x_1, x_2) = (p_1 \circ d_{G_1 \otimes G_2}(x_1, x_2), p_2 \circ d_{G_1 \otimes G_2}(x_1, x_2), \ldots, p_m \circ d_{G_1 \otimes G_2}(x_1, x_2))$ and is defined by

$$\begin{aligned}
p_i \circ d_{G_1 \otimes G_2}(x_1, x_2) = & \sum_{(x_1,x_2)(y_1,y_2) \in E} p_i \circ (B_1 \otimes B_2)((x_1, x_2)(y_1, y_2)) \\
= & \sum_{x_1 = y_1, x_2 y_2 \in E_2} p_i \circ A_1(x_1) \wedge p_i \circ B_2(x_2 y_2) \\
+ & \sum_{x_2 = y_2, x_1 y_1 \in E_1} p_i \circ A_2(x_2) \wedge p_i \circ B_1(x_1 y_1) \\
+ & \sum_{x_1 y_1 \in E_1, x_2 y_2 \in E_2} p_i \circ B_1(x_1 y_1) \wedge p_i \circ B_2(x_2 y_2) \text{ for } i = 1, 2, \ldots, m.
\end{aligned}$$

Theorem 3.4.8. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs. If $B_2 \subseteq A_1$, $B_1 \subseteq A_2$ and $B_1 \subseteq B_2$, then $d_{G_1 \otimes G_2}(x_1, x_2) = |V_2| d_{G_1}(x_1) + d_{G_2}(x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$.*

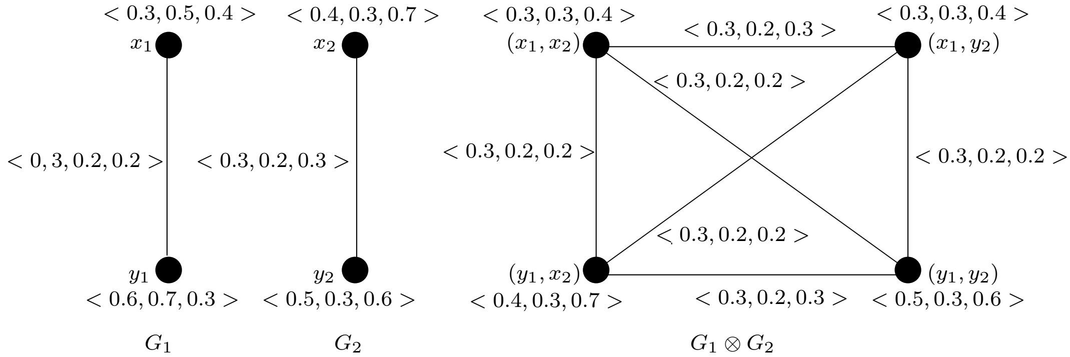

Figure 3.9: Strong product of G_1 and G_2

Proof. For $i = 1, 2, \ldots, m$ and $(x_1, x_2) \in V_1 \times V_2$ we have,

$$\begin{aligned}
p_i \circ d_{G_1 \otimes G_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E} p_i \circ (B_1 \otimes B_2)((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1 = y_1, x_2y_2 \in E_2} p_i \circ A_1(x_1) \wedge p_i \circ B_2(x_2y_2) \\
&+ \sum_{x_2 = y_2, x_1y_1 \in E_1} p_i \circ A_2(x_2) \wedge p_i \circ B_1(x_1y_1) \\
&+ \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} p_i \circ B_1(x_1y_1) \wedge p_i \circ B_2(x_2y_2) \\
&= \sum_{x_2y_2 \in E_2} p_i \circ B_2(x_2y_2) + \sum_{x_2 = y_2, x_1y_1 \in E_1} p_i \circ B_1(x_1y_1) \\
&+ \sum_{x_1y_1 \in E_1} p_i \circ B_1(x_1y_1) \\
&= p_i \circ d_{G_2}(x_2) + |V_2| p_i \circ d_{G_1}(x_1).
\end{aligned}$$

This shows that, $d_{G_1 \otimes G_2}(x_1, x_2) = |V_2| d_{G_1}(x_1) + d_{G_2}(x_2)$. □

Example 3.4.5. *Let us consider the 3-polar fuzzy graphs G_1, G_2 and their strong product $G_1 \otimes G_2$ (see Fig. 3.9). Here, $p_i \circ A_1 \geq p_i \circ B_2$, $p_i \circ A_2 \geq p_i \circ B_1$ and $p_i \circ B_1 \leq p_i \circ B_2$ for $i = 1, 2, 3$, i.e. $B_2 \subseteq A_1$, $B_1 \subseteq A_2$ and $B_1 \subseteq B_2$. Hence by Theorem 3.4.8, we have*

$p_1 \circ d_{G_1 \otimes G_2}(x_1, x_2) = p_1 \circ d_{G_2}(x_2) + |V_2| p_1 \circ d_{G_1}(x_1) = 0.3 + 2 \times 0.3 = 0.9,$

$p_2 \circ d_{G_1 \otimes G_2}(x_1, x_2) = p_2 \circ d_{G_2}(x_2) + |V_2| p_2 \circ d_{G_1}(x_1) = 0.2 + 2 \times 0.2 = 0.6,$

$p_3 \circ d_{G_1 \otimes G_2}(x_1, x_2) = p_3 \circ d_{G_2}(x_2) + |V_2| p_3 \circ d_{G_1}(x_1) = 0.3 + 2 \times 0.2 = 0.7.$

So, $d_{G_1 \otimes G_2}(x_1, x_2) = (0.9, 0.6, 0.7)$. Again, from the Fig. 3.9 we see that,

$d_{G_1 \otimes G_2}(x_1, x_2) = (0.3 + 0.3 + 0.3, 0.2 + 0.2 + 0.2, 0.3 + 0.2 + 0.2) = (0.9, 0.6, 0.7).$

Similarly, we can find the degrees of all vertices in the strong product from the Theorem 3.4.8 as well as from the Fig. 3.9 directly.

3.5 3-polar fuzzy influence graphs

A directed graph (or digraph) is a graph whose edges have direction and called arcs (edges). Arrows on arcs are used to encode the directional information: an arc from the vertex x to the vertex y indicates that one may move from x to y but not from y to x. We write $xy \in E$ to mean $x \to y \in E$, and if $e = xy \in E$, we say x and y are adjacent such that x is a starting node and y is an ending node.

Definition 3.5.1. *An m-polar fuzzy digraph of a digraph $G^* = (V, E)$ is a pair $G = (V, A, B)$, where $A : V \to [0,1]^m$ is an m-polar fuzzy set on V and $B : \widetilde{V^2} \to [0,1]^m$ is an m-polar fuzzy set in $\widetilde{V^2}$ such that $p_i \circ B(xy) \leq min\{p_i \circ A(x), p_i \circ A(y)\}$ for all $xy \in \widetilde{V^2}$, for each $i = 1, 2, \ldots, m$ and $B(xy) = \mathbf{0}$ for all $xy \in (\widetilde{V^2} - E)$. B need not be symmetric, i.e. $B(xy) \neq B(yx)$.*

Graph models have broad application in many disciplines of mathematics, social sciences, natural sciences and computer sciences. In studies of group behavior, it is inspected that many people can influence thinking of others. A digraph can be use to model such behavior and this graph is called an influence graph. We will present the influence of a person in a social group on Gtalk.

Let V={Asit, Sankar, Kartik, Prabir, Shakti} be the set of five persons in a social group. The influence degree depends on the legitimate prevailing, unity building, and appealing to values. Then, we have a 3-polar fuzzy influence graph $G = (V, A, B)$, where vertices represent the person of a social group and edges represent the influence of a person on other. From the above Fig. 3.10, we see that Kartik influence Asit, Sankar and Prabir. Kartik's 60% hold on Asit is due to legitimate prevailing, 40% is due to unity building, 50% is due to appealing to values. His 70% hold on Sankar is due to legitimate prevailing, 60% is due to unity building, 50% is due to appealing to values. Similarly, for Prabir also. Asit influence Sankar, Sankar influence Shakti and Prabir. So, we observe that Kartik is the most influential person in the group.

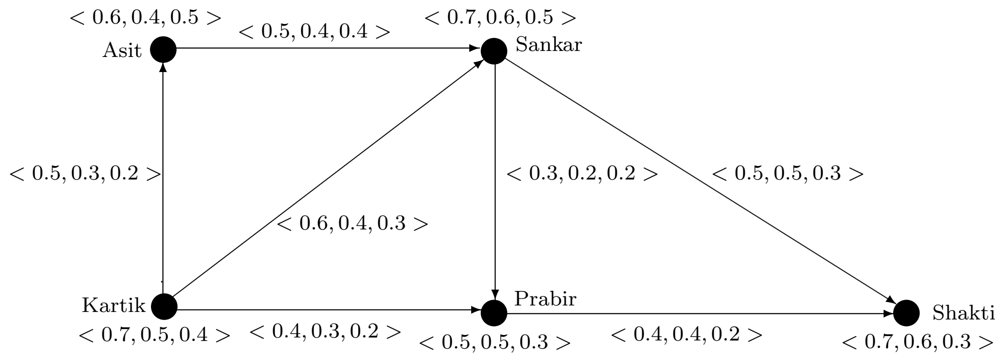

Figure 3.10: 3-polar fuzzy influence graph

3.6 Summary

The main goal of this chapter is to define three new operations on m-polar fuzzy graph such as direct product, semi-strong product and strong product, and study their properties. Some subclasses of m-polar fuzzy graphs, namely the product m-polar fuzzy graphs are also introduced here. Some operations like union, direct product, ring sum are defined to construct new product m-polar fuzzy graphs. We have calculated the degree of vertices in $G_1 \times G_2, G_1 \circ G_2, G_1 \otimes G_2$ and $G_1 \bullet G_2$ in terms of the degree of vertices of the graphs G_1 and G_2 under some conditions. This will be helpful when the graphs are very large. The degrees and edges of any graph are very important parameters. The number of edges is not evaluated in this chapter. Finally, 3-polar fuzzy influence graph is introduced as an applications.

Chapter 4

Density of m-polar fuzzy graphs

4.1 Introduction

The density of a crisp graph $G^* = (V, E)$ is defined by $D(G^*) = \frac{2\sum|E|}{|V|(|V|-1)}$. This gives the number of edges per unit vertex. $D(G^*)$ is non-negative for any graph G^* and its maximum value is 1, when G^* is complete. Thus, $0 \leq D(G^*) \leq 1$. Higher value of $D(G^*)$ represent more edges in G^*. If G^* has no edges, then $D(G^*)$ is 0. Density $D(G^*)$ of a graph G^* is concerned with the patterns of connections of the entire networks. Graphs for which $D(H) \leq D(G)$ for all subgraph H of G, are called balanced graph. Balanced graph first arose in the study of random graphs and balanced m-polar fuzzy graphs defined here is based on density functions. A graph with maximum density is complete and graph with minimum density is a null graph. There are several papers written on balanced extension of graph which has tremendous applications in artificial intelligence, signal processing, robotics, computer networks and decision making. Al-Hawary [10] first introduced the concept of balanced fuzzy graphs. In this chapter, the density of an m-polar fuzzy graph is defined and studied the notion of balanced m-polar fuzzy graph and established necessary and sufficient conditions for the preceding products of two balanced m-polar fuzzy graphs to be balanced.

4.2 m-polar fuzzy graphs and its subgraphs

Definition 4.2.1. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of $G^* = (V, E)$. The m-polar fuzzy graph $H = (P, C, D)$ is called an m-polar fuzzy subgraph of G induced by P if $P \subseteq V$, $C(x) = A(x)$ for all $x \in P$ and $D(xy) = B(xy)$ for all $xy \in \widetilde{P^2}$.*

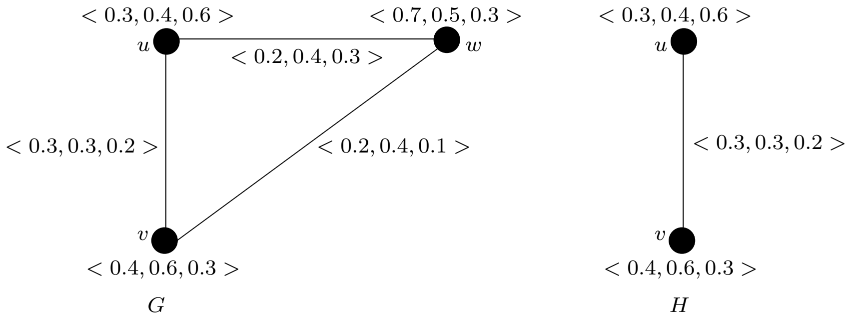

Figure 4.1: Example of 3-polar fuzzy subgraph of the graph G

Example 4.2.1. *H is a 3-polar fuzzy subgraph of G (see Fig. 4.1).*

4.3 Balanced m-polar fuzzy graphs

This section is began by defining the density of an m-polar fuzzy graph and balanced m-polar fuzzy graphs. Then it is proved that any complete m-polar fuzzy graph is balanced, but the converse is not true always.

Definition 4.3.1. *The density of an m-polar fuzzy graph $G = (V, A, B)$ of $G^* = (V, E)$ is $D(G) = (p_1 \circ D(G), p_2 \circ D(G), \ldots, p_m \circ D(G))$, where for each $i = 1, 2, \ldots, m$*

$$p_i \circ D(G) = \frac{2(\sum\limits_{u,v \in V} p_i \circ B(uv))}{\sum\limits_{u,v \in V} (p_i \circ A(u) \wedge p_i \circ A(v))}.$$

G is said to be balanced if $p_i \circ D(H) \leq p_i \circ D(G)$ for all non-empty subgraphs H of G, $i = 1, 2, \ldots, m$.

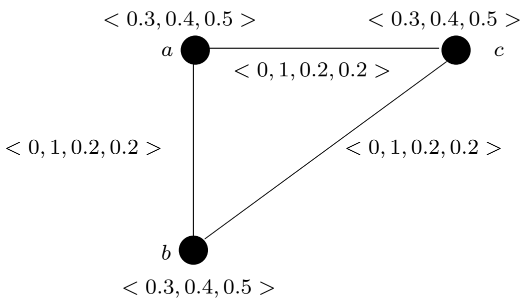

Figure 4.2: 3-polar fuzzy balanced graph G

Example 4.3.1. *Consider the 3-polar fuzzy graph $G = (V, A, B)$ of $G^* = (V, E)$ where $V = \{a, b, c\}$, $E = \{ab, bc, ca\}$, $A = \{\frac{<0.3,0.4,0.5>}{a}, \frac{<0.3,0.4,0.5>}{b}, \frac{<0.3,0.4,0.5>}{c}\}$,*

$B = \{\frac{<0.1,0.2,0.2>}{ab}, \frac{<0.1,0.2,0.2>}{bc}, \frac{<0.1,0.2,0.2>}{ca}\}$. We have,

$p_1 \circ D(G) = \frac{2(p_1 \circ B(ab) + p_1 \circ B(bc) + p_1 \circ B(ca))}{(p_1 \circ A(a) \wedge p_1 \circ A(b) + p_1 \circ A(b) \wedge p_1 \circ A(c) + p_1 \circ A(c) \wedge p_1 \circ A(a))} = \frac{2(0.1+0.1+0.1)}{0.3+0.3+0.3} = 0.67.$

Similarly, $p_2 \circ D(G) = 1$ *and* $p_3 \circ D(G) = 0.8$. *Hence,* $D(G) = (0.67, 1, 0.8)$.

The non-empty subgraphs of G *are* $H_1 = \{a, b\}$, $H_2 = \{b, c\}$ *and* $H_3 = \{c, a\}$.

Then $D(H_1) = (\frac{2\times 0.1}{0.3}, \frac{2\times 0.2}{0.4}, \frac{2\times 0.2}{0.5}) = (0.67, 1, 0.8)$,

$D(H_2) = (\frac{2\times 0.1}{0.3}, \frac{2\times 0.2}{0.4}, \frac{2\times 0.2}{0.5}) = (0.67, 1, 0.8)$ *and*

$D(H_3) = (\frac{2\times 0.1}{0.3}, \frac{2\times 0.2}{0.4}, \frac{2\times 0.2}{0.5}) = (0.67, 1, 0.8)$.

We see that, $D(H_1) = D(H_2) = D(H_3) = D(G) = (0.67, 1, 0.8)$.

Hence, G *is a balanced 3-polar fuzzy graph (see Fig. 4.2).*

Definition 4.3.2. *An* m*-polar fuzzy graph* G *is said to be strictly balanced if* $p_i \circ D(H) = p_i \circ D(G)$ *for all non-empty subgraphs* H *of* G, $i = 1, 2, \ldots, m$.

Example 4.3.2. *The 3-polar fuzzy graph of Fig. 4.2 is actually a strictly balanced 3-polar fuzzy graph.*

Theorem 4.3.1. *Any complete* m*-polar fuzzy graph is balanced.*

Proof. Let $G = (V, A, B)$ be a complete m-polar fuzzy graph and H be a non-empty subgraph of G. Then for each $i = 1, 2, \ldots, m$

$$p_i \circ D(G) = \frac{2(\sum\limits_{u,v\in V} p_i \circ B(uv))}{\sum\limits_{u,v\in V}(p_i \circ A(u) \wedge p_i \circ A(v))} = \frac{2(\sum\limits_{u,v\in V} p_i \circ A(u) \wedge p_\circ A(v))}{\sum\limits_{u,v\in V}(p_i \circ A(u) \wedge p_i \circ A(v))} = 2$$

and

$$p_i \circ D(H) = \frac{2(\sum\limits_{u,v\in V(H)} p_i \circ B(uv))}{\sum\limits_{u,v\in V(H)}(p_i \circ A(u) \wedge p_i \circ A(v))} \leq \frac{2(\sum\limits_{u,v\in V(H)} p_i \circ A(u) \wedge p_i \circ A(v))}{\sum\limits_{u,v\in V(H)}(p_i \circ A(u) \wedge p_i \circ A(v))} = 2$$

(where $V(H)$ represents the vertex of H). This shows that G is balanced. □

The converse of the above theorem is not true always. For example, the 3-polar fuzzy graph in Fig. 4.2 is balanced but not complete.

Below we will discuss two types of m-polar fuzzy graphs each with density equal to $\mathbf{1} = (1, 1, \ldots, 1)$.

Theorem 4.3.2. *Every self-complementary strong* m*-polar fuzzy graph has density equal to* $\mathbf{1} = (1, 1, \ldots, 1)$.

Proof. Let $G = (V, A, B)$ be a self-complementary strong m-polar fuzzy graph of $G^* = (V, E)$. Then by Proposition 6.12 of [45], we have for each $i = 1, 2, \ldots, m$ and $xy \in \widetilde{V^2}$,

$$\sum_{x \neq y} p_i \circ B(xy) = \frac{1}{2} \sum_{x \neq y} (p_i \circ A(x) \wedge p_i \circ A(y)).$$

Hence,

$$p_i \circ D(G) = \frac{2(\sum\limits_{u,v \in V} p_i \circ B(uv))}{\sum\limits_{u,v \in V} (p_i \circ A(u) \wedge p_i \circ A(v))} = 1(\text{by the above})$$

for each $i = 1, 2, \ldots, m$. Thus, $D(G) = \mathbf{1}$. □

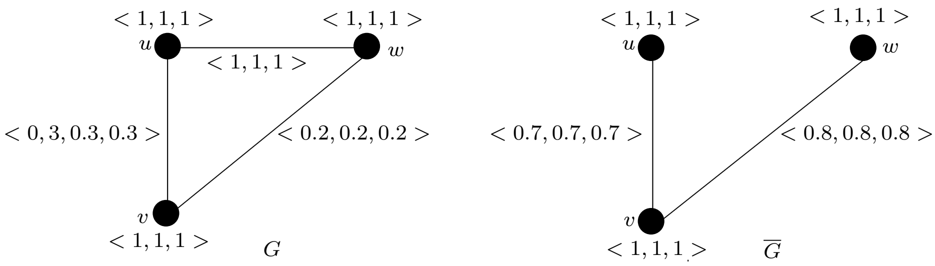

Figure 4.3: G is a 3-polar fuzzy graph with density $(1, 1, 1)$ but not self-complementary and strong

The converse of this theorem is not true in general. For example, the 3-polar fuzzy graph in Fig. 4.3 has density equal to (1,1,1), but it is not self-complementary strong. Here, we see that $D(G) = (1, 1, 1)$ but $G \not\cong \overline{G}$.

Theorem 4.3.3. *Let $G = (V, A, B)$ be a strictly balanced m-polar fuzzy graph and let $\overline{G} = (V, \overline{A}, \overline{B})$ be its complement. Then, $D(G) + D(\overline{G}) = (2, 2, \ldots, 2)$.*

Proof. Let H be any nonempty subgraph of G.

Since G is strictly balanced $p_i \circ D(H) = p_i \circ D(G)$ for every $H \subseteq G$, $i = 1, 2, \ldots, m$.

Now, for all $uv \in \widetilde{V^2}$ and $i = 1, 2, \ldots, m$ we have

$p_i \circ \overline{B}(uv) = p_i \circ A(u) \wedge p_i \circ A(v) - p_i \circ B(uv)$,

i.e. $\frac{p_i \circ \overline{B}(uv)}{p_i \circ A(u) \wedge p_i \circ A(v)} = 1 - \frac{p_i \circ B(uv)}{p_i \circ A(u) \wedge p_i \circ A(v)}$,

i.e. $\sum\limits_{u,v \in V} \frac{p_i \circ \overline{B}(uv)}{p_i \circ A(u) \wedge p_i \circ A(v)} = 1 - \sum\limits_{u,v \in V} \frac{p_i \circ B(uv)}{p_i \circ A(u) \wedge p_i \circ A(v)}$,

i.e. $2(\sum\limits_{u,v \in V} \frac{p_i \circ \overline{B}(uv)}{p_i \circ A(u) \wedge p_i \circ A(v)}) = 2 - 2(\sum\limits_{u,v \in V} \frac{p_i \circ B(uv)}{p_i \circ A(u) \wedge p_i \circ A(v)})$,

i.e. $p_i \circ D(\overline{G}) = 2 - p_i \circ D(G)$, i.e. $p_i \circ D(\overline{G}) + p_i \circ D(G) = 2$,

i.e. $D(\overline{G}) + D(G) = (2, 2, \ldots, 2)$.

This completes the proof. □

Theorem 4.3.4. *The complement of strictly balanced m-polar fuzzy graph is strictly balanced.*

Proof. Follows from the definition. □

Theorem 4.3.5. *Let $G = (V, A, B)$ be a strong m-polar fuzzy graph such that for each $i = 1, 2, \ldots, m$ and $uv \in \widetilde{V^2}$, $p_i \circ B(uv) = \frac{1}{2}(p_i \circ A(u) \wedge p_i \circ A(v))$. Then, $D(G) = \mathbf{1} = (1, 1, \ldots, 1)$.*

Proof. Since $G = (V, A, B)$ is a strong m-polar fuzzy graph such that for each $i = 1, 2, \ldots, m$ and $uv \in \widetilde{V^2}$, $p_i \circ B(uv) = \frac{1}{2}(p_i \circ A(u) \wedge p_i \circ A(v))$, therefore by Proposition 6.13 of [45], we have G is self-complementary. Hence, by Theorem 4.3.2, it follows that $D(G) = \mathbf{1}$. □

Next, necessary and sufficient conditions are established for the direct product, semi-strong product and strong product of two m-polar fuzzy graphs to be balanced.

Theorem 4.3.6. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. Then, $D(G_k) \leq D(G_1 \sqcap G_2)$ for $k = 1, 2$ if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.*

Proof. Let $D(G_k) \leq D(G_1 \sqcap G_2)$ for $k = 1, 2$. Then for $i = 1, 2, \ldots, m$

$$p_i \circ D(G_1) = \frac{2(\sum\limits_{u_1, u_2 \in V_1} p_i \circ B_1(u_1 u_2))}{\sum\limits_{u_1, u_2 \in V_1} (p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2))}$$

$$\geq \frac{2(\sum\limits_{\substack{u_1, u_2 \in V_1 \\ v_1, v_2 \in V_2}} p_i \circ B_1(u_1 u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2))}{\sum\limits_{\substack{u_1, u_2 \in V_1 \\ v_1, v_2 \in V_2}} (p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2))}$$

$$= \frac{2(\sum\limits_{\substack{u_1, u_2 \in V_1 \\ v_1, v_2 \in V_2}} p_i \circ B_1(u_1 u_2) \wedge p_i \circ B_2(v_1 v_2))}{\sum\limits_{\substack{u_1, u_2 \in V_1 \\ v_1, v_2 \in V_2}} (p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2))}$$

$$= \frac{2(\sum\limits_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} p_i\circ(B_1\sqcap B_2)(u_1,v_1)(u_2,v_2))}{\sum\limits_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (p_i\circ(A_1\sqcap A_2)(u_1,v_1)\wedge p_i\circ(A_1\sqcap A_2)(u_2,v_2))} = p_i\circ D(G_1\sqcap G_2).$$

Hence, $p_i\circ D(G_1)\geq p_i\circ D(G_1\sqcap G_2)$ for each $i=1,2,\dots,m$,

i.e $D(G_1)\geq D(G_1\sqcap G_2)$.

Similarly, $D(G_2)\geq D(G_1\sqcap G_2)$.

Therefore, $D(G_1)=D(G_2)=D(G_1\sqcap G_2)$. □

Theorem 4.3.7. *Let $G_1=(V_1,A_1,B_1)$ and $G_2=(V_2,A_2,B_2)$ be two balanced m-polar fuzzy graphs. Then, $G_1\sqcap G_2$ is balanced if and only if $D(G_1)=D(G_2)=D(G_1\sqcap G_2)$.*

Proof. Suppose $D(G_1\sqcap G_2)$ is balanced.

Then $D(G_k)\leq D(G_1\sqcap G_2)$ for $k=1,2$ and by Theorem 4.3.6, $D(G_1)=D(G_2)=D(G_1\sqcap G_2)$.

Conversely, let $D(G_1)=D(G_2)=D(G_1\sqcap G_2)$ and H be a non-empty subgraph of $G_1\sqcap G_2$. Then there exist subgraphs H_1 of G_1 and H_2 of G_2.

Let $p_i\circ D(G_1)=p_i\circ D(G_2)=\frac{q_i}{r_i}$,

$p_i\circ D(H_1)=\frac{s_i}{t_i}$ and

$p_i\circ D(H_2)=\frac{a_i}{b_i}$ for $i=1,2,\dots,m$ and $a_i,b_i,q_i,r_i,s_i,t_i\in\mathbb{R}$.

Since G_1 and G_2 are balanced, therefore for $i=1,2,\dots,m$

$p_i\circ D(H_1)=\frac{s_i}{t_i}\leq p_i\circ D(G_1)=\frac{q_i}{r_i}$ and

$p_i\circ D(H_2)=\frac{a_i}{b_i}\leq p_i\circ D(G_2)=\frac{q_i}{r_i}$.

Thus, $s_ir_i+a_ir_i\leq t_iq_i+b_iq_i$,

i.e $\frac{s_i+a_i}{t_i+b_i}\leq\frac{q_i}{r_i}$ for $i=1,2,\dots,m$.

Hence, $p_i\circ D(H)\leq\frac{s_i+a_i}{t_i+b_i}\leq\frac{q_i}{r_i}=p_i\circ D(G_1\sqcap G_2)$ for $i=1,2,\dots,m$.

Therefore, $G_1\sqcap G_2$ is balanced. □

Similarly, we have the following results.

Theorem 4.3.8. *Let $G_1=(V_1,A_1,B_1)$ and $G_2=(V_2,A_2,B_2)$ be two balanced m-polar fuzzy graphs. Then*

(i) $G_1\bullet G_2$ is balanced if and only if $D(G_1)=D(G_2)=D(G_1\bullet G_2)$.

(ii) $G_1\otimes G_2$ is balanced if and only if $D(G_1)=D(G_2)=D(G_1\otimes G_2)$.

We end this section by showing that isomorphism between m-polar fuzzy graphs preserve balanced.

Theorem 4.3.9. *Let G_1 and G_2 be two isomorphic m-polar fuzzy graphs. If G_2 is balanced, then G_1 is balanced.*

Proof. Since G_1 and G_2 are isomorphic therefore there exists a bijective mapping $\phi : V_1 \to V_2$ such that $p_i \circ A_1(u) = p_i \circ A_2(\phi(u))$ for all $u \in V_1$ and $p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v))$ for all $uv \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$.

Then, $\sum\limits_{u \in V_1} p_i \circ A_1(u)$
$= \sum\limits_{\phi(u) \in V_2} p_i \circ A_2(\phi(u))$ and
$\sum\limits_{u,v \in V_1} p_i \circ B_1(uv)$
$= \sum\limits_{\phi(u),\phi(v) \in V_2} p_i \circ B_2(\phi(u)\phi(v))$.

Let H_1 and H_2 be two nonempty subgraphs of G_1 and G_2 respectively.

Then, $p_i \circ A_1(u) = p_i \circ A_2(\phi(u))$ and $p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v))$ for all $u, v \in V_1(H_1)$, $i = 1, 2, \ldots, m$. Here, $V_1(H_1)$ represents the vertices of H_1.

Since G_2 is balanced, therefore for $i = 1, 2, \ldots, m$

$p_i \circ D(H_2) \leq p_i \circ D(G_2)$,

i.e. $2 \sum\limits_{u,v \in V_2(H_2)} \frac{p_i \circ B_2(uv)}{p_i \circ A_2(u) \wedge p_i \circ A_2(v)} \leq 2 \sum\limits_{u,v \in V_2} \frac{p_i \circ B_2(uv)}{p_i \circ A_2(u) \wedge p_i \circ A_2(v)}$,

i.e. $2 \sum\limits_{u,v \in V_1(H_1)} \frac{p_i \circ B_1(uv)}{p_i \circ A_1(u) \wedge p_i \circ A_1(v)} \leq 2 \sum\limits_{u,v \in V_1} \frac{p_i \circ B_1(uv)}{p_i \circ A_1(u) \wedge p_i \circ A_1(v)}$,

i.e. $p_i \circ D(H_1) \leq p_i \circ D(G_1)$,

i.e. G_1 is balanced. □

4.4 Summary

The main purpose of this chapter is to define density of an m-polar fuzzy graph. This chapter deals with the significant properties of balanced m-polar fuzzy graphs. Density of some special m-polar fuzzy graphs are calculated using the formula. Necessary and sufficient conditions are established for the products of two m-polar fuzzy balanced graphs to be balanced. Finally, it has been shown that isomorphic m-polar fuzzy graphs preserves the property of balanced.

Chapter 5

m-polar fuzzy planar graphs and its dual*

5.1 Introduction

Graph model can be used to represent electrical circuits. Minimizing the non-overlapping circuit is the main objective in such system. In a city planning, subway tunnels, pipelines, metro lines, etc. are all essential. There are chances of accident due to crossing. Routes without crossing is preferable, but due to the lack of space crossing of such lines are allowed. Crossing between congested and non-congested routes are more preferable than the crossing between two congested routes. The term "congested" has no definite meaning. We generally use "congested", "very congested", "highly congested" routes, etc. These terms are called linguistic terms and they have some membership values. A congested route may be termed as strong route and low congested route may be termed as weak route. Thus, crossing between strong and weak route may be allowed in a city planning with certain amount of safety. The terms "strong route" and "weak route" lead to strong edge and weak edge of an m-polar fuzzy graph respectively and the permission of crossing between strong and weak edges leads to the concept of m-polar fuzzy planar graphs. Abdul et. al [1] introduced the concept of fuzzy planar graph. Samanta and Pal [114,116] defined fuzzy planar graph assuming crossing of edges. In this chapter, m-polar fuzzy planar graphs, m-polar fuzzy dual

*A part of the work presented in this chapter is published in *Int. J. of Computing Science and Mathematics*, **7**(3) 283-292 (2016) and *Journal of Intelligent and Fuzzy Systems*, **31**(3) 2043-2049 (2016).

graphs are defined and some important properties are established. Here, the "degree of planarity" is used to measure the nature of planarity of an m-polar fuzzy planar graph. Also, we introduced some terms like m-polar fuzzy multiset, m-polar fuzzy multigraphs, m-polar fuzzy dual graph. Some theorems have been proved on degree of planarity. Depending on the degree of planarity, the considerable edge has been introduced.

5.2 m-polar fuzzy multiset and m-polar fuzzy multigraph

Yager [135] first discussed fuzzy multiset, although he used the term "fuzzy bag". Fuzzy multiset over a non-empty set V is a mapping $\widetilde{C} : V \times [0,1] \to \mathbb{N}$. m-polar fuzzy multiset is another generalization of multiset which is defined below.

Definition 5.2.1. (m-polar fuzzy multiset) *Let V be a nonempty set and $A^j : V \to [0,1]^m$, $j = 1, 2, \ldots, p$ be the mappings. The m-polar fuzzy multiset on V is denoted by A and is defined as $\{(v, A^j(v)) : v \in V, j = 1, 2, \ldots, p\}$.*

Example 5.2.1. *Let $V = \{a, b, c, d\}$. Then one of the 3-polar fuzzy multisets on V is $(a, < 0.3, 0.4, 0.5 >)$, $(a, < 0.5, 0.7, 0.8 >)$, $(a, < 0.4, 0.2, 0.3 >)$, $(b, < 0.7, 0.8, 0.5 >)$, $(b, < 0.5, 0.3, 0.4 >)$, $(c, < 0.4, 0.5, 0.7 >)$, $(d, < 0.3, 0.9, 0.2 >)$, $(d, < 0.6, 0.3, 0.2 >)$.*

The concept of m-polar fuzzy multigraph is introduced using the notion of m-polar fuzzy multiset.

Definition 5.2.2. *Let V be a nonempty set and let A be an m-polar fuzzy set on V. Let $B = \{((u, v), B^j(u, v)), j = 1, 2, \ldots, p : (u, v) \in V \times V\}$ be an m-polar fuzzy multiset of $V \times V$. Then $G = (V, A, B)$ is said to be m-polar fuzzy multigraph if $p_i \circ B^j(u, v) \leq min\{p_i \circ A(u), p_i \circ A(v)\}$ for all $u, v \in V$, $j = 1, 2, \ldots, p$ and $i = 1, 2, \ldots, m$.*

Here, $A(u)$ and $B(u, v)$ represent the membership value of the vertex u and of the edge (u, v) in G respectively. It may be noted that there may be more than one edge between the vertices u and v. $B^j(u, v)$ denotes membership value of the j-th edge between the vertices u and v and p represents the number of edges between the vertices u and v.

An example of 3-polar fuzzy multigraph is given in Fig. 5.1.

A special type of m-polar fuzzy multigraph is defined below.

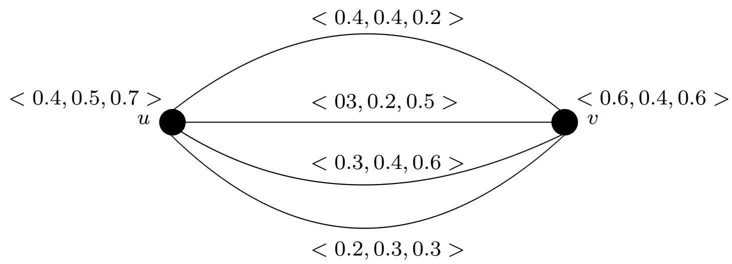

Figure 5.1: Example of 3-polar fuzzy multigraph.

5.3 m-polar fuzzy planar graphs

Let $G = (V, A, B)$ be an m-polar fuzzy multigraph and for a certain geometrical representation, the graph has only one crossing between the edges $((w, x), B(w, x))$ and $((y, z), B(y, z))$. If $B(w, x) = \mathbf{1}$ and $B(y, z) = \mathbf{0}$, then we say that the graph has no crossing. Similarly, if $B(w, x)$ has value near to $\mathbf{1}$ and $B(y, z)$ has value near to $\mathbf{0}$, the crossing will not be important for the planarity. If $B(w, x)$, $B(y, z)$ have value near to $\mathbf{1}$, then the crossing becomes very important for the planarity. So, if there is a crossing at a point between two edges, a value is assigned corresponding to that point, called the intersecting value.

5.3.1 Intersecting value in m-polar fuzzy multigraph

Let $G = (V, A, B)$ be an m-polar fuzzy multigraph where $B = \{((u, v), B^j(u, v)), j = 1, 2, \ldots, p : (u, v) \in V \times V\}$. G is called m-polar fuzzy complete multigraph if $p_i \circ B^j(u, v) = min\{p_i \circ A(u), p_i \circ A(v)\}$ for all $u, v \in V$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$.

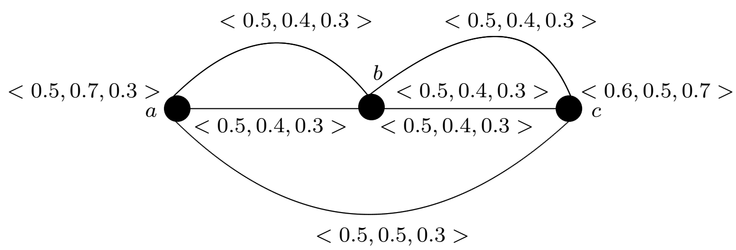

Figure 5.2: 3-polar fuzzy complete multigraph

Example 5.3.1. *It is easy to see that G is a 3-polar fuzzy complete multigraph as shown in Fig. 5.2.*

The strength of an edge $((u, v), B^j(u, v))$ is defined by a value

$$I_{(u,v)} = (I^1_{(u,v)}, I^2_{(u,v)}, \ldots, I^m_{(u,v)})$$

where

$$I^i_{(u,v)} = \frac{p_i \circ B^j(u, v)}{min\{p_i \circ A(u), p_i \circ A(v)\}}, i = 1, 2, \ldots, m.$$

Definition 5.3.1. *Let $G = (V, A, B)$ be an m-polar fuzzy multigraph. An edge (u, v) in G is said to be m-polar fuzzy strong if $I^i_{(u,v)} \geq 0.5$ for each $i = 1, 2, \ldots, m$. Otherwise, it is called m-polar fuzzy weak edge.*

In m-polar fuzzy multigraph, when two edges intersect at a point, a value is assigned to that point in the following way.

Let in an m-polar fuzzy multigraph $G = (V, A, B)$, B contains two edges $((u_1, v_1), B^j(u_1, v_1))$ and $((u_2, v_2), B^k(u_2, v_2))$ which intersect at a point P, where j and k are fixed integers. The intersecting value at the point P is given by

$$\mathcal{I}_P = (\mathcal{I}^1_P, \mathcal{I}^2_P, \ldots, \mathcal{I}^m_P)$$

where

$$\mathcal{I}^i_P = \frac{I^i_{(u_1,v_1)} + I^i_{(u_2,v_2)}}{2}, i = 1, 2, \ldots, m.$$

If the number of points of intersection in an m-polar fuzzy multigraph increases, the ‘planarity’ decreases. Using these concept, the notion of m-polar fuzzy planar graph is introduced below.

Definition 5.3.2. (Planarity of m-polar fuzzy multigraph) *Let $G = (V, A, B)$ be an m-polar fuzzy multigraph and for a certain geometrical representation $P_1, P_2, \ldots, P_k$ be the points of intersections between the edges. Then G is said to be m-polar fuzzy planar graph with m-polar fuzzy planarity value*

$$\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m)$$

where

$$\mathcal{P}_i = \frac{1}{1 + \{\mathcal{I}^i_{P_1} + \mathcal{I}^i_{P_2} + \ldots + \mathcal{I}^i_{P_k}\}}, i = 1, 2, \ldots, m.$$

$\mathcal{P}$ is bounded, since $0 < \mathcal{P}_i \leq 1$ for each $i = 1, 2, \ldots, m$.

Example 5.3.2. *Let us consider a 3-polar fuzzy multigraph with two point of intersections P_1 and P_2 (see Fig. 5.3). P_1 is a point between the edges $((a,b), < 0.3, 0.6, 0.3 >)$ and $((c,d), < 0.5, 0.6, 0.2 >)$, P_2 is a point between the edges $((a,b), < 0.4, 0.5, 0.3 >)$ and $((c,d), < 0.5, 0.6, 0.2 >)$.*

Now, for the edge $((a,b), < 0.3, 0.6, 0.3 >)$, $I_{(a,b)} = (0.75, 0.85, 0.75)$,

for the edge $((a,b), < 0.4, 0.5, 0.3 >)$, $I_{(a,b)} = (1, 0.71, 0.75)$ and

for the edge $((c,d), < 0.5, 0.6, 0.2 >)$, $I_{(c,d)} = (1, 0.85, 0.66)$.

The intersecting values are $\mathcal{I}_{P_1} = (0.875, 0.85, 0.705)$ and $\mathcal{I}_{P_2} = (1, 0.78, 0.705)$.

So, planarity value for the 3-polar fuzzy multigraph is $(0.35, 0.38, 0.41)$.

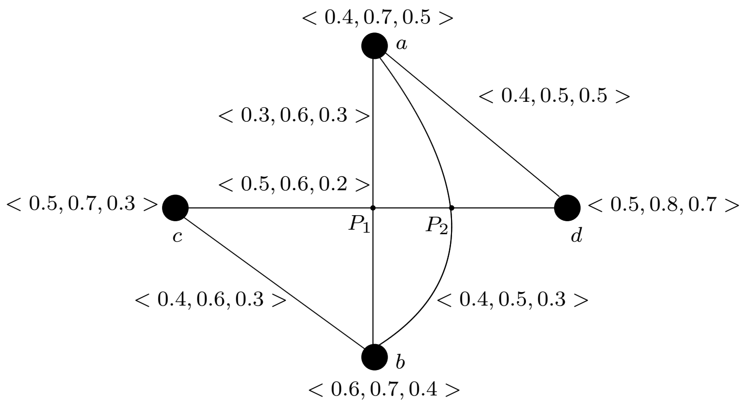

Figure 5.3: 3-polar fuzzy planar graph with 3-polar fuzzy planarity value $(0.35, 0.38, 0.41)$

Now consider an m-polar fuzzy complete multigraph whose m-polar fuzzy planarity value is given by the following theorem.

Theorem 5.3.1. *Let $G = (V, A, B)$ be an m-polar fuzzy complete multigraph. The m-polar fuzzy planarity value $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m)$ is given by $\mathcal{P}_i = \frac{1}{1+n_k}$, $i = 1, 2, \dots, m$ where n_k is the number of points of intersection between the edges in G.*

Proof. Since G is complete, we have,

$p_i \circ B^j(u,v) = min\{p_i \circ A(u), p_i \circ A(v)\}$ for all $u, v \in V$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, p$.

Let $P_1, P_2, \dots, P_k$ be the points of intersection between the edges in G.

For an edge (u,v) in G, $I^i_{(u,v)} = \frac{p_i \circ B^j(u,v)}{min\{p_i \circ A(u), p_i \circ A(v)\}} = 1$, $i = 1, 2, \dots, m$.

Therefore, for the point P_1 which is the point of intersection between the edges (a,b) and (c,d), the intersecting value is $\mathcal{I}_{P_1} = (1, 1, \dots, 1)$.

Hence, $\mathcal{I}_{P_i} = (1, 1, \ldots, 1)$ for $i = 1, 2, \ldots, k$.

Now for $i = 1, 2, \ldots, m$,

$\mathcal{P}_i = \frac{1}{1+(\mathcal{I}^i_{P_1}+\mathcal{I}^i_{P_2}+\ldots+\mathcal{I}^i_{P_k})} = \frac{1}{1+(1+1+\ldots+1)} = \frac{1}{1+n_k}$.

Therefore, the m-polar fuzzy planarity $\mathcal{P}$ is given by $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m)$ where $\mathcal{P}_i = \frac{1}{1+n_k}$, $i = 1, 2, \ldots, m$. □

Theorem 5.3.2. *Let $G = (V, A, B)$ be an m-polar fuzzy planar graph with m-polar fuzzy planarity $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m)$ is such that $\mathcal{P}_i > 0.5$ for $i = 1, 2, \ldots, m$. Then the number of points of intersection between m-polar fuzzy strong edges in G is at most one.*

Proof. If possible, let G has at least two points of intersection P_1 and P_2 between two m-polar fuzzy strong edges in G.

For any m-polar fuzzy strong edge $((u, v), B^j(u, v))$, $I^i_{(u,v)} \geq 0.5$, $i = 1, 2, \ldots, m$.

Thus, for any two intersecting strong edges $((u, v), B^j(u, v))$ and $((w, x), B^k(w, x))$, $\frac{I^i_{(u,v)}+I^i_{(w,x)}}{2} \geq 0.5$ for $i = 1, 2, \ldots, m$,

i.e. $\mathcal{I}^i_{P_1} \geq 0.5$.

Similarly, $\mathcal{I}^i_{P_2} \geq 0.5$.

Then, $1 + \mathcal{I}^i_{P_1} + \mathcal{I}^i_{P_2} \geq 2$,

i.e. $\mathcal{P}_i = \frac{1}{1+\mathcal{I}^i_{P_1}+\mathcal{I}^i_{P_2}} \leq 0.5$.

This is a contradiction, since $\mathcal{P}_i > 0.5$ for $i = 1, 2, \ldots, m$.

Hence, the number of points cannot be two.

Clearly, if the number of point of intersection between m-polar fuzzy strong edges increases, then the planarity value decreases. Similarly, if the number of point of intersection is one, then the planarity value $\mathcal{P}$ is such that $\mathcal{P}_i > 0.5$, $i = 1, 2, \ldots, m$. Any m-polar fuzzy planar graph without any crossing between edges has m-polar fuzzy planarity value $\mathcal{P}$ where $\mathcal{P}_i > 0.5$. Hence, the proof. □

Theorem 5.3.3. *Let $G = (V, A, B)$ be an m-polar fuzzy planar graph with m-polar fuzzy planarity value $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m)$. If $\mathcal{P}_i \geq 0.67$, $i = 1, 2, \ldots, m$, then G does not contain any point of intersection between two m-polar fuzzy strong edges.*

Proof. If possible, let P be a point of intersection between two m-polar fuzzy strong edges $((u, v), B^j(u, v))$ and $((w, x), B^k(w, x))$.

For any m-polar fuzzy strong edge $((u,v), B^j(u,v))$, we have $I^i_{(u,v)} \geq 0.5$, $i = 1, 2, \ldots, m$.

For the minimum value of $I^i_{(u,v)}$ and $I^i_{(w,x)}$, $\mathcal{I}^i_P = 0.5$, $i = 1, 2, \ldots, m$.

Then, $\mathcal{P}_i = \frac{1}{1+0.5} < 0.67$ for $i = 1, 2, \ldots, m$, a contradiction.

Hence, G does not contain any point of intersection between two m-polar fuzzy strong edges. □

Next the definition of strong m-polar fuzzy planar graph is given below.

Definition 5.3.3. *An m-polar fuzzy planar graph G is said to be strong m-polar fuzzy planar graph if the m-polar fuzzy planarity value $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m)$ of the graph is such that $\mathcal{P}_i \geq 0.67$, $i = 1, 2, \ldots, m$.*

Strength of an edge has an important role to model some kind of projects. Edges of small strength may be ignored. So, the edges with sufficient strengths are very useful to design such projects. These edges are called considerable edges which is defined below.

Definition 5.3.4. *Let $G = (V, A, B)$ be an m-polar fuzzy planar graph. Let $0 < c < 0.5$ be a rational number. An edge $((u,v), B(u,v))$ is said to be considerable edge if $\frac{p_i \circ B(u,v)}{min\{p_i \circ A(u), p_i \circ A(v)\}} \geq c$ for $i = 1, 2, \ldots, m$. Otherwise, it is called non-considerable edge.*

For an m-polar fuzzy multigraph G, a multi-edge $((u,v), B^j(u,v))$ is said to be considerable edge if $\frac{p_i \circ B^j(u,v)}{min\{p_i \circ A(u), p_i \circ A(v)\}} \geq c$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$.

Theorem 5.3.4. *Let G be an m-polar fuzzy planar graph with m-polar fuzzy planarity value $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m)$ be such that $\mathcal{P}_i > 0.5$ for $i = 1, 2, \ldots, m$ and considerable number c. Then the number of point of intersection between considerable edges in G is at most $[\frac{1}{c}]$ or $\frac{1}{c} - 1$ according as $\frac{1}{c}$ is not an integer or an integer respectively.*

Proof. Let $G = (V, A, B)$ be an m-polar fuzzy planar graph where

$B = \{((u,v), B^j(u,v)), j = 1, 2, \ldots, p : (u,v) \in V \times V\}$.

Let $0 < c < 0.5$ be the considerable number.

For any considerable edge $((u,v), B^j(u,v))$, we have,

$p_i \circ B^j(u,v) \geq c\ min\{p_i \circ A(u), p_i \circ A(v)\}$, $i = 1, 2, \ldots, m$.

This implies that, $I^i_{(u,v)} \geq c$ for $i = 1, 2, \ldots, m$.

Let $P_1, P_2, \ldots, P_l$ be the l-points of intersection between the considerable edges.

Also let, P_1 be the point of intersection between the considerable edges $((u_1, v_1), B^j(u_1, v_1))$ and $((u_2, v_2), B^k(u_2, v_2))$.

Then, $\mathcal{I}^i_{P_1} = \frac{I^i_{(u_1,v_1)}+I^i_{(u_2,v_2)}}{2} \geq c$.

So, $\sum\limits_{n=1}^{l} \mathcal{I}^i_{P_n} \geq lc$.

Hence, $\mathcal{P}_i \leq \frac{1}{1+lc}$.

This imply that $0.5 \leq \mathcal{P}_i \leq \frac{1}{1+lc}$,

i.e. $0.5 < \frac{1}{1+lc}$,

i.e. $l < \frac{1}{c}$.

Hence, the values of l are given by $l = \begin{cases} [\frac{1}{c}] & \text{if } \frac{1}{c} \text{ is not an integer} \\ \frac{1}{c} - 1 & \text{if } \frac{1}{c} \text{ is an integer} \end{cases}$

This completes the proof. □

Theorem 5.3.5. *Any complete m-polar fuzzy graph of five vertices or complete bipartite m-polar fuzzy graph of six vertices are not strong m-polar fuzzy planar graph.*

Proof. Let $G = (V, A, B)$ be a complete m-polar fuzzy graph of five vertices where $V = \{u, v, w, x, y\}$ and $B = \{((u, v), B(u, v) : (u, v) \in V \times V\}$.

For all $u, v \in V$, we have,

$p_i \circ B(u, v) = min\{p_i \circ A(u), p_i \circ A(v)\}$, $i = 1, 2, \ldots, m$.

By Theorem 5.3.1, the m-polar fuzzy planarity value of a complete m-polar fuzzy graph is $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m)$, where $\mathcal{P}_i = \frac{1}{1+n_p}$, n_p being the number of point of intersection of the edges in G.

We know that the geometric representation of the underlying crisp graph of an m-polar fuzzy complete graph of five vertices is non-planar and one point of intersection cannot be avoided for any representation.

So, $\mathcal{P}_i = 0.5$, $i = 1, 2, \ldots, m$.

Hence, G is not strong m-polar fuzzy planar graph.

Similarly, it can be proved that the complete bipartite m-polar fuzzy graph of six vertices is not strong m-polar fuzzy planar graph. □

5.4 Faces of m-polar fuzzy planar graph

Face of m-polar fuzzy planar graph is an important parameter. Face of an m-polar fuzzy planar graph is a region bounded by m-polar fuzzy edges. Every m-polar fuzzy face is characterized by m-polar fuzzy edges in its boundary. If all the edges in the boundary of m-polar fuzzy face have membership values 1, it becomes crisp face. If one such edges is removed, the m-polar fuzzy face does not exist. So, the existence of an m-polar fuzzy face depends on the minimum value of strength of m-polar fuzzy edges in its boundary. m-polar fuzzy face and its membership values of an m-polar fuzzy graph are defined below.

Definition 5.4.1. *Let $G = (V, A, B)$ be an m-polar fuzzy planar graph and $B = \{((u, v), B^j(u, v)), j = 1, 2, \ldots, p : (u, v) \in V \times V\}$. An m-polar fuzzy face of G is a region bounded by the set of m-polar fuzzy edges $E' \subseteq V \times V$, of a geometric representation of G. The strength of the face is $(S_F^1, S_F^2, \ldots, S_F^m)$, where $S_F^i = min\{I^i_{(u,v)} : (u, v) \in E'\}$, $i = 1, 2, \ldots, m$.*

Definition 5.4.2. *An m-polar fuzzy face is called strong m-polar fuzzy face if $S_F^i > 0.5$ for $i = 1, 2, \ldots, m$ and weak m-polar fuzzy face otherwise. Every m-polar fuzzy planar graph has an infinite region which is called outer m-polar fuzzy face. Other faces are called inner m-polar fuzzy faces.*

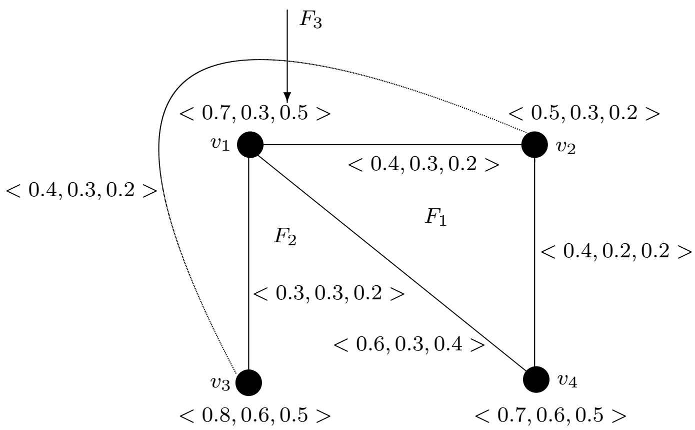

Figure 5.4: 3-polar fuzzy planar graph with three 3-polar fuzzy faces

Example 5.4.1. *Let us consider the 3-polar fuzzy planar graph as shown in the Fig. 5.4. Here, F_1, F_2, F_3 are three 3-polar fuzzy faces. The 3-polar fuzzy face F_1 is bounded by the edges $\big((v_1, v_2), < 0.4, 0.3, 0.2 >\big)$, $\big((v_2, v_4), < 0.4, 0.2, 0.2 >\big)$ and $\big((v_4, v_1), < 0.6, 0.3, 0.4 >\big)$ with strength $(0.8, 0.67, 0.8)$.*

Similarly, F_3 is a face bounded by the edges $\big((v_1, v_3), < 0.3, 0.3, 0.2 >\big)$, $\big((v_1, v_2), < 0.4, 0.3, 0.2 >\big)$ and $\big((v_2, v_3), < 0.4, 0.2, 0.2 >\big)$ with strength $(0.42, 0.67, 0.4)$. F_2 is the outer face with strength $(0.42, 1, 0.4)$. So, F_1 is strong 3-polar fuzzy face while F_2, F_3 are weak 3-polar fuzzy faces.

5.5 m-polar fuzzy dual graph

In this section, we introduce the concept of dual of an m-polar fuzzy planar graph. In m-polar fuzzy dual graph, vertices are corresponding to the strong m-polar fuzzy faces and each edge in dual graph between two vertices is corresponding to each edge in the boundary between two m-polar fuzzy faces of m-polar fuzzy planar graph. The definition is given below.

Definition 5.5.1. *Let $G = (V, A, B)$ be an m-polar fuzzy planar graph where $B = \{((u, v), B^j(u, v)), j = 1, 2, \ldots, p : (u, v) \in V \times V\}$. Let $F_1, F_2, \ldots, F_k$ be the strong m-polar fuzzy faces of G. The m-polar fuzzy dual graph of G is an m-polar fuzzy planar graph $G_1 = (V_1, A_1, B_1)$ where $V_1 = \{x_q, q = 1, 2, \ldots, k\}$, the vertex x_q of G_1 is correspond to the face F_q of G. The membership value of vertices are given by the mapping $A_1 : V_1 \to [0, 1]^m$ such that $p_i \circ A_1(x_q) = max\{p_i \circ B^j(u, v), j = 1, 2, \ldots, l : (u, v)$ is an edge of the boundary of the m-polar fuzzy face $F_q\}$.*

There may exist more than one common edge between two m-polar fuzzy faces F_i and F_j of G. Thus there may be more than one edge between two vertices x_i and x_j in the m-polar fuzzy dual graph G_1. Let $B^l(x_i, x_j)$ denote the membership value of the l-th edge between x_i and x_j. The membership value of the edges of the m-polar fuzzy dual graph are given by ${B_1}^l(x_i, x_j) = B^l(u, v)$ where (u, v) is a common edge between two m-polar fuzzy faces F_i and F_j and $l = 1, 2, \ldots, t$; t being the number of common edges in the boundary between F_i and F_j or the number the edges between x_i and x_j.

If there is any strong pendant edge in the m-polar fuzzy planar graph, then there will be a self-loop in G_1 corresponding to this pendant edge. The edge membership value of the self-loop is equal to the membership value of the pendant edge. m-polar fuzzy dual graph of m-polar fuzzy planar graph does not contain any point of intersection of edges for a certain representation, so it is an m-polar fuzzy planar graph with m-polar fuzzy planarity value $(1, 1, \ldots, 1)$.

Next, we give an example of an m-polar fuzzy dual graph of an m-polar fuzzy planar graph which are shown in Fig. 5.5. We assume that the black filled circles and the lines represent the vertices and edges of the m-polar fuzzy planar graph while the empty circles and the dotted lines represent the vertices and edges of m-polar fuzzy dual graph corresponding to the m-polar fuzzy planar graph.

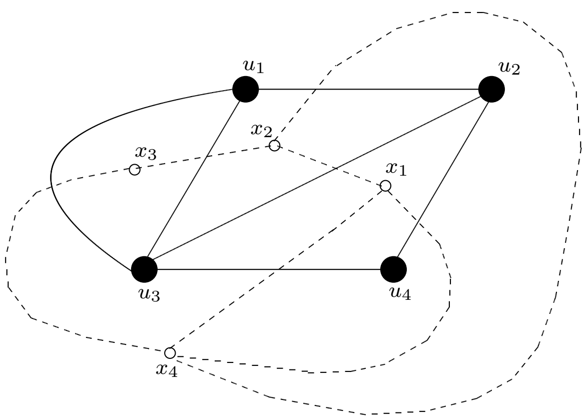

Figure 5.5: 3-polar fuzzy graph and it's 3-polar fuzzy dual graph

Example 5.5.1. *Let us now consider a 3-polar fuzzy planar graph* $G = (V, A, B)$ *as shown in Fig. 5.5, where* $V = \{u_1, u_2, u_3, u_4\}$*,* $A = \{\big(u_1, < 0.8, 0.7, 0.6 >\big),$ $\big(u_2, < 0.6, 0.5, 0.7 >\big), \big(u_3, < 0.8, 0.8, 0.9 >\big), \big(u_4, < 0.9, 0.7, 0.6 >\big)\}$*, and*

$B = \{\big((u_1, u_2), < 0.5, 0.4, 0.5 >\big), \big((u_2, u_4), < 0.6, 0.4, 0.5 >\big), \big((u_3, u_4), < 0.7, 0.7, 0.5 >\big), \big((u_1, u_3), < 0.7, 0.6, 0.6 >\big), \big((u_1, u_3), < 0.7, 0.5, 0.4 >\big), \big((u_2, u_3), < 0.5, 0.5, 0.6 >\big)\}$.

The 3-polar fuzzy planar graph has the following faces:

(i) the 3-polar fuzzy face F_1 *is bounded by* $\big((u_2, u_3), < 0.5, 0.5, 0.6 >\big)$, $\big((u_3, u_4), < 0.7, 0.7, 0.5 >\big)$, $\big((u_2, u_4), < 0.6, 0.4, 0.5 >\big)$,

(ii) the 3-polar fuzzy face F_2 *is bounded by* $\big((u_1, u_2), < 0.5, 0.4, 0.5 >\big)$, $\big((u_1, u_3), < 0.7, 0.6, 0.6 >\big)$, $\big((u_2, u_3), < 0.5, 0.5, 0.6 >\big)$,

(iii) the 3-polar fuzzy face F_3 *is bounded by* $\big((u_1, u_3), < 0.7, 0.6, 0.6 >\big)$, $\big((u_1, u_3), < 0.7, 0.5, 0.4 >\big)$*, and*

(iv) the outer 3-polar fuzzy face F_4 *is surrounded by* $\big((u_1, u_2), < 0.5, 0.4, 0.5 >\big)$, $\big((u_2, u_4), < 0.6, 0.4, 0.5 >\big)$, $\big((u_1, u_3), < 0.7, 0.6, 0.6 >\big)$, $\big((u_3, u_4), < 0.7, 0.7, 0.5 >\big)$.

The strengths of the faces F_1*,* F_2*,* F_3*,* F_4 *are* $(0.83, 0.8, 0.83)$, $(0.62, 0.8, 0.83)$, $(0.87, 0.71, 0.66)$, $(0.83, 0.71, 0.66)$ *respectively. Since all the faces are strong 3-polar fuzzy faces, for each strong 3-polar fuzzy faces, we consider a vertex for the 3-polar fuzzy dual graph. Thus the vertex set* V_1 *of the 3-polar fuzzy dual graph is* $V_1 = \{x_1, x_2, x_3, x_4\}$*, where the vertex* x_i *corresponds to the strong 3-polar fuzzy face* F_i*,* $i = 1, 2, 3, 4$*. Now, the membership value of the vertex set* V_1 *is calculated below:*

$A_1(x_1) =< 0.7, 0.7, 0.6 >$, $A_1(x_2) =< 0.7, 0.6, 0.6 >$,

$A_1(x_3) =< 0.7, 0.6, 0.6 >$, $A_1(x_4) =< 0.7, 0.7, 0.5 >$.

There are two common edges (u_2, u_4) and (u_3, u_4) between the faces F_1 and F_4 in G. Therefore, there exist two edges between x_1 and x_4 in the 3-polar fuzzy dual graph. The membership values of these edges are given by

$B_1(x_1, x_4) = B(u_3, u_4) =< 0.7, 0.7, 0.5 >$, $B_1(x_1, x_4) = B(u_2, u_4) =< 0.6, 0.4, 0.5 >$.

The membership values of other edges of the 3-polar fuzzy dual graph are calculated as $B_1(x_1, x_2) = B(u_2, u_3) =< 0.5, 0.5, 0.6 >$, $B_1(x_2, x_3) = B(u_1, u_3) =< 0.7, 0.6, 0.6 >$,

$B_1(x_2, x_4) = B(u_1, u_2) =< 0.5, 0.4, 0.5 >$, $B_1(x_3, x_4) = B(u_1, u_3) =< 0.7, 0.5, 0.4 >$.

Thus, the edge set of the 3-polar fuzzy dual graph is

$B_1 = \{\big((x_1, x_2), < 0.5, 0.5, 0.6 >\big), \big((x_2, x_3), < 0.7, 0.6, 0.6 >\big), \big((x_2, x_4), < 0.5, 0.4, 0.5 >\big), \big((x_3, x_4), < 0.7, 0.5, 0.4 >\big), \big((x_1, x_4), < 0.7, 0.7, 0.5 >\big), \big((x_1, x_4), < 0.6, 0.4, 0.5 >\big)\}$.

Now, we have the following observations.

Theorem 5.5.1. *Let $G = (V, A, B)$ be an m-polar fuzzy planar graph whose number of vertices, number of edges and number of strong m-polar fuzzy faces denoted by n, e and f respectively. Let G_1 be the m-polar fuzzy dual graph of G. Then*

(i) the number of vertices of G_1 is equal to f,

(ii) the number of edges of G_1 is equal to e,

(iii) the number of m-polar fuzzy faces of G_1 is equal to n.

Proof. Proof of (i), (ii) and (iii) follows from the definition of m-polar fuzzy dual graph. □

Theorem 5.5.2. *Let $G_1 = (V_1, A_1, B_1)$ be an m-polar fuzzy dual graph of the strong m-polar fuzzy planar graph $G = (V, A, B)$. Then the number of strong m-polar fuzzy faces in G_1 is less than or equal to the number of vertices of G.*

Proof. Since all the faces of the m-polar fuzzy dual graph G_1 may not be strong m-polar fuzzy faces, therefore the result holds from the (iii)-rd part of the Theorem 5.5.1. □

Theorem 5.5.3. *Let $G = (V, A, B)$ be a strong m-polar fuzzy planar graph having no weak m-polar fuzzy edges and G_1 be the m-polar fuzzy dual graph of G. Then the* *membership value of the m-polar fuzzy edges of G_1 is equal to the membership value of the m-polar fuzzy edges of G.*

Proof. The dual graph G_1 of G is a strong m-polar fuzzy planar graph as there is no point of intersection between any edges. Let $\{F_1, F_2, \ldots, F_k\}$ be the set of strong faces of G.

From the definition of m-polar fuzzy dual graph we know that ${B_1}^l(x_i, x_j) = B^l(u, v)$ where (u, v) is a common edge between two strong m-polar fuzzy faces F_i and F_j and $l = 1, 2, \ldots, t$; t being the number of common edges in the boundary between F_i and F_j. The number of m-polar fuzzy edges of the two graphs G and G_1 are same as G has no weak edges. Hence, for each m-polar fuzzy edge of G there is an m-polar fuzzy edge in G_1 with the same membership value. $\square$

5.6 Isomorphism on m-polar fuzzy planar graphs

In this section, we introduce the notion of isomorphism between m-polar fuzzy graphs. The related definitions are given as follows.

Definition 5.6.1. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy planar graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. An isomorphism between G_1 and G_2 is a bijective mapping $\phi : V_1 \to V_2$ such that for each $i = 1, 2, \ldots, m$*

(i) $p_i \circ A_1(x_1) = p_i \circ A_2(\phi(x_1))$ *for all* $x_1 \in V_1$,

(ii) $p_i \circ B_1(x_1, y_1) = p_i \circ B_2(\phi(x_1), \phi(y_1))$ *for all* $(x_1, y_1) \in \widetilde{V_1^2}$.

In this case, we write $G_1 \cong G_2$.

Definition 5.6.2. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy planar graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. A weak isomorphism between G_1 and G_2 is a bijective mapping $\phi : V_1 \to V_2$ which satisfies the following conditions:*

(i) ϕ *is a homomorphism,*

(ii) $p_i \circ A_1(x_1) = p_i \circ A_2(\phi(x_1))$ *for all* $x_1 \in V_1$ *and* $i = 1, 2, \ldots, m$.

Definition 5.6.3. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy planar graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. A co-weak* *isomorphism between G_1 and G_2 is a bijective mapping $\phi : V_1 \to V_2$ which satisfies the following:*

(i) ϕ is a homomorphism,

(ii) $p_i \circ B_1(x_1, y_1) = p_i \circ B_2(\phi(x_1 y_1))$ for all $(x_1, y_1) \in \widetilde{V_1^2}$ and $i = 1, 2, \ldots, m$.

Isomorphism between m-polar fuzzy graphs is an equivalence relation. But, if there is an isomorphism between two m-polar fuzzy graphs and one is m-polar fuzzy planar graph, then the other also be an m-polar fuzzy planar graph. This can be proved as follows.

Theorem 5.6.1. *Let G_1 be an m-polar fuzzy planar graph. If there exists an isomorphism $\phi : G_1 \to G_2$ where G_2 is an m-polar fuzzy graph, then G_2 can be drawn as an m-polar fuzzy planar graph with same planarity value as of G_1.*

Proof. Isomorphism preserves edge and vertex weights. Also, the order and size of isomorphic m-polar fuzzy graphs are preserved [46]. So, the order and size of G_2 will be equal to G_1. Then, G_2 can be drawn similarly as G_1. Hence, the number of points of intersections between edges and planarity value of G_2 will be same as G_1. This means that G_2 can be drawn as m-polar fuzzy planar graph with same planarity value as of G_1. □

Theorem 5.6.2. *Let G_2 be the m-polar fuzzy dual graph of m-polar fuzzy dual graph of a strong m-polar fuzzy dual graph G without weak edges. Then there exists a co-weak isomorphism between G and G_2.*

Proof. Let G_1 be the m-polar fuzzy dual graph of G and G_2 be the m-polar fuzzy dual graph of G_1. Now, the number of vertices of G_2 is equal to the strong m-polar fuzzy faces of G_1 and the number of strong m-polar fuzzy faces in G_1 is equal to the number of vertices in G. Hence, the number of vertices of G_2 and G are same. Also, the number of edges of an m-polar fuzzy planar graph and its dual are same. By the definition of m-polar fuzzy dual graph, the edge membership value of an edge in dual graph is equal to the edge membership value of an edge in m-polar fuzzy graph. Thus, we can construct a co-weak isomorphism between G and G_2. Hence the result. □

Theorem 5.6.3. *Let G_1 and G_2 be two isomorphic m-polar fuzzy graphs with m-polar fuzzy planarity values $\mathcal{P}_{G_1}$ and $\mathcal{P}_{G_2}$ respectively. Then, $\mathcal{P}_{G_1} = \mathcal{P}_{G_2}$.*

Proof. Follows from Theorem 5.6.1. □

Next, we state the following without proof.

Theorem 5.6.4. *Let G_1 and G_2 be two weak isomorphic m-polar fuzzy graphs with m-polar fuzzy planarity values $\mathcal{P}_{G_1}$ and $\mathcal{P}_{G_2}$ respectively. Then, $\mathcal{P}_{G_1} = \mathcal{P}_{G_2}$ if the edge membership values of corresponding intersecting edges are same.*

Theorem 5.6.5. *Let G_1 and G_2 be two co-weak isomorphic m-polar fuzzy graphs with m-polar fuzzy planarity values $\mathcal{P}_{G_1}$ and $\mathcal{P}_{G_2}$ respectively. Then, $\mathcal{P}_{G_1} = \mathcal{P}_{G_2}$ if the minimum of end vertex membership values of corresponding intersecting edges are same.*

5.7 Summary

Crossing may be allowed in connecting the wire lines, gas lines, water lines, printed circuit designs, etc. These graph theoretic problems may be uncertain in some aspects. It is quiet natural to deal with the vagueness and uncertainty using the concepts of m-polar fuzzy sets compared to fuzzy sets. Therefore, the concept of m-polar fuzzy sets is applied to multigraph and planar graphs. m-polar fuzzy planar graph is a very important subclass of m-polar fuzzy graph and m-polar fuzzy multigraph is a generalization of m-polar fuzzy graph. In this chapter, we define both these graphs and study several properties. The m-polar fuzzy planar graph is defined in a very interesting way along with a parameter " m-polar fuzzy planarity value". This parameter measures the planarity of an m-polar fuzzy graph. The other relevant terms such as considerable edges, m-polar fuzzy faces, m-polar fuzzy strong faces are defined here. A very close association of m-polar fuzzy planar graph is m-polar fuzzy dual graph. This is also defined and several properties of it are studied. m-polar fuzzy planar graph and m-polar fuzzy dual graph have many applications in different fields including design of subway tunnel or routes, gas or oil pipelines, image segmentation, etc.

Chapter 6

Isomorphic properties of m-polar fuzzy graphs*

6.1 Introduction

In this chapter, weak self complement m-polar fuzzy graphs is defined and a necessary condition is mentioned for an m-polar fuzzy graph to be weak self complement. Some properties of self complement and weak self complement m-polar fuzzy graphs are discussed. The order, size, busy vertices and free vertices of an m-polar fuzzy graphs are also defined and proved that isomorphic m-polar fuzzy graphs have same order, size and degree. Also, we have proved some results of busy vertices in isomorphic and weak isomorphic m-polar fuzzy graphs. A relative study of complement and operations on m-polar fuzzy graphs have been made. Finally, we have modeled some real life situations in terms of m-polar fuzzy graphs as an application.

6.2 Weak self complement m-polar fuzzy graphs

Self complement m-polar fuzzy graphs have many important role in the theory of m-polar fuzzy graphs. If an m-polar fuzzy graph is not self complement, then also we can say that it is self complement in some weaker sense. Simultaneously, we can establish some results with this graph. This motivates to define weak self complement m-polar fuzzy graphs.

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Definition 6.2.1. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of the crisp graph $G^* = (V, E)$. The complement of G is an m-polar fuzzy graph $\overline{G} = (V, \overline{A}, \overline{B})$ of $\overline{G^*} = (V, \widetilde{V^2})$ such that $\overline{A} = A$ and $\overline{B}$ is defined by*

$p_i \circ \overline{B}(xy) = min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy)$ *for all* $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.

Example 6.2.1. *Let $G = (V, A, B)$ be a 3-polar fuzzy graph of the graph $G^* = (V, E)$ where $V = \{u, v, w, x\}$, $E = \{uv, vw, wu, ux\}$,*

$A = \{\frac{<0.2,0.3,0.5>}{u}, \frac{<0.5,0.6,0.3>}{v}, \frac{<0.7,0.2,0.3>}{w}, \frac{<0.2,0.5,0.7>}{x}\}$,

$B = \{\frac{<0.2,0.3,0.3>}{uv}, \frac{<0.4,0.1,0.1>}{vw}, \frac{<0.1,0.1,0.1>}{wu}, \frac{<0.1,0.2,0.4>}{xu}, \frac{<0,0,0>}{xv}, \frac{<0,0,0>}{wx}\}$.

Then by Definition 6.2.1, we have constructed the complement $\overline{G}$ of G which is shown in Fig. 6.1.

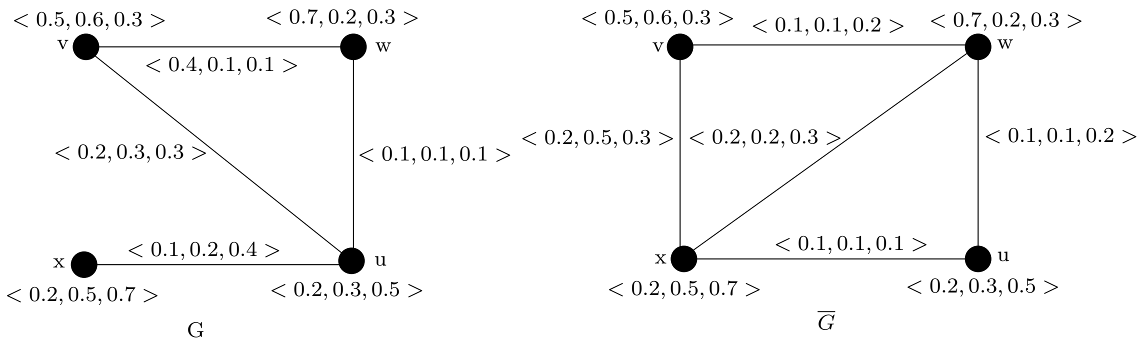

Figure 6.1: G and it's complement $\overline{G}$

Remark 6.2.1. *Let $\overline{\overline{G}} = (V, \overline{\overline{A}}, \overline{\overline{B}})$ be the complement of $\overline{G}$ where $\overline{\overline{A}} = \overline{A} = A$ and*

$p_i \circ \overline{\overline{B}}(uv) = min\{p_i \circ \overline{A}(u), p_i \circ \overline{A}(v)\} - p_i \circ \overline{B}(uv)$

$= min\{p_i \circ A(u), p_i \circ A(v)\} - \{min\{p_i \circ A(u), p_i \circ A(v)\} - p_i \circ B(uv)\}$

$= p_i \circ B(uv)$ *for* $uv \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.

Hence, $\overline{\overline{G}} = G$.

Definition 6.2.2. *The m-polar fuzzy graph $G = (V, A, B)$ is said to be weak self complement if there is a weak isomorphism between G onto $\overline{G}$. In other words, there exist a bijective homomorphism $\phi : G \to \overline{G}$ such that for $i = 1, 2, \ldots, m$*

(i) $p_i \circ A(u) = p_i \circ \overline{A}(\phi(u))$ *for all* $u \in V$,

(ii) $p_i \circ B(uv) \leq p_i \circ \overline{B}(\phi(u)\phi(v))$ *for all* $uv \in \widetilde{V^2}$.

Example 6.2.2. *Let $G = (V, A, B)$ be a 3-polar fuzzy graph of the graph $G^* = (V, E)$ where $V = \{u, v, w\}$, $E = \{uv, vw\}$, $A = \{\frac{<0.3,0.4,0.4>}{u}, \frac{<0.2,0.5,0.7>}{v}, \frac{<0.3,0.6,0.7>}{w}\}$,*

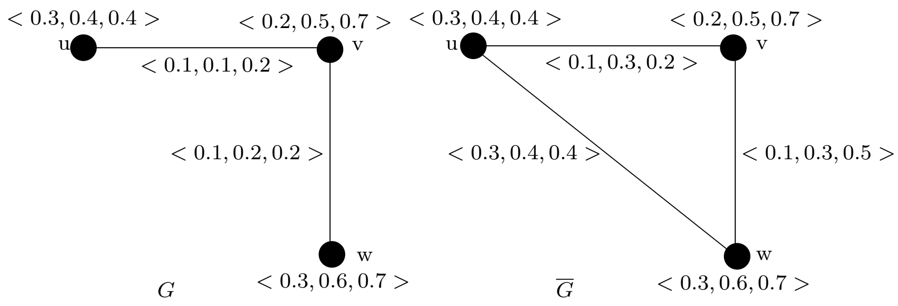

Figure 6.2: Weak self complement 3-polar fuzzy graphs

$B = \{\frac{<0.1,0.1,0.2>}{uv}, \frac{<0.1,0.2,0.2>}{vw}, \frac{<0,0,0>}{wu}\}$.

Then $\overline{G} = (V, \overline{A}, \overline{B})$ *is also a 3-polar fuzzy graph where* $\overline{A} = A$ *and*

$\overline{B} = \{\frac{<0.1,0.3,0.2>}{uv}, \frac{<0.1,0.3,0.5>}{vw}, \frac{<0.3,0.4,0.4>}{wu}\}$. *We can easily verify that, the identity map is an weak isomorphism from* G *onto* $\overline{G}$ *(see Fig. 6.2). Hence,* G *is weak self complement.*

In [45], Ghorai and Pal proved that if G is a self complementary strong m-polar fuzzy graph then for all $xy \in \widetilde{V^2}$ and $i = 1, 2, \ldots, m$

$$\sum_{x \neq y} p_i \circ B(xy) = \frac{1}{2} \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\}.$$

The converse of the above result does not hold always.

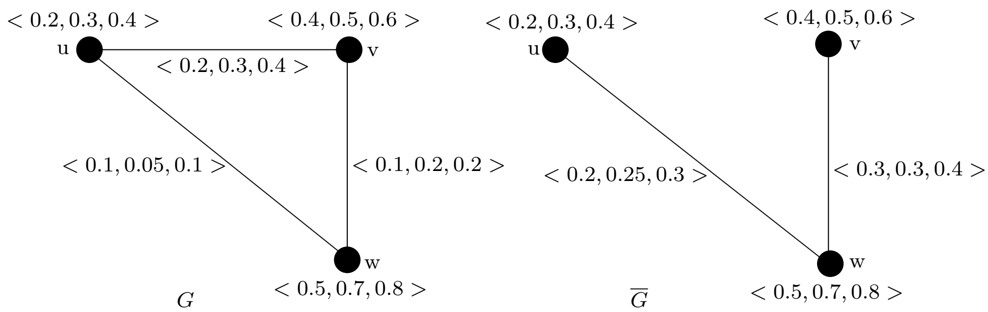

Figure 6.3: Example of 3-polar fuzzy graph G which is not self complement

Example 6.2.3. *For example, let us consider a 3-polar fuzzy graph* $G = (V, A, B)$ *of* $G^* = (V, E)$ *where* $V = \{u, v, w\}$, $E = \{uv, vw, wu\}$,

$A = \{\frac{<0.2,0.3,0.4>}{u}, \frac{<0.4,0.5,0.6>}{v}, \frac{<0.5,0.7,0.8>}{w}\}$,

$B = \{\frac{<0.2,0.3,0.4>}{uv}, \frac{<0.1,0.2,0.2>}{vw}, \frac{<0.1,0.05,0.1>}{wu}\}$.

Then, we have the following:

$p_1 \circ B(uv) + p_1 \circ B(vw) + p_1 \circ B(wu) = 0.2 + 0.1 + 0.1 = 0.4$ *and*

$\frac{1}{2}[min\{p_1 \circ A(u), p_1 \circ A(v)\} + min\{p_1 \circ A(v), p_1 \circ A(w)\} + min\{p_1 \circ A(w), p_1 \circ A(u)\}]$
$= \frac{1}{2}[min\{0.2, 0.4\} + min\{0.4, 0.5\} + min\{0.5, 0.2\}] = \frac{1}{2}(0.2 + 0.4 + 0.2) = 0.4.$

So,

$$\sum_{u \neq v} p_1 \circ B(uv) = 0.4 = \frac{1}{2} \sum_{u \neq v} min\{p_1 \circ A(u), p_i \circ A(v)\}.$$

Similarly,

$$\sum_{u \neq v} p_2 \circ B(uv) = 0.55 = \frac{1}{2} \sum_{u \neq v} min\{p_2 \circ A(u), p_2 \circ A(v)\}$$

and

$$\sum_{u \neq v} p_3 \circ B(uv) = 0.7 = \frac{1}{2} \sum_{u \neq v} min\{p_3 \circ A(u), p_3 \circ A(v)\}.$$

Hence, for $i = 1, 2, 3$ we have,

$$\sum_{u \neq v} p_i \circ B(uv) = \frac{1}{2} \sum_{u \neq v} min\{p_i \circ A(u), p_i \circ A(v)\}.$$

But, G is not self complementary as there exists no isomorphism from G onto $\overline{G}$ (see Fig. 6.3).

Now suppose an m-polar fuzzy graph $G = (V, A, B)$ is a weak self complement. Then the following inequality holds.

Theorem 6.2.1. *Let $G = (V, A, B)$ be a weak self complement m-polar fuzzy graph of G^*. Then for $i = 1, 2, \ldots, m$*

$$\sum_{x \neq y} p_i \circ B(xy) \leq \frac{1}{2} \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\}.$$

Proof. Since G is weak self complement, therefore there exists a weak isomorphism $\phi : V \to V$ such that

$p_i \circ A(x) = p_i \circ \overline{A}(\phi(x))$ for all $x \in V$ and

$p_i \circ B(xy) \leq p_i \circ \overline{B}(\phi(x)\phi(y))$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.

Using the above we have,

$p_i \circ B(xy) \leq p_i \circ \overline{B}(\phi(x)\phi(y)) = min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(\phi(x)\phi(y)),$

i.e. $p_i \circ B(xy) + p_i \circ B(\phi(x)\phi(y)) \leq min\{p_i \circ A(\phi(x)), p_i \circ A(\phi(y))\}.$

Therefore, for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$

$$\sum_{x \neq y} p_i \circ B(xy) + \sum_{x \neq y} p_i \circ B(\phi(x)\phi(y)$$

$$\leq \sum_{x \neq y} min\{p_i \circ A(\phi(x)), p_i \circ A(\phi(y))\}$$
$$= \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\},$$

i.e.

$$2 \sum_{x \neq y} p_i \circ B(xy) \leq \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\},$$

i.e.

$$\sum_{x \neq y} p_i \circ B(xy) \leq \frac{1}{2} \sum_{x \neq y} min\{p_i \circ A(x), p_i \circ A(y)\}.$$

□

Remark 6.2.2. *The converse of the above theorem is not true in general. For example, consider the 3-polar fuzzy graph of Fig. 6.3. We see that for the 3-polar fuzzy graph G, the condition of Theorem 6.2.1 is satisfied. But, G is not weak self complementary as there is no weak isomorphism from G onto $\overline{G}$.*

Theorem 6.2.2. *If $p_i \circ B(xy) \leq \frac{1}{2} min\{p_i \circ A(x), p_i \circ A(y)\}$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$, then G is a weak self complement m-polar fuzzy graph.*

Proof. Let $\overline{G} = (V, \overline{A}, \overline{B})$ be the complement of G where

$\overline{A}(x) = A(x)$ for all $x \in V$ and

$p_i \circ \overline{B}(xy) = min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy)$ for $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.

Let us now consider the identity map $I : V \to V$.

Then $A(x) = A(I(x)) = \overline{A}(I(x))$ for all $x \in V$ and

$p_i \circ \overline{B}(I(x)I(y)) = p_i \circ \overline{B}(xy)$

$= min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy)$

$\geq min\{p_i \circ A(x), p_i \circ A(y)\} - \frac{1}{2} min\{p_i \circ A(x), p_i \circ A(y)\}$

$= \frac{1}{2} min\{p_i \circ A(x), p_i \circ A(y)\} \geq p_i \circ B(xy)$.

So, $p_i \circ B(xy) \leq p_i \circ \overline{B}(I(x)I(y))$ for $i = 1, 2, \ldots, m$ and $xy \in \widetilde{V^2}$.

Hence, $I : V \to V$ is a weak isomorphism. □

Example 6.2.4. *Consider the 3-polar fuzzy graph $G = (V, A, B)$ of $G^* = (V, E)$ where $V = \{u, v, w\}$, $E = \{uv, vw, wu\}$,*

$A = \{\frac{<0.2,0.3,0.4>}{u}, \frac{<0.4,0.5,0.6>}{v}, \frac{<0.5,0.7,0.9>}{w}\}$, $B = \{\frac{<0.1,0.1,0.2>}{uv}, \frac{<0.2,0.2,0.3>}{vw}, \frac{<0.1,0.1,0.2>}{wu}\}$.

We see that for each $i = 1, 2, 3$ and $xy \in \widetilde{V^2}$,

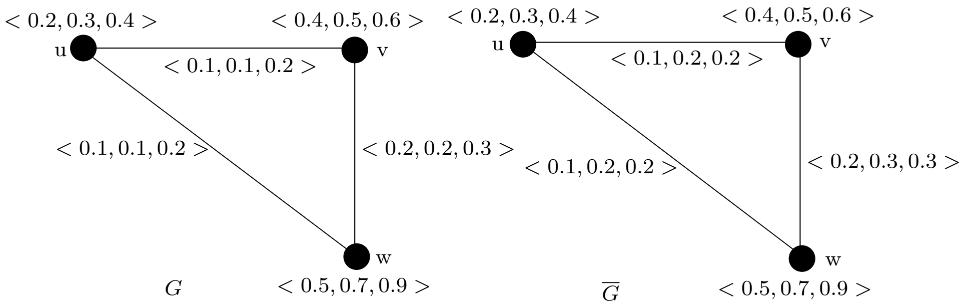

Figure 6.4: Example of 3-polar fuzzy graph G which is weak self complement

$p_i \circ B(xy) \leq \frac{1}{2} min\{p_i \circ A(x), p_i \circ A(y)\}$.

Also, consider the complement of G of Fig. 6.4.

Let us now consider the identity mapping $I : G \to \overline{G}$ such that $I(u) = u$ for all $u \in V$. Then, I is the required weak isomorphism from G onto $\overline{G}$. Hence, G is weak self complementary.

6.3 Order, size and busy value of vertices of m-polar fuzzy graphs

In this section, the order, size, busy value of vertices of an m-polar fuzzy graph is defined.

Definition 6.3.1. *The order of the m-polar fuzzy graph $G = (V, A, B)$ is denoted by $|V|$ (or $O(G)$) where*

$$O(G) = |V| = \sum_{x \in V} \frac{1 + \sum\limits_{i=1}^{m} p_i \circ A(x)}{2}.$$

The size of G is denoted by $|E|$ (or $S(G)$) where

$$S(G) = |E| = \sum_{xy \in E} \frac{1 + \sum\limits_{i=1}^{m} p_i \circ B(xy)}{2}.$$

Theorem 6.3.1. *Two isomorphic m-polar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ have same order and size.*

Proof. Let ϕ be an isomorphism from G_1 onto G_2.

Then $A_1(x) = A_2(\phi(x))$ for all $x \in V_1$ and

$p_i \circ B_1(xy) = p_i \circ B_2(\phi(x)\phi(y))$ for all $xy \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$.

Now,

$$O(G_1) = |V_1| = \sum_{x\in V_1} \frac{1+\sum\limits_{i=1}^{m} p_i \circ A_1(x)}{2}$$

$$= \sum_{\phi(x)\in V_2} \frac{1+\sum\limits_{i=1}^{m} p_i \circ A_2(\phi(x))}{2} = O(G_2)$$

and

$$S(G_1) = |E_1| = \sum_{xy\in E_1} \frac{1+\sum\limits_{i=1}^{m} p_i \circ B_1(xy)}{2}$$

$$= \sum_{\phi(x)\phi(y)\in E_2} \frac{1+\sum\limits_{i=1}^{m} p_i \circ B_2(\phi(x)\phi(y))}{2} = S(G_2).$$

□

Definition 6.3.2. *The busy value of a vertex u of an m-polar fuzzy graph G is denoted as $D(u) = (p_1 \circ D(u), p_2 \circ D(u), \ldots, p_m \circ D(u))$, where $p_i \circ D(u) = \sum\limits_k min\{p_i \circ A(u), p_i \circ A(u_k)\}$; u_k are the neighbors of u. The busy value of G is denoted as $D(G)$ where $D(G) = \sum\limits_k D(u_k)$, $u_k \in V$.*

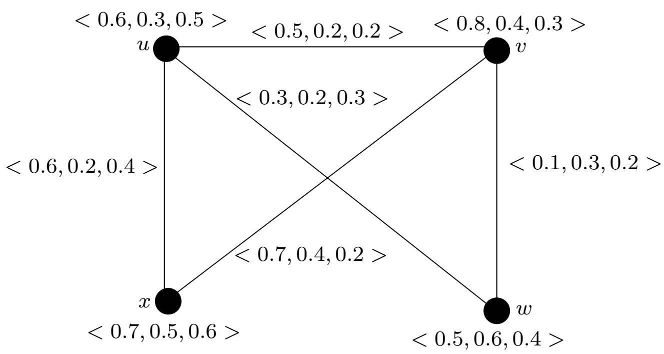

Figure 6.5: 3-polar fuzzy graph G and busy value of its vertices

Example 6.3.1. *Consider the 3-polar fuzzy graph $G = (V, A, B)$ of $G^* = (V, E)$ where $V = \{u, v, w, x\}$, $E = \{uv, vw, ux, uw, vx\}$,*

$A = \{\frac{<0.6,0.3,0.5>}{u}, \frac{<0.8,0.4,0.3>}{v}, \frac{<0.5,0.6,0.4>}{w}, \frac{<0.7,0.5,0.6>}{x}\}$ *and*

$B = \{\frac{<0.5,0.2,0.2>}{uv}, \frac{<0.1,0.3,0.2>}{vw}, \frac{<0.6,0.2,0.4>}{ux}, \frac{<0.3,0.2,0.3>}{uw}, \frac{<0.7,0.4,0.2>}{vx}\}$.

Then we have from Fig. 6.5,

$p_1 \circ D(u) = 1.7$, $p_2 \circ D(u) = 0.9$, $p_3 \circ D(u) = 1.2$,

$p_1 \circ D(v) = 1.8$, $p_2 \circ D(v) = 1.1$, $p_3 \circ D(v) = 0.9$,

$p_1 \circ D(w) = 1,\ p_2 \circ D(w) = 0.7,\ p_3 \circ D(w) = 0.7,$

$p_1 \circ D(x) = 1.3,\ p_2 \circ D(x) = 0.7,\ p_3 \circ D(x) = 0.8.$

So, $D(u) = (1.7, 0.9, 1.2),\ D(v) = (1.8, 1.1, 0.9),$

$D(w) = (1, 0.7, 0.7),\ D(x) = (1.3, 0.7, 0.8).$

Definition 6.3.3. *If $p_i \circ A(u) \leq p_i \circ deg(u)$ for $i = 1, 2, \ldots, m$, then the vertex u of G is called a busy vertex. Otherwise, it is a free vertex.*

Definition 6.3.4. *If $p_i \circ B(u_1v_1) = min\{p_i \circ A(u_1), p_i \circ A(v_1)\}$, $i = 1, 2, \ldots, m$ for $u_1v_1 \in E$, then it is called an effective edge of G.*

Definition 6.3.5. *Let $u \in V$ be a vertex of the m-polar fuzzy graph $G = (V, A, B)$.*

(i) u is called a partial free vertex if it is a free vertex of G and $\overline{G}$.

(ii) u is called a fully free vertex if it is a free vertex of G and it is a busy vertex of $\overline{G}$.

(iii) u is called a partial busy vertex if it is a busy vertex of G and $\overline{G}$.

(iv) u is called a fully busy vertex if it is a busy vertex in G and it is a free vertex of $\overline{G}$.

Theorem 6.3.2. *Let ϕ be an isomorphism from $G_1 = (V_1, A_1, B_1)$ onto $G_2 = (V_2, A_2, B_2)$. Then, $deg(u) = deg(\phi(u))$ for all $u \in V_1$.*

Proof. Since ϕ is an isomorphism between G_1 and G_2, we have

$p_i \circ A_1(u) = p_i \circ A_2(\phi(u))$ for all $u \in V_1$ and

$p_i \circ B_1(x_1y_1) = p_i \circ B_2(\phi(x_1)\phi(y_1))$ for all $x_1y_1 \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$.

Therefore,

$$p_i \circ deg(u) = \sum_{\substack{u \neq v \\ uv \in E_1}} p_i \circ B_1(uv) = \sum_{\substack{\phi(u) \neq \phi(v) \\ \phi(u)\phi(v) \in E_2}} p_i \circ B_2(\phi(u)\phi(v)) = p_i \circ deg(\phi(u))$$

for $u \in V_1$, $i = 1, 2, \ldots, m$. Hence, $deg(u) = deg(\phi(u))$ for all $u \in V_1$. □

Theorem 6.3.3. *If ϕ is an isomorphism from G_1 onto G_2 and u is a busy vertex of G_1, then $\phi(u)$ is a busy vertex of G_2.*

Proof. Since ϕ is an isomorphism between G_1 and G_2 we have,

$p_i \circ A_1(u) = p_i \circ A_2(\phi(u))$ for $u \in V_1$ and

$p_i \circ B_1(x_1y_1) = p_i \circ B_2(\phi(x_1)\phi(y_1))$ for $x_1y_1 \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$.

If u is a busy vertex of G_1, then $p_i \circ A_1(u) \leq p_i \circ deg(u)$ for $i = 1, 2, \ldots, m$.

Then, by the above and Theorem 6.3.2,

$p_i \circ A_2(\phi(u)) = p_i \circ A_1(u) \leq p_i \circ deg(u) = p_i \circ deg(\phi(u))$ for $i = 1, 2, \ldots, m$.

Hence, $\phi(u)$ is a busy vertex in G_2. □

Theorem 6.3.4. *Let the two m-polar fuzzy graphs G_1 and G_2 be weak isomorphic. If $u \in V_1$ is a busy vertex of G_1, then the image of u under the weak isomorphism is also busy in G_2.*

Proof. Let $\phi : V_1 \to V_2$ be a weak isomorphism between G_1 and G_2.

Then, $p_i \circ A_1(x) = p_i \circ A_2(\phi(x))$ for all $x \in V_1$ and

$p_i \circ B_1(x_1y_1) \leq p_i \circ B_2(\phi(x_1)\phi(y_1))$ for all $x_1y_1 \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$.

Let $u \in V_1$ be a busy vertex.

Then, $p_i \circ A_1(u) \leq p_i \circ deg(u)$ for $i = 1, 2, \ldots, m$.

Now, by the above for $i = 1, 2, \ldots, m$

$$p_i \circ A_2(\phi(u)) = p_i \circ A_1(u)$$
$$\leq p_i \circ deg(u) = \sum_{\substack{u \neq v \\ uv \in E_1}} p_i \circ B_1(uv) \leq \sum_{\substack{\phi(u) \neq \phi(v) \\ \phi(u)\phi(v) \in E_2}} p_i \circ B_2(\phi(u)\phi(v))$$
$$= p_i \circ deg(\phi(u)).$$

Hence, $\phi(u)$ is a busy vertex in G_2. □

6.4 Complement and isomorphism in m-polar fuzzy graphs

In this section, some important properties of isomorphism, weak isomorphism, co weak isomorphism related with complement are discussed.

Theorem 6.4.1. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. If $G_1 \cong G_2$, then $\overline{G_1} \cong \overline{G_2}$.*

Proof. Let $G_1 \cong G_2$.

Then there exists an isomorphism $\phi : V_1 \to V_2$ such that

$A_1(x) = A_2(\phi(x))$ for all $x \in V_1$ and

$p_i \circ B_1(xy) = p_i \circ B_2(\phi(x)\phi(y))$ for each $i = 1, 2, \ldots, m$ and $xy \in \widetilde{V_1^2}$.

Now, $\overline{A_1}(x) = A_1(x) = A_2(\phi(x)) = \overline{A_2}(\phi(x))$ for all $x \in V_1$.

Also, for $i = 1, 2, \ldots, m$ and $xy \in \widetilde{V_1^2}$ we have,

$p_i \circ \overline{B_1}(xy)$

$= min\{p_i \circ A_1(x), p_i \circ A_1(y)\} - p_i \circ B_1(xy)$

$= min\{p_i \circ A_2(\phi(x), p_i \circ A_2(\phi(y)\} - p_i \circ B_2(\phi(x)\phi(y))$

$= p_i \circ \overline{B_2}(\phi(x)\phi(y))$.

Hence, ϕ is an isomorphism between $\overline{G_1}$ and $\overline{G_2}$, i.e. $\overline{G_1} \cong \overline{G_2}$. □

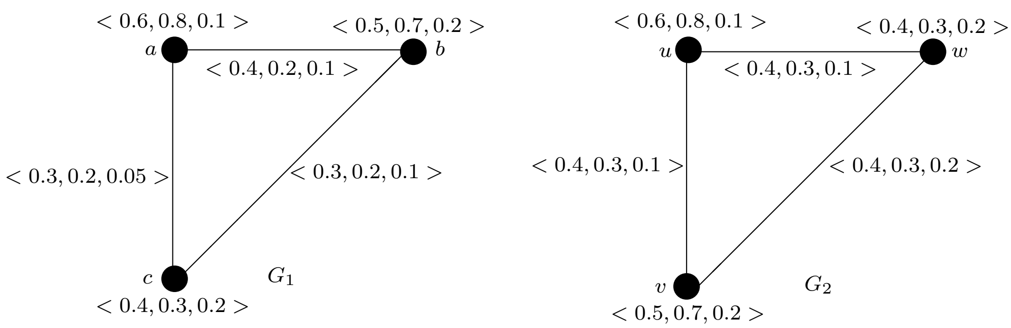

Figure 6.6: Weak isomorphic 3-polar fuzzy graphs G_1 and G_2

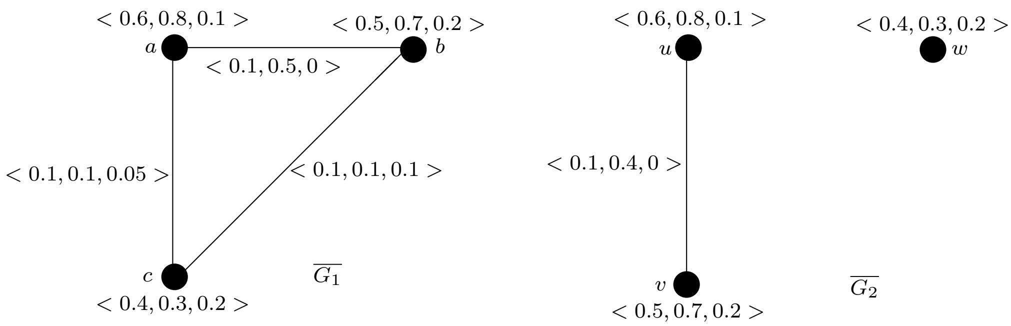

Figure 6.7: Example of weak isomorphic graphs whose complement is not weak isomorphic

Remark 6.4.1. *Suppose there is a weak isomorphism between two m-polar fuzzy graphs G_1 and G_2. Then there may not be a weak isomorphism between $\overline{G_1}$ and $\overline{G_2}$.*

For example, consider two 3-polar fuzzy graphs G_1 and G_2 of Fig. 6.6.

Let us now define a mapping $\phi : V_1 \to V_2$ such that $\phi(a) = u$, $\phi(b) = v$, $\phi(c) = w$. Then, ϕ is a weak isomorphism from G_1 onto G_2. But, there is no weak isomorphism from $\overline{G_1}$ onto $\overline{G_2}$ (see Fig. 6.7) because

$\overline{B_2}(uw = \phi(a)\phi(c)) = \boldsymbol{0} = (0, 0, \ldots, 0) < \overline{B_1}(ac) = (0.1, 0.1, 0.05)$ *and*

$\overline{B_2}(vw = \phi(b)\phi(c)) = \boldsymbol{0} = (0, 0, \ldots, 0) < \overline{B_1}(bc) = (0.1, 0.1, 0.1)$.

Remark 6.4.2. *In a similar way, we can construct example to show that if there is a co-weak isomorphism between two m-polar fuzzy graphs G_1 and G_2 then there may not be a co-weak isomorphism between $\overline{G_1}$ and $\overline{G_2}$.*

Theorem 6.4.2. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 \cap V_2 = \emptyset$. Then $\overline{G_1 + G_2} \cong \overline{G_1} \cup \overline{G_2}$.*

Proof. To show that $\overline{G_1 + G_2} \cong \overline{G_1} \cup \overline{G_2}$, we need to show that there exists an isomorphism between $\overline{G_1 + G_2}$ and $\overline{G_1} \cup \overline{G_2}$.

We will show that the identity map $I : V_1 \cup V_2 \to V_1 \cup V_2$ is the required isomorphism between them. For this, we will show the following:

$\overline{(A_1 + A_2)}(x) = (\overline{A_1} \cup \overline{A_2})(x)$ for all $x \in V_1 \cup V_2$ and

$p_i \circ \overline{(B_1 + B_2)}(xy) = p_i \circ (\overline{B_1} \cup \overline{B_2})(xy)$ for all $xy \in \widetilde{V_1 \times V_2}^2$, $i = 1, 2, \ldots, m$.

Let $x \in V_1 \cup V_2$.

Then $\overline{(A_1 + A_2)}(x)$

$= (A_1 + A_2)(x)$

$= (A_1 \cup A_2)(x)$

$$= \begin{cases} A_1(x) & if \quad x \in V_1 - V_2 \\ A_2(x) & if \quad x \in V_2 - V_1 \end{cases}$$

$$= \begin{cases} \overline{A_1}(x) & if \quad x \in V_1 - V_2 \\ \overline{A_2}(x) & if \quad x \in V_2 - V_1 \end{cases}$$

$= (\overline{A_1} \cup \overline{A_2})(x)$.

Now, for each $i = 1, 2, \ldots, m$ and $xy \in \widetilde{V_1 \times V_2}^2$ we have,

$p_i \circ \overline{(B_1 + B_2)}(xy)$

$= min\{p_i \circ (A_1 + A_2)(x), p_i \circ (A_1 + A_2)(y)\} - p_i \circ (B_1 + B_2)(xy)$

$$= \begin{cases} min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\} - p_i \circ (B_1 \cup B_2)(xy), \text{if } xy \in E_1 \cup E_2 \\ min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\} - min\{p_i \circ (A_1)(x), p_i \circ (A_2)(y)\}, \\ \text{if } xy \in E' \end{cases}$$

$$= \begin{cases} min\{p_i \circ A_1(x), p_i \circ A_1(y)\} - p_i \circ B_1(xy), \text{if } xy \in E_1 - E_2 \\ min\{p_i \circ A_2(x), p_i \circ A_2(y)\} - p_i \circ B_2(xy), \text{if } xy \in E_2 - E_1 \\ min\{p_i \circ A_1(x), p_i \circ A_2(y)\} - min\{p_i \circ (A_1)(x), p_i \circ (A_2)(y)\}, \text{if } xy \in E' \end{cases}$$

$$= \begin{cases} p_i \circ \overline{B_1}(xy), & if \quad xy \in E_1 - E_2 \\ p_i \circ \overline{B_2}(xy), & if \quad xy \in E_2 - E_1 \\ 0, & if \quad xy \in E' \end{cases}$$

$= p_i \circ (\overline{B_1} \cup \overline{B_2})(xy).$ □

Theorem 6.4.3. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 \cap V_2 = \emptyset$. Then $\overline{G_1 \cup G_2} \cong \overline{G_1} + \overline{G_2}$.*

Proof. Consider the identity map $I : V_1 \cup V_2 \to V_1 \cup V_2$. We will show that I is the required isomorphism between $\overline{G_1 \cup G_2}$ and $\overline{G_1} + \overline{G_2}$.

For this, we will show the following:

$\overline{(A_1 \cup A_2)}(x) = (\overline{A_1} + \overline{A_2})(x)$, for all $x \in V_1 \cup V_2$ and

$p_i \circ \overline{(B_1 \cup B_2)}(xy) = p_i \circ (\overline{B_1} + \overline{B_2})(xy)$ for all $xy \in \widetilde{V_1 \times V_2}^{2}$, $i = 1, 2, \ldots, m$.

Let $x \in V_1 \cup V_2$.

Then $\overline{A_1 \cup A_2}(x)$

$= (A_1 \cup A_2)(x)$

$$= \begin{cases} A_1(x), & if \quad x \in V_1 - V_2 \\ A_2(x), & if \quad x \in V_2 - V_1 \end{cases}$$

$$= \begin{cases} \overline{A_1}(x), & if \quad x \in V_1 - V_2 \\ \overline{A_2}(x), & if \quad x \in V_2 - V_1 \end{cases}$$

$= (\overline{A_1} \cup \overline{A_2})(x)$

and

$p_i \circ \overline{(B_1 \cup B_2)}(xy)$

$= min\{p_i \circ (A_1 \cup A_2)(x), p_i \circ (A_1 \cup A_2)(y)\} - p_i \circ (B_1 \cup B_2)(xy)$

$$= \begin{cases} min\{p_i \circ A_1(x), p_i \circ A_1(y)\} - p_i \circ B_1(xy), & if \quad xy \in E_1 - E_2 \\ min\{p_i \circ A_2(x), p_i \circ A_2(y)\} - p_i \circ B_2(xy), & if \quad xy \in E_2 - E_1 \\ min\{p_i \circ A_1(x), p_i \circ A_2(y)\} - 0, & if \quad x \in V_1, y \in V_2 \end{cases}$$

$$= \begin{cases} p_i \circ \overline{B_1}(xy), & if \quad xy \in E_1 - E_2 \\ p_i \circ \overline{B_2}(xy), & if \quad xy \in E_2 - E_1 \\ min\{p_i \circ A_1(x), p_i \circ A_2(y)\} - 0, & if \quad x \in V_1, y \in V_2 \end{cases}$$

$$= \begin{cases} p_i \circ \overline{B_1}(xy), & if \quad xy \in E_1 - E_2 \\ p_i \circ \overline{B_2}(xy), & if \quad xy \in E_2 - E_1 \\ min\{p_i \circ A_1(x), p_i \circ A_2(y)\} - 0, & if \quad xy \in E' \end{cases}$$

$=p_i \circ (\overline{B_1} + \overline{B_2})(xy)$ for $i = 1, 2, \dots, m$, $xy \in \widetilde{V_1 \times V_2}^{2}$. This completes the proof. $\square$

Theorem 6.4.4. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two strong m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. Then $\overline{G_1 \circ G_2} \cong \overline{G_1} \circ \overline{G_2}$.*

Proof. Let $G_1 \circ G_2 = (V_1 \times V_2, A_1 \circ A_2, B_1 \circ B_2)$ be an m-polar fuzzy graph of the graph $G^* = (V, E)$ where $V = V_1 \times V_2$ and

$E = \{(x, x_2)(x, y_2) : x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) : z \in V_2, x_1y_1 \in E_1\} \cup \{(x_1, x_2)(y_1, y_2) : x_1y_1 \in E_1, x_2 \neq y_2\}$.

We show that the identity map I is the required isomorphism between the graphs $\overline{G_1 \circ G_2}$ and $\overline{G_1} \circ \overline{G_2}$.

Let us consider the identity map $I : V_1 \times V_2 \to V_1 \times V_2$.

In order to show that I is the required isomorphism, we show that

$p_i \circ \overline{(B_1 \circ B_2)}(xy) = p_i \circ (\overline{B_1} \circ \overline{B_2})(xy)$ for all $xy \in \widetilde{V_1 \times V_2}^{2}$, $i = 1, 2, \dots, m$.

Several cases may arise.

Case(i): Let $e = (x, x_2)(x, y_2)$ where $x \in V_1$, $x_2y_2 \in E_2$. Then $e \in E$.

Since $G_1 \circ G_2$ is strong m-polar fuzzy graph, we have for each $i = 1, 2, \dots, m$

$p_i \circ \overline{(B_1 \circ B_2)}(e) = 0$ and

$p_i \circ (\overline{B_1} \circ \overline{B_2})(e) = min\{p_i \circ A_1(x), p_i \circ \overline{B_2}(x_2y_2)\} = 0$

(since G_2 is strong and $x_2y_2 \in E_2$, therefore $p_i \circ \overline{B_2}(x_2y_2) = 0$ for each $i = 1, 2, \dots, m$).

Case(ii): Let $e = (x, x_2)(x, y_2)$ where $x_2 \neq y_2$, $x_2y_2 \notin E_2$.

Then $e \notin E$.

So for each $i = 1, 2, \dots, m$

$p_i \circ (B_1 \circ B_2)(e) = 0$ and

$p_i \circ \overline{(B_1 \circ B_2)}(e)$

$= min\{p_i \circ (A_1 \circ A_2)(x, x_2), p_i \circ (A_1 \circ A_2)(x, y_2)\}$

$= min\{p_i \circ A_1(x), p_i \circ A_2(x_2), p_i \circ A_2(y_2)\}$.

Again, since $x_2y_2 \in \overline{E_2}$, therefore

$p_i \circ (\overline{B_1} \circ \overline{B_2})(e)$

$= min\{p_i \circ A_1(x), p_i \circ \overline{B_2}(x_2y_2)\}$

$= min\{p_i \circ A_1(x), p_i \circ A_2(x_2), p_i \circ A_2(y_2)\}$ for each $i = 1, 2, \dots, m$.

Case(iii): Let $e = (x_1, z)(y_1, z)$ where $x_1y_1 \in E_1$, $z \in V_2$.

Then $e \in E$.

So, $p_i \circ \overline{(B_1 \circ B_2)}(e) = 0$ for each $i = 1, 2, \dots, m$ as in Case(i).

Also, since $x_1y_1 \notin \overline{E_1}$, therefore $p_i \circ (\overline{B_1} \circ \overline{B_2})(e) = 0$ for each $i = 1, 2, \dots, m$.

Case(iv): Let $e = (x_1, z)(y_1, z)$ where $x_1y_1 \notin E_1$, $z \in V_2$.

Then $e \notin E$.

Hence, $p_i \circ (B_1 \circ B_2)(e) = 0$ for each $i = 1, 2, \dots, m$.

$p_i \circ \overline{(B_1 \circ B_2)}(e)$

$= min\{p_i \circ (A_1 \circ A_2)(x_1, z), p_i \circ (A_1 \circ A_2)(y_1, z)\}$

$= min\{p_i \circ A_1(x_1), p_i \circ A_1(y_1), p_i \circ A_2(z)\}$ and

$p_i \circ (\overline{B_1} \circ \overline{B_2})(e)$

$= min\{p_i \circ A_2(z), p_i \circ \overline{B_1}(x_1y_1)\}$

$= min\{p_i \circ A_1(x_1), p_i \circ A_1(y_1), p_i \circ A_2(z)\}$ (G_1 being strong).

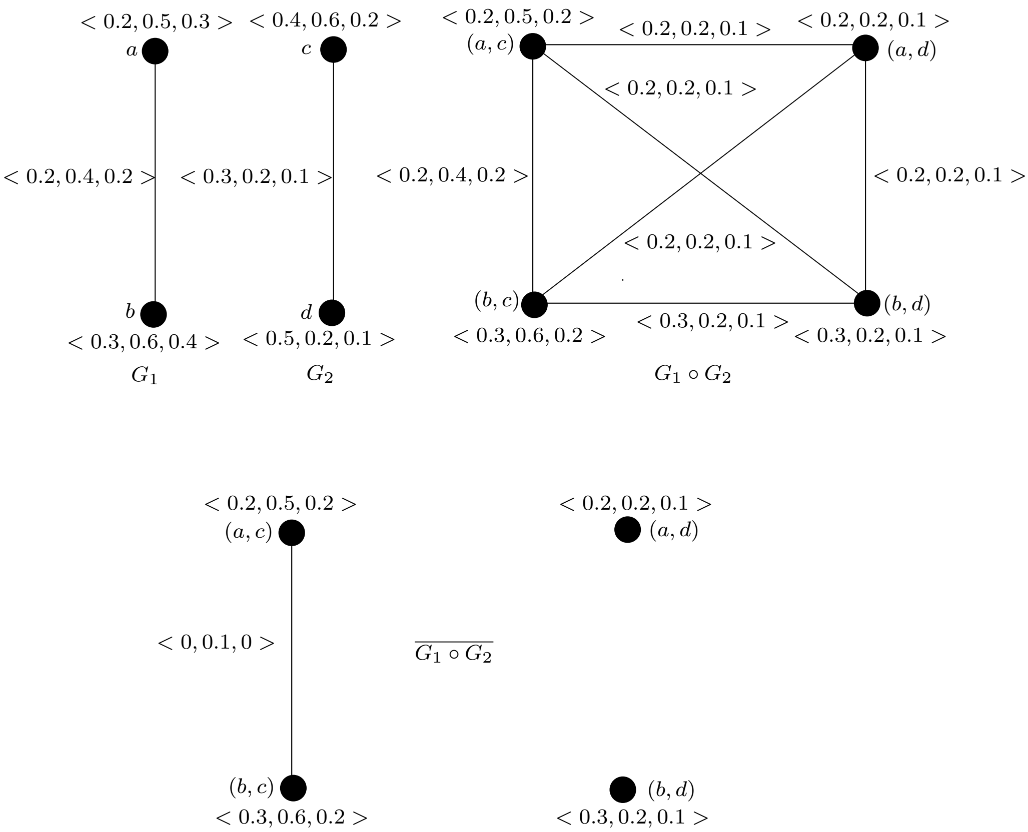

Figure 6.8: G_1, G_2, $G_1 \circ G_2$ and $\overline{G_1 \circ G_2}$

Case(v): Let $e = (x_1, x_2)(y_1, y_2)$ where $x_1y_1 \in E_1$, $x_2 \neq y_2$.

Then $e \in E$.

So, we have $p_i \circ \overline{(B_1 \circ B_2)}(e) = 0$ for each $i = 1, 2, \dots, m$ as in Case(i).

Also, since $x_1y_1 \in E_1$, we have $p_i \circ (\overline{B_1} \circ \overline{B_2})(e) = 0$ for each $i = 1, 2, \ldots, m$.

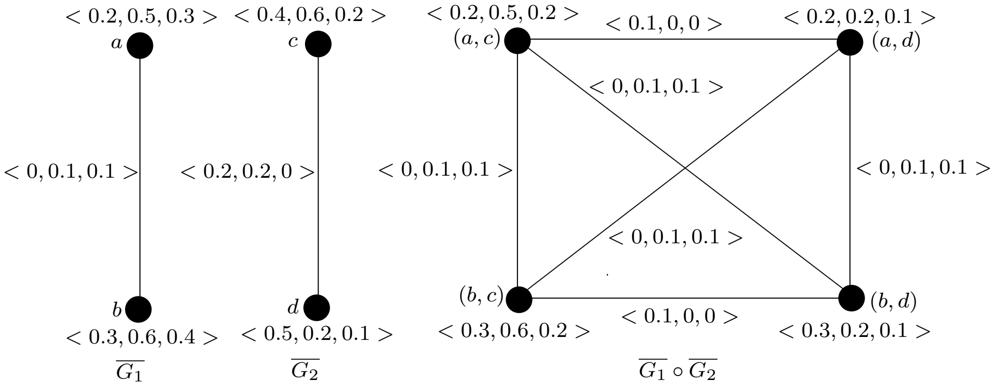

Figure 6.9: Example of 3-polar fuzzy graphs G_1 and G_2, where $\overline{G_1 \circ G_2} \ncong \overline{G_1} \circ \overline{G_2}$

Case(vi): Let $e = (x_1, x_2)(y_1, y_2)$ where $x_1y_1 \notin E_1$, $x_2 \neq y_2$.

Then $e \notin E$ and hence for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \circ B_2)(e) = 0,$

$p_i \circ \overline{(B_1 \circ B_2)}(e)$

$= min\{p_i \circ (A_1 \circ A_2)(x_1, x_2), p_i \circ (A_1 \circ A_2)(y_1, y_2)\}$

$= min\{p_i \circ A_1(x_1), p_i \circ A_1(y_1), p_i \circ A_2(x_2), p_i \circ A_2(y_2)\}$ and since $x_1y_1 \in \overline{E_1}$,

$p_i \circ (\overline{B_1} \circ \overline{B_2})(e)$

$= min\{p_i \circ A_2(x_2), p_i \circ A_2(y_2), p_i \circ \overline{B_1}(x_1y_1)\}$

$= min\{p_i \circ A_1(x_1), p_i \circ A_1(y_1), p_i \circ A_2(x_2), p_i \circ A_2(y_2)\}$ ($\overline{G_1}$ being strong by [45]).

Case(vii): Finally, let $e = (x_1, x_2)(y_1, y_2)$ where $x_1y_1 \notin E_1$, $x_2y_2 \notin E_2$.

Then $e \notin E$ and hence for each $i = 1, 2, \ldots, m$

$p_i \circ (B_1 \circ B_2)(e) = 0,$

$p_i \circ \overline{(B_1 \circ B_2)}(e) = min\{p_i \circ (A_1 \circ A_2)(x_1, x_2), p_i \circ (A_1 \circ A_2)(y_1, y_2)\}.$

Now, $x_1y_1 \in \overline{E_1}$ and if $x_2 = y_2 = z$, then we have the Case(iv).

Again, if $x_1y_1 \in \overline{E_1}$ and if $x_2 \neq y_2$, then we have Case(vi).

Thus combining all the cases we have,

$p_i \circ \overline{(B_1 \circ B_2)}(xy) = p_i \circ (\overline{B_1} \circ \overline{B_2})(xy)$ for all $xy \in \widetilde{V_1 \times V_2}^2$, $i = 1, 2, \ldots, m$. $\square$

Remark 6.4.3. *If G_1 and G_2 are not strong, then $\overline{G_1 \circ G_2} \ncong \overline{G_1} \circ \overline{G_2}$ always. For example, consider two 3-polar fuzzy graphs G_1 and G_2 which are not strong (see Fig. 6.8). From the Fig. 6.8 and Fig. 6.9, we see that, $\overline{G_1 \circ G_2} \ncong \overline{G_1} \circ \overline{G_2}$.*

6.5 Applications

Now a days, fuzzy graphs and bipolar fuzzy graphs are most familiar graphs to us and they can also be thought of as 1-polar and 2-polar fuzzy graphs respectively. These graphs have many important applications in social networks, medical diagnosis, computer networks, database theory, expert system, neural networks, artificial intelligence, signal processing, pattern recognition, engineering science, cluster analysis, etc. The concepts of bipolar fuzzy graphs can be generalized to m-polar fuzzy graphs. For example, consider the sorting of mangoes and guavas. Now the different characteristics of a given fruit can change the decision in sorting process more towards the decision mango or vice versa. There are two poles present in this case. One is 100% sure mango and the other is 100% sure guava. This shows that the situation is bipolar. This situation can be generalized further by adding a new fruit, for example sweet lemon into the sorting process.

6.5.1 Graphical representation of tug of war

Consider the another example of tug of war where two people pull the rope in opposite directions. Here, who uses the larger force, the center of the rope will move in the respective direction of their pulling. The situation is symmetric in this case. We present an example where m people pull a special rope in m different directions. We use this example to represent it as an m-polar fuzzy graph. We assume that O is the origin and there are m straight paths leading from O. We also assume that there is a wall in between these paths. In this setting, we have the special rope with one node at O and m endings going out from this nodes- one end corresponding to each of the paths. Suppose on every path there is a man standing and pulling the rope in the direction of the path on which he is standing. This situation can be represented as an m-polar fuzzy graph by considering the nodes as m-polar fuzzy set and edges between them as m-polar fuzzy relations, which is shown in Fig. 6.10. In this context, one can ask the question what is the strength require in order to pull the node O from the center into one of the paths (assuming no friction)? The answer to this is that if the corresponding forces which are pulling the rope are F_k, $k = 1, 2, \ldots, m$, then the node O will move to the jth path if $F_j > \sum\limits_{\substack{k=1,2,\ldots,m \\ k\neq j}} F_k$.

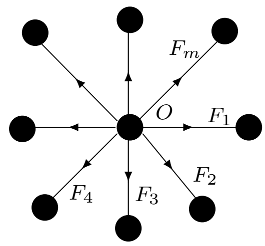

Figure 6.10: Graphical representation of tug of war

6.5.2 Evaluation graph corresponding to the teacher's evaluation by the students

In this section, we present the model of m-polar fuzzy graph which is used in evaluating the teachers by the students of 4th semester of a department in an university during the session 2015-2016. Here the nodes represent the teachers of the corresponding department and edges represent the relationship between two teachers. Suppose the department has six teachers denoted as $T = \{t_1, t_2, t_3, t_4, t_5, t_6\}$. The membership value of each node represents the corresponding teachers feedback response of the students depending on the following: $\{$*regularity of classes, style of presentation, quality of lectures, generation of interest and encouraging future reading among students, updated information*$\}$. Since all the above characteristics of a teacher according to the different students are uncertain in real life, therefore we consider 5-polar fuzzy subset of the vertex set T.

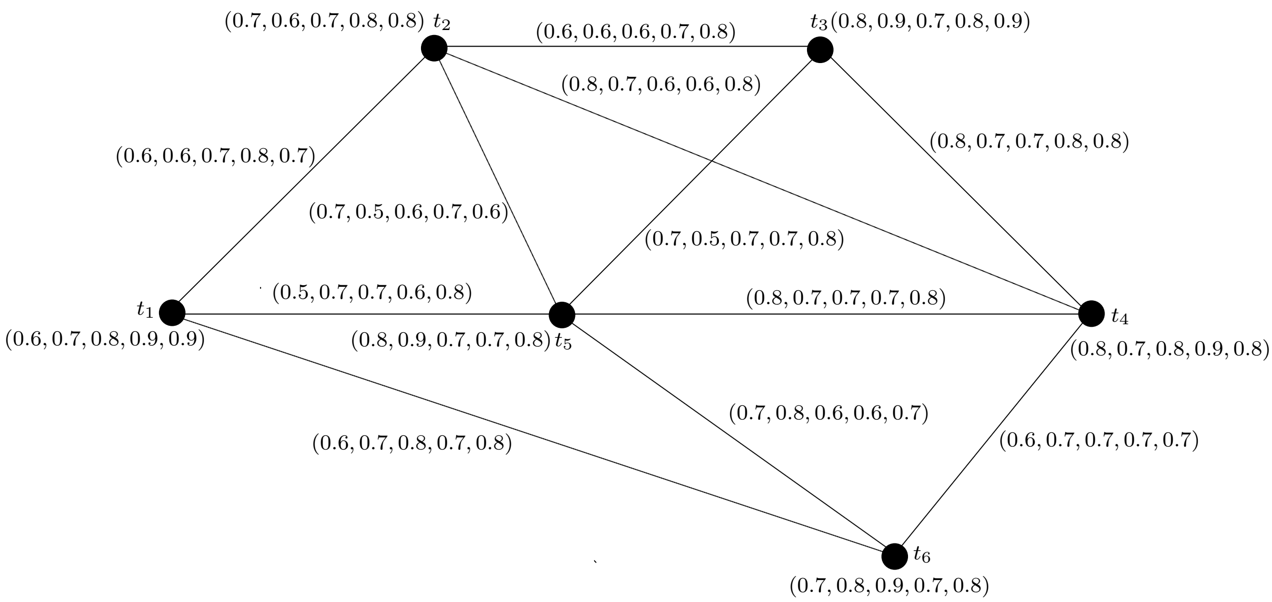

Figure 6.11: 5-polar fuzzy evaluation graph corresponding to the teacher's evaluation by students

In the Table 6.1, the membership values of the teacher's are given which is according to the evaluation of the students.

Edge membership values which represent the relationship between the teachers can be calculated by using the relation $p_i \circ B(uv) \leq min\{p_i \circ A(u), p_i \circ A(v)\}$ for all $u, v \in T$, $i = 1, 2, \ldots, 5$. These values are given in the Table 6.2.

Table 6.1: 5-polar fuzzy set A of T

	t_1	t_2	t_3	t_4	t_5	t_6
$p_1 \circ A$	0.6	0.7	0.8	0.8	0.8	0.7
$p_2 \circ A$	0.7	0.6	0.9	0.7	0.9	0.8
$p_3 \circ A$	0.8	0.7	0.7	0.8	0.7	0.9
$p_4 \circ A$	0.9	0.8	0.8	0.9	0.7	0.7
$p_5 \circ A$	0.9	0.8	0.9	0.8	0.8	0.8

Table 6.2: 5-polar fuzzy relation B on A

	t_1t_2	t_1t_5	t_1t_6	t_2t_3	t_2t_4	t_2t_5	t_3t_4	t_3t_5	t_4t_5	t_4t_6	t_5t_6
$p_1 \circ B$	0.6	0.5	0.6	0.6	0.8	0.7	0.8	0.7	0.8	0.6	0.7
$p_2 \circ B$	0.6	0.7	0.7	0.6	0.7	0.5	0.7	0.5	0.7	0.7	0.8
$p_3 \circ B$	0.7	0.7	0.8	0.6	0.6	0.6	0.7	0.7	0.7	0.7	0.6
$p_4 \circ B$	0.8	0.6	0.7	0.7	0.6	0.7	0.8	0.7	0.7	0.7	0.6
$p_5 \circ B$	0.7	0.8	0.8	0.8	0.8	0.6	0.8	0.8	0.8	0.7	0.7

Table 6.3: Average response score of the teachers

$Teachers$	t_1	t_2	t_3	t_4	t_5	t_6
$Scores$	0.78	0.72	0.82	0.8	0.78	0.78

We rank the teacher's performance according the following:

Teacher's average response score $< 60\%$, teacher's performance according to the students is **Average**.

Teacher's average response score $\geq 60\%$ and $< 70\%$, teacher's performance according to the students is **Good**.

Teacher's average response score $\geq 70\%$ and $< 80\%$, teacher's performance according to the students is **Very Good**.

Teacher's average response score is $\geq 80\%$, teacher's performance according to the students is **Excellent**.

From the Table 6.3, we see that the performance of the teachers t_1, t_2, t_5, t_6 are very good whereas the performance of the teachers t_3 and t_4 are excellent. Among these teachers, teacher t_3 is the best teacher according the response score of the students of the department during the session 2015-2016.

6.6 Summary

The theory of fuzzy graphs play an important role in many fields including decision makings, computer networking and management sciences. An m-polar fuzzy graph can be used to represent real world problems which involve multi-agent, multi-attribute, multi-object, multi-index, multi-polar information with uncertainty. In this chapter, we have introduced weak self complement m-polar fuzzy graph in some weaker sense and studied the properties of self complement and weak self complement m-polar fuzzy graphs. Then order, size, busy vertices and free vertices in m-polar fuzzy graphs are defined. A relative study of complement and operations have been made. Some real life situations like tug of war and evaluation of teachers by students have been modeled in terms of m-polar fuzzy graphs as applications.

Chapter 7

Edge regularity of m-polar fuzzy graphs*

7.1 Introduction

Regular graphs play a central role in combinatorics and theoretical computer science. Strongly regular graph form an important class of graphs which is highly structured. Strongly regular graph was first defined by Bose [35]. Nagoorgani et al. [85, 88] introduced regular and irregular fuzzy graphs. Radha and Kumaravel [100] introduced the concept of strongly regular fuzzy graph. In this chapter, the concept of edge regular, strongly regular and biregular m-polar fuzzy graph are introduced. Some properties of them are studied. Also, the concept of partially edge regular m-polar fuzzy graph and fully edge regular m-polar fuzzy graph are introduced. Finally, we introduce the notion of strongly edge irregular and strongly edge totally irregular m-polar fuzzy graphs. Some properties of them are also studied to characterize strongly edge irregular and strongly edge totally irregular m-polar fuzzy graphs.

7.2 Some preliminaries

Definition 7.2.1. *[7] Let $G = (V, A, B)$ be an m-polar fuzzy graph of $G^* = (V, E)$.*

(i) The neighborhood degree of a vertex v is defined as

$d_N(v) = (d_N^1(v), d_N^2(v), \ldots, d_N^m(v))$ *where* $d_N^i(v) = \sum\limits_{u \in N(v)} p_i \circ A(u)$, $i = 1, 2, \ldots, m$.

*A part of the work presented in this chapter is published in *International Journal of Applied and Computational Mathematics*, DOI:10.1007/s40819-016-0296-y, (2016).

(ii) *The degree of a vertex v in G is defined by* $d_G(v) = (d_G^1(v), d_G^2(v), \ldots, d_G^m(v))$, *where* $d_G^i(v) = \sum\limits_{\substack{u \neq v \\ uv \in E}} p_i \circ B(uv)$, $i = 1, 2, \ldots, m$. *If all the vertices of G have the same degree, then G is called regular m-polar fuzzy graph.*

(iii) *The closed degree of a vertex v is defined by* $d_G[v] = (d_G^1[v], d_G^2[v], \ldots, d_G^m[v])$, *where* $d_G^i[v] = d_G^i(v) + p_i \circ A(v)$, $i = 1, 2, \ldots, m$. *If each vertex of G has the same closed degree, then G is called totally regular m-polar fuzzy graph.*

(iv) *The order of G is defined as* $O(G) = (O^1(G), O^2(G), \ldots, O^m(G))$ *where* $O^i(G) = \sum\limits_{v \in V} p_i \circ A(v)$, $i = 1, 2, \ldots, m$.

The size of G is defined as $S(G) = (S^1(G), S^2(G), \ldots, S^m(G))$ *where* $S^i(G) = \sum\limits_{uv \in E} p_i \circ B(uv)$, $i = 1, 2, \ldots, m$.

Definition 7.2.2. *[102] Let $G^* = (V, E)$ be a crisp graph and let $e = v_iv_j$ be an edge in G^*. Then the degree of the edge $e = v_iv_j \in E$ is defined as* $d_{G^*}(v_iv_j) = d_{G^*}(v_i) + d_{G^*}(v_j) - 2$.

7.3 Edge regularity in m-polar fuzzy graphs

In this section, edge regular, strongly regular and biregular m-polar fuzzy graphs are defined and some properties of them are given. Then, the necessary and sufficient condition for an m-polar fuzzy graph to be strongly regular is given. Also, partially edge regular m-polar fuzzy graph and fully edge regular m-polar fuzzy graph are defined.

Definition 7.3.1. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of $G^* = (V, E)$.*

(i) *The degree of an edge $e_{jk} = v_jv_k \in E$ is denoted as*

$d_G(e_{jk}) = (d^1(e_{jk}), d^2(e_{jk}), \ldots, d^m(e_{jk}))$ *and is defined as*

$d^i(e_{jk}) = d_G^i(v_j) + d_G^i(v_k) - 2p_i \circ B(v_jv_k)$

or, $d^i(e_{jk}) = \sum\limits_{\substack{v_jv_l \in E \\ l \neq k}} p_i \circ B(v_jv_l) + \sum\limits_{\substack{v_lv_k \in E \\ l \neq j}} p_i \circ B(v_lv_k)$ *for* $i = 1, 2, \ldots, m$.

(ii) *The minimum edge degree of G is denoted as* $\delta_E(G) = (\delta_1(G), \delta_2(G), \ldots, \delta_m(G))$, *where* $\delta_i(G) = \wedge\{d^i(e_{jk}) | e_{jk} \in E\}$, $i = 1, 2, \ldots, m$.

(iii) *The maximum edge degree of G is denoted as* $\Delta_E(G) = (\Delta_1(G), \Delta_2(G), \ldots, \Delta_m(G))$, *where* $\Delta_i(G) = \vee\{d^i(e_{jk}) | e_{jk} \in E\}$, $i = 1, 2, \ldots, m$.

(iv) The total edge degree of an edge $e_{jk} \in E$ *is denoted as*

$td_G(e_{jk}) = (td^1(e_{jk}), td^2(e_{jk}), \ldots, td^m(e_{jk}))$ *where*

$$td^i(e_{jk}) = \sum_{\substack{v_jv_l \in E \\ l \neq k}} p_i \circ B(v_jv_l) + \sum_{\substack{v_lv_k \in E \\ l \neq j}} p_i \circ B(v_lv_k) + p_i \circ B(e_{jk}) \text{ for } i = 1, 2, \ldots, m.$$

Example 7.3.1. *Let us consider the 4-polar fuzzy graph* $G = (V, A, B)$ *(see Fig. 7.1) of* $G^* = (V, E)$ *where* $V = \{v_1, v_2, v_3, v_4\}$ *and* $E = \{v_1v_2, v_1v_4, v_1v_3, v_2v_3, v_3v_4\}$*. Then,* $d_G(e_{12}) = (1.3, 1, 0.9, 1.2)$*. Hence,* $td_G(e_{12}) = (1.3 + 0.4, 1 + 0.5, 0.9 + 0.6, 1.2 + 0.3) = (1.7, 1.5, 1.5, 1.5)$ *(Here,* $e_{ij} = v_iv_j$*).*

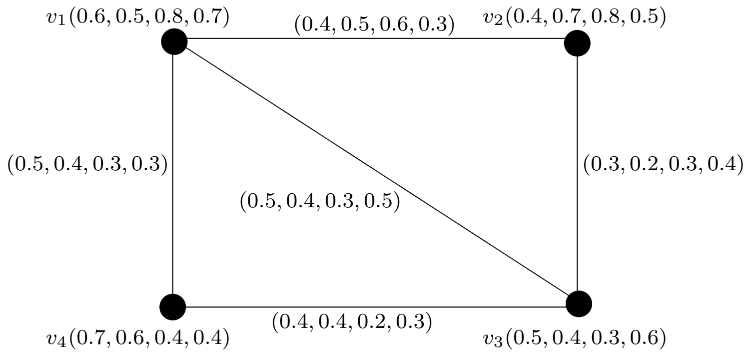

Figure 7.1: 4-polar fuzzy graph G

Definition 7.3.2. *Let* $G = (V, A, B)$ *be an m-polar fuzzy graph.*

(i) If all the edges in G *has the same degree* $(l_1, l_2, \ldots, l_m)$*, then* G *is said to be an edge regular m-polar fuzzy graph.*

(ii) If all the edges in G *has the same total degree* $(t_1, t_2, \ldots, t_m)$*, then* G *is said to be a totally edge regular m-polar fuzzy graph.*

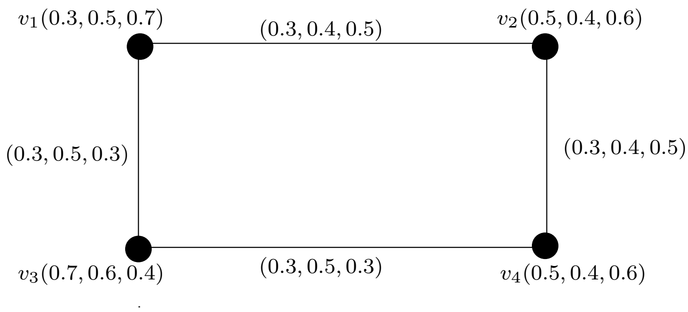

Figure 7.2: $(0.6, 0.9, 0.8)$-edge regular 3-polar fuzzy graph G

Example 7.3.2. *Consider the 3-polar fuzzy graph* $G = (V, A, B)$ *of the graph* $G^* = (V, E)$ *(see Fig. 7.2) where* $V = \{v_1, v_2, v_3, v_4\}$ *and* $E = \{v_1v_2, v_1v_3, v_3v_4, v_2v_4\}$*. Then,* $d_G(e_{12}) = d_G(e_{24}) = d_G(e_{13}) = d_G(e_{34}) = (0.6, 0.9, 0.8)$*. Hence,* G *is* $(0.6, 0.9, 0.8)$*-edge regular 3-polar fuzzy graph.*

Theorem 7.3.1. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of a cycle $G^* = (V, E)$. Then* $\sum\limits_{v_j \in V} d_G(v_j) = \sum\limits_{v_j v_k \in E} d_G(v_j v_k)$, *i.e.* $\sum\limits_{v_j \in V} d^i_G(v_j) = \sum\limits_{v_j v_k \in E} d^i(v_j v_k)$, $i = 1, 2, \ldots, m$.

Proof. Let $G = (V, A, B)$ be an m-polar fuzzy graph and G^* be a cycle $v_1v_2v_3 \ldots v_nv_1$.

Then $\sum\limits_{j=1}^{n} d_G(v_jv_{j+1}) = \big(\sum\limits_{j=1}^{n} d^1(v_jv_{j+1}), \sum\limits_{j=1}^{n} d^2(v_jv_{j+1}), \ldots, \sum\limits_{j=1}^{n} d^m(v_jv_{j+1})\big)$.

Now for $i = 1, 2, \ldots, m$

$$\begin{aligned}
\sum_{j=1}^{n} d^i(v_jv_{j+1}) &= d^i(v_1v_2) + d^i(v_2v_3) + \ldots + d^i(v_nv_1) \text{ where } v_{n+1} = v_1 \\
&= d^i_G(v_1) + d^i_G(v_2) - 2p_i \circ B(v_1v_2) + d^i_G(v_2) + d^i_G(v_3) \\
&\quad -2p_i \circ B(v_2v_3) + \ldots + d^i_G(v_n) + d^i_G(v_1) - 2p_i \circ B(v_nv_1) \\
&= 2d^i_G(v_1) + 2d^i_G(v_2) + \ldots + 2d^i_G(v_n) \\
&\quad -2(p_i \circ B(v_1v_2) + p_i \circ B(v_2v_3) + \ldots + p_i \circ B(v_nv_1)) \\
&= 2\sum_{v_j \in V} d^i_G(v_j) - 2\sum_{j=1}^{n} p_i \circ B(v_jv_{j+1}) \\
&= \sum_{v_j \in V} d^i_G(v_j) + 2\sum_{j=1}^{n} p_i \circ B(v_jv_{j+1}) - 2\sum_{j=1}^{n} p_i \circ B(v_jv_{j+1}) \\
&= \sum_{v_j \in V} d^i_G(v_j).
\end{aligned}$$

Hence, $\sum\limits_{j=1}^{n} d_G(v_jv_{j+1}) = \big(\sum\limits_{v_j \in V} d^1_G(v_j), \sum\limits_{v_j \in V} d^2_G(v_j), \ldots, \sum\limits_{v_j \in V} d^m_G(v_j)\big) = \sum\limits_{v_j \in V} d_G(v_j)$. □

Remark 7.3.1. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of the graph G^*. Then,* $\sum\limits_{v_jv_k \in E} d_G(v_jv_k) = \big(\sum\limits_{v_jv_k \in E} d_{G^*}(v_jv_k)p_1 \circ B(v_jv_k), \sum\limits_{v_jv_k \in E} d_{G^*}(v_jv_k)p_2 \circ B(v_jv_k), \ldots, \sum\limits_{v_jv_k \in E} d_{G^*}(v_jv_k)p_m \circ B(v_jv_k)\big)$, *where $d_{G^*}(v_jv_k) = d_{G^*}(v_j) + d_{G^*}(v_k) - 2$ for all $v_jv_k \in E$.*

Theorem 7.3.2. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of the p-regular crisp graph G^*. Then,*

$$\sum_{v_jv_k \in E} d_G(v_jv_k) = \big((p-1)\sum_{v_j \in V} d^1_G(v_j), (p-1)\sum_{v_j \in V} d^2_G(v_j), \ldots, (p-1)\sum_{v_j \in V} d^m_G(v_j)\big).$$

Proof. By Remark 7.3.1, we have

$\sum\limits_{v_jv_k \in E} d_G(v_jv_k)$

$= \big(\sum\limits_{v_jv_k \in E} d_{G^*}(v_jv_k)p_1 \circ B(v_jv_k), \sum\limits_{v_jv_k \in E} d_{G^*}(v_jv_k)p_2 \circ B(v_jv_k), \ldots, \sum\limits_{v_jv_k \in E} d_{G^*}(v_jv_k)p_m \circ B(v_jv_k)\big)$

Now, $\sum\limits_{v_jv_k \in E} d_{G^*}(v_jv_k)p_i \circ B(v_jv_k) = \sum\limits_{v_jv_k \in E} \big(d_{G^*}(v_j) + d_{G^*}(v_k) - 2\big)p_i \circ B(v_jv_k)$.

Since G^* is a p-regular crisp graph, therefore $d_{G^*}(v_j) = p$, for all $v_j \in V$.

So, $\sum\limits_{v_jv_k \in E} d_{G^*}(v_jv_k)p_i \circ B(v_jv_k)$

$= \sum_{v_jv_k \in E} (p + p - 2)p_i \circ B(v_jv_k)$
$= 2(p - 1) \sum_{v_jv_k \in E} p_i \circ B(v_jv_k)$
$= (p - 1) \sum_{v_j \in V} d^i_G(v_j)$.
Hence, $\sum_{v_jv_k \in E} d_G(v_jv_k) = \big((p-1) \sum_{v_j \in V} d^1_G(v_j), (p-1) \sum_{v_j \in V} d^2_G(v_j), \ldots, (p-1) \sum_{v_j \in V} d^m_G(v_j)\big)$.
□

Theorem 7.3.3. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of the crisp graph G^*. Then, $\sum_{v_jv_k \in E} td_G(v_jv_k) = \sum_{v_jv_k \in E} d_{G^*}(v_jv_k)B(v_jv_k) + \sum_{v_jv_k \in E} B(v_jv_k)$.*

Proof. From definition of total edge degree, we have for $i = 1, 2, \ldots, m$
$\sum_{v_jv_k \in E} td^i(v_jv_k) = \sum_{v_jv_k \in E} \big(d^i(v_jv_k)+p_i \circ B(v_jv_k)\big) = \sum_{v_jv_k \in E} d^i(v_jv_k) + \sum_{v_jv_k \in E} p_i \circ B(v_jv_k)$.
From Remark 7.3.1, we have
$\sum_{v_jv_k \in E} td^i(v_jv_k) = \sum_{v_jv_k \in E} d_{G^*}(v_jv_k)p_i \circ B(v_jv_k) + \sum_{v_jv_k \in E} p_i \circ B(v_jv_k)$ for $i = 1, 2, \ldots, m$.
Hence the proof. □

Theorem 7.3.4. *Let $G = (V, A, B)$ be an m-polar fuzzy graph. Then B is a constant function if and only if the following are equivalent:*

(i) G is edge regular m-polar fuzzy graph.
(ii) G is totally edge regular m-polar fuzzy graph.

Proof. Let us assume that B be a constant function.

Then $B(v_iv_j) = (c_1, c_2, \ldots, c_m)$ for all $v_iv_j \in E$, where $c_1, c_2, \ldots, c_m \in [0, 1]$ are constants.

Let G be an $(r_1, r_2, \ldots, r_m)$-edge regular m-polar fuzzy graph.

Then, $d_G(v_iv_j) = (r_1, r_2, \ldots, r_m)$ for all $v_iv_j \in E$ and

$td_G(v_iv_j) = \big(d^1(v_iv_j)+p_1 \circ B(v_iv_j), d^2(v_iv_j)+p_2 \circ B(v_iv_j), \ldots, d^m(v_iv_j)+p_m \circ B(v_iv_j)\big)$
$= (r_1 + c_1, r_2 + c_2, \ldots, r_m + c_m)$ for all $v_iv_j \in E$.

Hence, G is a $(r_1+c_1, r_2+c_2, \ldots, r_m+c_m)$-totally edge regular m-polar fuzzy graph.

Now, let G be a $(t_1, t_2, \ldots, t_m)$-totally edge regular m-polar fuzzy graph.

Then $td_G(v_iv_j) = (t_1, t_2, \ldots, t_m)$ for all $v_iv_j \in E$,

i.e. $\big(d^1(v_iv_j) + p_1 \circ B(v_iv_j), d^2(v_iv_j) + p_2 \circ B(v_iv_j), \ldots, d^m(v_iv_j) + p_m \circ B(v_iv_j)\big)$
$= (t_1, t_2, \ldots, t_m)$,

i.e. $\big(d^1(v_iv_j) + c_1, d^2(v_iv_j) + c_2, \ldots, d^m(v_iv_j) + c_m\big) = (t_1, t_2, \ldots, t_m)$,

i.e. $(d^1(v_iv_j), d^2(v_iv_j), \ldots, d^m(v_iv_j)) = (t_1 - c_1, t_2 - c_2, \ldots, t_m - c_m)$, i.e. G is $(t_1 - c_1, t_2 - c_2, \ldots, t_m - c_m)$-edge regular m-polar fuzzy graph.

Conversely, we assume that the statements (i) and (ii) are equivalent.

We need to show that B is a constant function.

If possible, let B be a non constant function.

Then there exists $v_jv_k, v_lv_r \in E$ such that $B(v_jv_k) \neq B(v_lv_r)$.

Let G be a $(r_1, r_2, \ldots, r_m)$-edge regular m-polar fuzzy graph.

Then for $v_jv_k, v_lv_r \in E$, we have

$td_G(v_jv_k)$

$= (d^1(v_jv_k) + p_1 \circ B(v_jv_k), d^2(v_jv_k) + p_2 \circ B(v_jv_k), \ldots, d^m(v_jv_k) + p_m \circ B(v_jv_k))$

$= (r_1 + p_1 \circ B(v_jv_k), r_2 + p_2 \circ B(v_jv_k), \ldots, r_m + p_m \circ B(v_jv_k))$ and

$td_G(v_lv_r)$

$= (d^1(v_lv_r) + p_1 \circ B(v_lv_r), d^2(v_lv_r) + p_2 \circ B(v_lv_r), \ldots, d^m(v_lv_r) + p_m \circ B(v_lv_r))$

$= (r_1 + p_1 \circ B(v_lv_r), r_2 + p_2 \circ B(v_lv_r), \ldots, r_m + p_m \circ B(v_lv_r))$.

Since $B(v_jv_k) \neq B(v_lv_r)$, therefore $td_G(v_jv_k) \neq td_G(v_lv_r)$.

Hence, G is not totally edge regular, which is a contradiction. So, B is constant. $\square$

Theorem 7.3.5. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of a p-regular crisp graph G^*. Then B is constant if and only if G is both regular and totally edge regular m-polar fuzzy graph.*

Proof. Let B be a constant function.

Let $B(u, v) = (c_1, c_2, \ldots, c_m)$ for all $uv \in E$ where c_i's are constants.

Then

$d_G(v) = (d_G^1(v), d_G^2(v), \ldots, d_G^m(v))$

$= \big(\sum\limits_{\substack{u \neq v \\ uv \in E}} p_1 \circ B(uv), \sum\limits_{\substack{u \neq v \\ uv \in E}} p_2 \circ B(uv), \ldots, \sum\limits_{\substack{u \neq v \\ uv \in E}} p_m \circ B(uv)\big)$

$= \big(\sum\limits_{\substack{u \neq v \\ uv \in E}} c_1, \sum\limits_{\substack{u \neq v \\ uv \in E}} c_2, \ldots, \sum\limits_{\substack{u \neq v \\ uv \in E}} c_m\big) = (pc_1, pc_2, \ldots, pc_m)$ for all $v \in V$.

Hence, G is $(pc_1, pc_2, \ldots, pc_m)$-regular m-polar fuzzy graph.

Again, $td_G(v_jv_k) = (td^1(v_jv_k), td^2(v_jv_k), \ldots, td^m(v_jv_k))$ where for $i = 1, 2, \ldots, m$

$td^i(v_jv_k) = \sum\limits_{\substack{v_jv_l \in E \\ l \neq k}} p_i \circ B(v_jv_l) + \sum\limits_{\substack{v_lv_k \in E \\ l \neq j}} p_i \circ B(v_lv_k) + p_i \circ B(v_jv_k)$

$= \sum\limits_{\substack{v_jv_l \in E \\ l \neq k}} c_i + \sum\limits_{\substack{v_lv_k \in E \\ l \neq j}} c_i + c_i = c_i(p-1) + c_i(p-1) + c_i = c_i(2p-1)$.

Hence, G is $\big((2p-1)c_1, (2p-1)c_2, \ldots, (2p-1)c_m\big)$-totally edge regular graph.

Conversely, let G be $(r_1, r_2, \ldots, r_m)$-regular and $(t_1, t_2, \ldots, t_m)$-totally edge regular m-polar fuzzy graph. We will prove that B is a constant function.

Now, $d_G(v) = (r_1, r_2, \ldots, r_m)$ for all $v \in V$ and

$td_G(v_jv_k) = (t_1, t_2, \ldots, t_m)$ for all $v_jv_k \in E$.

Again, $td_G(v_jv_k) = (td^1(v_jv_k), td^2(v_jv_k), \ldots, td^m(v_jv_k))$ where

$td^i(v_jv_k) = d^i_G(v_j) + d^i_G(v_k) - p_i \circ B(v_jv_k)$ for all $v_jv_k \in E$.

This implies for all $v_jv_k \in E$, $i = 1, 2, \ldots, m$

$c_i + c_i - p_i \circ B(v_jv_k) = t_i$,

i.e. $p_i \circ B(v_jv_k) = 2c_i - t_i$.

Hence, $B(v_jv_k) = (2c_1 - t_1, 2c_2 - t_2, \ldots, 2c_m - t_m)$ for all $v_jv_k \in E$,

i.e. B is constant. □

Definition 7.3.3. *A finite m-polar fuzzy graph $G = (V, A, B)$ is said to be strongly regular if it satisfies the following conditions:*

(i) G is $r = (r_1, r_2, \ldots, r_m)$-regular m-polar fuzzy graph,

(ii) The sum of the membership values of the common neighborhood vertices of any pair of adjacent vertices and non adjacent vertices of G has the same weight and is denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, $\delta = (\delta_1, \delta_2, \ldots, \delta_m)$, respectively.

A strong regular graph G is denoted by $G = (n, r, \lambda, \delta)$, where n is the number of vertices in G.

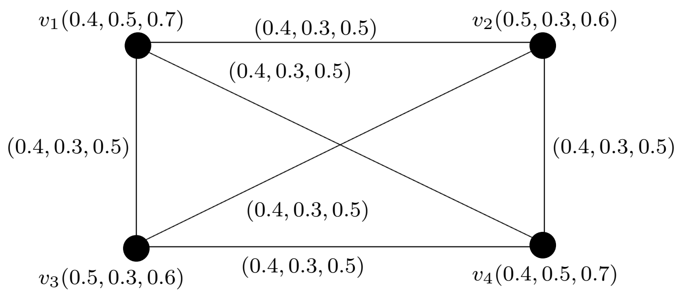

Figure 7.3: Strongly regular 3-polar fuzzy graph G

Example 7.3.3. *Let us consider the 3-polar fuzzy graph G of Fig. 7.3. Here, $n = 4$, $r = (1.2, 0.9, 1.5)$, $\lambda = (0.9, 0.8, 1.3)$, $\delta = (0, 0, 0)$. Hence, G is strongly regular 3-polar fuzzy graph.*

Theorem 7.3.6. *Let $G = (V, A, B)$ be a complete m-polar fuzzy graph where A and B are constant functions. Then, G is strongly regular m-polar fuzzy graph.*

Proof. Let $G = (V, A, B)$ be a complete m-polar fuzzy graph where $V = \{v_1, v_2, \ldots, v_n\}$.

Let $A(v_i) = (a_1, a_2, \ldots, a_m)$ for all $v_i \in V$ and $B(v_jv_k) = (c_1, c_2, \ldots, c_m)$ for all $v_jv_k \in E$ where a_i's and c_i's are constants.

Since G is complete,

$d_G(v_j) = \big(\sum\limits_{\substack{v_j \neq v_k \\ v_jv_k \in E}} p_1 \circ B(v_jv_k), \sum\limits_{\substack{v_j \neq v_k \\ v_jv_k \in E}} p_2 \circ B(v_jv_k), \ldots, \sum\limits_{\substack{v_j \neq v_k \\ v_jv_k \in E}} p_m \circ B(v_jv_k)\big)$

$= ((n-1)c_1, (n-1)c_2, \ldots, (n-1)c_m)$ for $v_j \in V$.

Hence, G is $((n-1)c_1, (n-1)c_2, \ldots, (n-1)c_m)$-regular m-polar fuzzy graph. Again since G is complete, therefore the sum of the membership values of common neighborhood vertices of any pair of adjacent vertices has the same weight $\lambda = ((n-2)a_1, (n-2)a_2, \ldots, (n-2)a_m)$ and the sum of the membership values of common neighborhood vertices of any pair of non adjacent vertices has the same $\delta = \mathbf{0}$. So, G is strongly regular m-polar fuzzy graph. □

Remark 7.3.2. *If G is strongly regular and disconnected m-polar fuzzy graph then $\delta = \mathbf{0}$.*

Definition 7.3.4. *An m-polar fuzzy graph $G = (V, A, B)$ is said to be a biregular m-polar fuzzy graph if it satisfies the following:*

(i) G is $r = (r_1, r_2, \ldots, r_m)$-regular m-polar fuzzy graph,

(ii) $V = V_1 \cup V_2$ be the bipartition of V and every vertex in V_1 has the same neighborhood degree $M = (M_1, M_2, \ldots, M_m)$ and every vertex in V_2 has the same neighborhood degree $N = (N_1, N_2, \ldots, N_m)$, where M and N are constants.

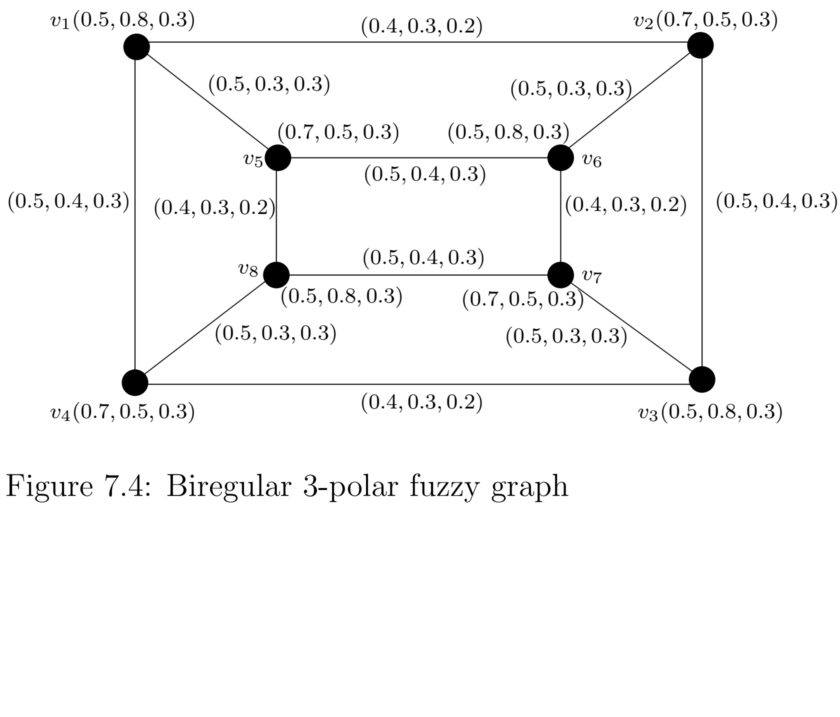

Figure 7.4: Biregular 3-polar fuzzy graph

Example 7.3.4. *Let G be a 3-polar fuzzy graph of the graph $G^* = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $E = \{v_1v_2, v_1v_4, v_1v_5, v_2v_6, v_2v_3, v_3v_4, v_3v_7, v_4v_8, v_5v_6, v_5v_8, v_6v_7, v_7v_8\}$ (see Fig. 7.4). Here, $n = 8$, $r = (1.4, 1, 0.8)$, $V_1 = \{v_1, v_3, v_6, v_8\}$, $V_2 = \{v_2, v_4, v_5, v_7\}$, $M = (2.1, 1.5, 0.9)$ and $N = (1.5, 2.4, 0.9)$. Hence, G is a biregular 3-polar fuzzy graph.*

Theorem 7.3.7. *If $G = (V, A, B)$ is a strongly regular m-polar fuzzy graph which is strong, then $\overline{G}$ is a $(r_1, r_2, \ldots, r_m)$-regular.*

Proof. Since G is strongly regular, therefore G is $(r_1, r_2, \ldots, r_m)$-regular. Again, since G is strong, therefore for $i = 1, 2, \ldots, m$

$$p_i \circ \overline{B}(v_jv_k) = \begin{cases} 0 & if \quad v_jv_k \in E \\ min\{p_i \circ A(v_j), p_i \circ A(v_k)\} & if \quad v_jv_k \notin E. \end{cases}$$

Now, the degree of a vertex v_j in $\overline{G}$ is $d_{\overline{G}}(v_j) = \big(d^1_{\overline{G}}(v_j), d^2_{\overline{G}}(v_j), \ldots, d^m_{\overline{G}}(v_j)\big)$, where $d^i_{\overline{G}}(v_j) = \sum\limits_{\substack{v_j \neq v_k \\ v_jv_k \in E}} p_i \circ \overline{B}(v_jv_k) = \sum\limits_{\substack{v_j \neq v_k \\ v_jv_k \in E}} p_i \circ A(v_j) \wedge p_i \circ A(v_k) = r_i$, for all $v_j \in V$, $i = 1, 2, \ldots, m$. Hence, $d_{\overline{G}}(v_j) = (r_1, r_2, \ldots, r_m)$ for all $v_j \in V$. So, $\overline{G}$ is $(r_1, r_2, \ldots, r_m)$ -regular m-polar fuzzy graph. □

Theorem 7.3.8. *Let $G = (V, A, B)$ be a strong m-polar fuzzy graph. Then, G is strongly regular if and only if $\overline{G}$ is strongly regular.*

Proof. Let G be a strongly regular m-polar fuzzy graph. Then, G is $(r_1, r_2, \ldots, r_m)$-regular and the adjacent vertices and non-adjacent vertices have the same common neighborhood weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\delta = (\delta_1, \delta_2, \ldots, \delta_m)$, respectively. Now, since G is strongly regular and strong, therefore by Theorem 7.3.7, $\overline{G}$ is $(r_1, r_2, \ldots, r_m)$-regular. Let S and T denote the set of all adjacent and non adjacent vertices of G; $\overline{S}$ and $\overline{T}$ denote the set of all adjacent and non adjacent vertices of $\overline{G}$.

So, $S = \{v_j, v_k | v_jv_k \in E\}$, where v_j and v_k have same common neighborhood weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $T = \{v_j, v_k | v_jv_k \notin E\}$, where v_j and v_k have same common neighborhood weight $\delta = (\delta_1, \delta_2, \ldots, \delta_m)$. Then, $\overline{S} = \{v_j, v_k | v_jv_k \in \overline{E}\}$, where v_j and v_k have same common neighborhood weight $\delta = (\delta_1, \delta_2, \ldots, \delta_m)$ and $\overline{T} = \{v_j, v_k | v_jv_k \notin \overline{E}\}$, where v_j and v_k have same common neighborhood weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. This shows that $\overline{G}$ is strongly regular. Similarly, we can show the converse part. □

Theorem 7.3.9. *A strongly regular m-polar fuzzy graph G is a biregular m-polar fuzzy graph if the adjacent vertices have the same common neighborhood weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \neq \mathbf{0}$ and the non adjacent vertices have the same common neighborhood weight $\delta = (\delta_1, \delta_2, \ldots, \delta_m) \neq \mathbf{0}$.*

Proof. Since G is strongly regular m-polar fuzzy graph, therefore G is $r = (r_1, r_2, \ldots, r_m)$-regular m-polar fuzzy graph. Let S be the set of all non adjacent vertices of G. Then S is a non-empty subset of V since the non adjacent vertices have the same common neighborhood weight $\delta = (\delta_1, \delta_2, \ldots, \delta_m) \neq \mathbf{0}$. So, $S = \{v_j, v_k | v_j$ is not adjacent to $v_k, j \neq k, v_j, v_k \in V\}$. Now the vertex partition of G is $V_1 = \{v_j | v_j \in S\}$ and $V_2 = \{v_k | v_k \in S\}$. Hence, G is biregular m-polar fuzzy graph. □

Definition 7.3.5. *(i) If the underlying graph G^* is an edge regular graph, then G is said to be a partially edge regular m-polar fuzzy graph.*

(ii) If G is both edge regular and partially edge regular m-polar fuzzy graph, then G is said to be a full edge regular m-polar fuzzy graph.

Theorem 7.3.10. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of G^* such that B is constant. If G is full regular, then G is full edge regular m-polar fuzzy graph.*

Proof. Let $B(v_j v_k) = (c_1, c_2, \ldots, c_m)$ for all $v_j v_k \in E$ where c_i's are constant.

Since G is full regular, therefore G and G^* is both regular, i.e. $d_G(v_j) = (r_1, r_2, \ldots, r_m)$ and $d_{G^*}(v_j) = p$ for all $v_j \in V$, where r_i and p are constants.

Now, $d_{G^*}(v_j v_k) = d_{G^*}(v_j) + d_{G^*}(v_k) - 2 = 2p - 2$ for all $v_j v_k \in E$.

This shows that, G^* is an edge regular graph, i.e. G is partially edge regular m-polar fuzzy graph.

Again, for all $v_j v_k \in E$, $i = 1, 2, \ldots, m$

$d^i_G(v_j v_k) = d^i_G(v_j) + d^i_G(v_k) - 2p_i \circ B(v_j v_k) = r_i + r_i - 2c_i = 2r_i - 2c_i$.

Hence, G is $(2r_1 - c_1, 2r_2 - 2c_2, \ldots, 2r_m - 2c_m)$-edge regular m-polar fuzzy graph. Therefore, G is fully edge regular m-polar fuzzy graph. □

Theorem 7.3.11. *Let $G = (V, A, B)$ be a $t = (t_1, t_2, \ldots, t_m)$-totally edge regular and p-partially edge regular m-polar fuzzy graph. Then, $S(G) = \frac{q}{1+p}(t_1, t_2, \ldots, t_m) = \frac{qt}{1+p}$ where $q = |E|$.*

Proof. The size of G is $S(G) = (S^1(G), S^2(G), \ldots, S^m(G))$ where

$S^i(G) = \sum\limits_{uv \in E} p_i \circ B(uv)$, $i = 1, 2, \ldots, m$. Now, from Theorem 7.3.3, we have

$$\sum_{v_jv_k \in E} td_G(v_jv_k)$$
$$= \big(\sum_{v_jv_k \in E} d_{G^*}(v_jv_k)p_1 \circ B(v_jv_k) + \sum_{v_jv_k \in E} p_1 \circ B(v_jv_k), \sum_{v_jv_k \in E} d_{G^*}(v_jv_k)p_2 \circ B(v_jv_k) + \sum_{v_jv_k \in E} p_2 \circ B(v_jv_k), \ldots, \sum_{v_jv_k \in E} d_{G^*}(v_jv_k)p_m \circ B(v_jv_k) + \sum_{v_jv_k \in E} p_m \circ B(v_jv_k)\big)$$
$$= \big(\sum_{v_jv_k \in E} d_{G^*}(v_jv_k)p_1 \circ B(v_jv_k), \sum_{v_jv_k \in E} d_{G^*}(v_jv_k)p_2 \circ B(v_jv_k), \ldots, \sum_{v_jv_k \in E} d_{G^*}(v_jv_k)p_m \circ B(v_jv_k)\big) + S(G),$$

i.e. $q(t_1, t_2, \ldots, t_m) = pS(G) + S(G)$, since $p = d_{G^*}(v_jv_k)$ for all $v_jv_k \in E$,

i.e. $qt = (1 + p)S(G)$,

i.e. $S(G) = \frac{qt}{1+p}$. □

7.4 Edge irregular m-polar fuzzy graphs

In this section, strongly edge irregular m-polar fuzzy graph and strongly edge totally irregular m-polar fuzzy graph are defined with examples. Some properties of them are studied.

Definition 7.4.1. *Let $G = (V, A, B)$ be an m-polar fuzzy graph. Then*

(i) G is said to be strongly edge irregular m-polar fuzzy graph if every pair of edges have distinct degrees, i.e. no two edges have the same degree.

(ii) G is said to be strongly edge totally irregular m-polar fuzzy graph if every pair of edges have distinct total degrees, i.e. no two edges have the same total degree.

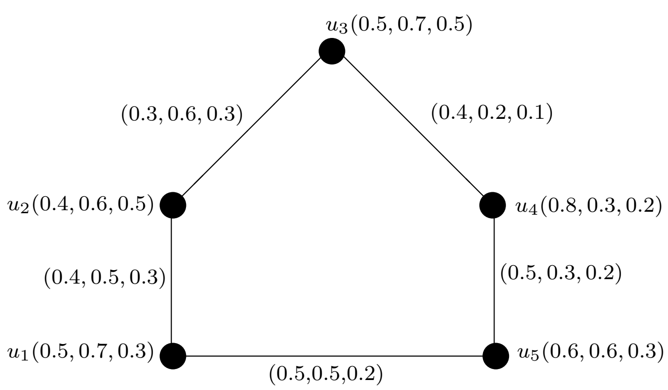

Figure 7.5: Example of 3-polar fuzzy graph which is both strongly edge irregular and strongly edge totally irregular

Example 7.4.1. *Here we will give an example of a 3-polar fuzzy graph which is both strongly edge irregular and strongly edge totally irregular. Let $G = (V, A, B)$ be a 3-polar fuzzy graph of $G^* = (V, E)$ (see Fig. 7.5). We have from the Fig. 7.5,*

$d_G(u_1u_2) = d_G(u_1) + d_G(u_2) - 2B(u_1u_2)$

$= (0.5 + 0.4 + 0.4 + 0.3 - 0.8, 0.5 + 0.5 + 0.5 + 0.6 - 1, 0.2 + 0.3 + 0.3 + 0.3 - 0.6)$

$= (0.8, 1.1, 0.5)$,

$d_G(u_2u_3) = d_G(u_2) + d_G(u_3) - 2B(u_2u_3) = (0.9, 1.2, 0.8)$,

$d_G(u_3u_4) = d_G(u_3) + d_G(u_4) - 2B(u_3u_4) = (0.8, 0.9, 0.5)$,

$d_G(u_4u_5) = d_G(u_4) + d_G(u_5) - 2B(u_4u_5) = (0.9, 0.7, 0.3)$,

$d_G(u_5u_1) = d_G(u_5) + d_G(u_1) - 2B(u_5u_1) = (0.9, 0.8, 0.5)$.

Since every pair of edges have different degrees, therefore G is a strongly edge irregular 3-polar fuzzy graph. Again,

$td_G(u_1u_2) = d_G(u_1u_2) + B(u_1u_2) = (0.8, 1.1, 0.5) + (0.4, 0.5, 0.3) = (1.2, 1.6, 0.8)$,

$td_G(u_2u_3) = d_G(u_2u_3) + B(u_2u_3) = (0.9, 1.2, 0.8) + (0.3, 0.6, 0.3) = (1.2, 1.8, 1.1)$,

$td_G(u_3u_4) = d_G(u_3u_4) + B(u_3u_4) = (0.8, 0.9, 0.5) + (0.4, 0.2, 0.1) = (1.2, 1.1, 0.6)$,

$td_G(u_4u_5) = d_G(u_4u_5) + B(u_4u_5) = (0.9, 0.7, 0.3) + (0.5, 0.3, 0.2) = (1.4, 1, 0.5)$,

$td_G(u_5u_1) = d_G(u_5u_1) + B(u_5u_1) = (0.9, 0.8, 0.5) + (0.5, 0.5, 0.2) = (1.4, 1.3, 0.7)$.

The total degrees of every pair of edges is distinct. So, G is strongly edge totally irregular 3-polar fuzzy graph. Hence, G is both strongly edge irregular and strongly edge totally irregular 3-polar fuzzy graph.

Example 7.4.2. *Here we show by example that strongly edge irregular m-polar fuzzy graphs need not be strongly edge totally irregular m-polar fuzzy graphs. For example, let us consider the 3-polar fuzzy graphs of Fig. 7.6. Then we have,*

$d_G(u_1u_2) = (0.8, 0.8, 0.4)$, $d_G(u_2u_3) = (0.5, 0.8, 0.4)$, $d_G(u_3u_1) = (0.9, 0.8, 0.4)$,

$td_G(u_1u_2) = (1.1, 1.2, 0.6)$, $td_G(u_2u_3) = (1.1, 1.2, 0.6)$, $td_G(u_3u_1) = (1.1, 1.2, 0.6)$.

This shows that G is strongly edge irregular 3-polar fuzzy graph and it is not strongly edge totally irregular 3-polar fuzzy graph. So, strongly edge irregular 3-polar fuzzy graphs may not be strongly edge totally irregular 3-polar fuzzy graphs.

Example 7.4.3. *Again strongly edge totally irregular m-polar fuzzy graphs need not be strongly edge irregular m-polar fuzzy graphs. For example, consider the 3-polar fuzzy graph of Fig. 7.7. We have, $d_G(v_1v_2) = (1.1, 0.8, 1.1)$, $d_G(v_2v_3) = (0.9, 0.7, 0.9)$,*

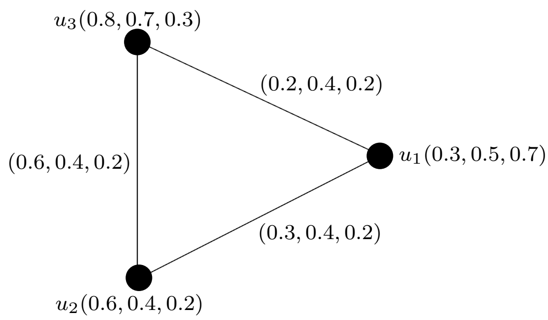

Figure 7.6: Example of 3-polar fuzzy graph which is strongly edge irregular but not strongly edge totally irregular

$d_G(v_3v_4) = (1.1, 0.8, 1.1)$, $d_G(v_4v_1) = (1.3, 0.7, 1.1)$. *Here,* $d_G(v_1v_2) = d_G(v_3v_4)$. *So, G is not strongly edge irregular 3-polar fuzzy graph. Also,* $td_G(v_1v_2) = (1.4, 1.1, 1.4)$, $td_G(v_2v_3) = (1.3, 1.2, 1.5)$, $td_G(v_3v_4) = (1.7, 1.2, 1.7)$, $td_G(v_4v_1) = (2, 1, 1.6)$. *Since* $td_G(v_1v_2) \neq td_G(v_2v_3) \neq td_G(v_3v_4) \neq td_G(v_4v_1)$, *therefore G is strongly edge totally irregular 3-polar fuzzy graph.*

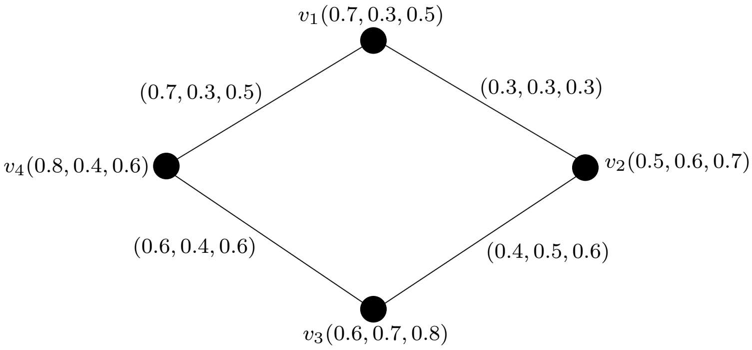

Figure 7.7: Example of 3-polar fuzzy graph which is strongly edge totally irregular but not strongly edge irregular

Theorem 7.4.1. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of $G^* = (V, E)$ where B is constant. Then G is strongly edge irregular m-polar fuzzy graph if and only if G is strongly edge totally irregular m-polar fuzzy graph.*

Proof. Let $B(uv) = (c_1, c_2, \dots, c_m)$ for all $uv \in E$, where $c_1, c_2, \dots, c_m \in [0, 1]$.

Let G be strongly edge irregular m-polar fuzzy graph.

$\Leftrightarrow d_G(u_1u_2) \neq d_G(v_1v_2)$ for all $u_1u_2, v_1v_2 \in E$

$\Leftrightarrow d_G(u_1u_2) + (c_1, c_2, \dots, c_m) \neq d_G(v_1v_2) + (c_1, c_2, \dots, c_m)$ for all $u_1u_2, v_1v_2 \in E$

$\Leftrightarrow d_G(u_1u_2) + B(u_1u_2) \neq d_G(v_1v_2) + B(v_1v_2)$ for all $u_1u_2, v_1v_2 \in E$

$\Leftrightarrow td_G(u_1u_2) \neq td_G(v_1v_2)$ for all $u_1u_2, v_1v_2 \in E$

$\Leftrightarrow G$ is strongly edge totally irregular m-polar fuzzy graph. $\square$

Remark 7.4.1. *If $G = (V, A, B)$ is both strongly edge irregular and strongly edge totally irregular m-polar fuzzy graph, then B may not be a constant function.*

For example, consider the 3-polar fuzzy graph of Fig. 7.5. Here B is not constant although G is both strongly edge irregular and strongly edge totally irregular.

Theorem 7.4.2. *If G is strongly edge irregular m-polar fuzzy graph, then G is neighborly edge irregular m-polar fuzzy graph.*

Proof. Since G is strongly edge irregular m-polar fuzzy graph, therefore every pair of edges in G have distinct degrees. So every pair of adjacent edges have distinct degrees. Hence, G is neighborly edge irregular m-polar fuzzy graph. □

Theorem 7.4.3. *If G is strongly edge totally irregular m-polar fuzzy graph, then G is neighborly edge totally irregular m-polar fuzzy graph.*

Proof. Let $G = (V, A, B)$ be an m-polar fuzzy graph which is strongly edge totally irregular. Then every pair of edges in G have distinct total degrees. So every pair of adjacent edges have distinct total degrees. Hence, G is neighborly edge totally irregular m-polar fuzzy graph. □

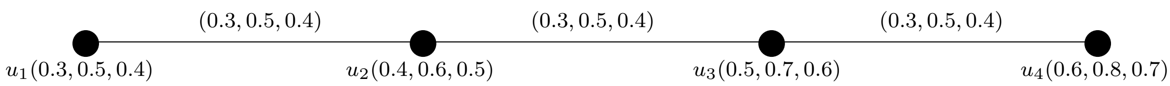

Figure 7.8: Example of 3-polar fuzzy graph which is both neighborly edge irregular and neighborly edge totally irregular but not strongly edge irregular and totally strongly edge irregular

Remark 7.4.2. *The converse of the above Theorems 7.4.2 and 7.4.3 may not be true. For example, see Fig. 7.8 of the 3-polar fuzzy graph G. Here,*

$d_G(u_1) = (0.3, 0.5, 0.4), d_G(u_2) = (0.6, 1, 0.8),$

$d_G(u_3) = (0.6, 1, 0.8), d_G(u_4) = (0.3, 0.5, 0.4),$

$d_G(u_1u_2) = (0.3, 0.5, 0.4), d_G(u_2u_3) = (0.6, 1, 0.8), d_G(u_3u_4) = (0.3, 0.5, 0.4)$ *and*

$td_G(u_1u_2) = (0.6, 1, 0.8), td_G(u_2u_3) = (0.9, 1.5, 1.2), td_G(u_3u_4) = (0.6, 1, 0.8).$

Note that, $d_G(u_1u_2) \neq d_G(u_2u_3)$, $d_G(u_2u_3) \neq d_G(u_3u_4)$ and $d_G(u_1u_2) = d_G(u_3u_4)$. Hence, we conclude that G is neighborly edge irregular 3-polar fuzzy graph, but G is not strongly edge irregular 3-polar fuzzy graph.

Again, $td_G(u_1u_2) \neq td_G(u_2u_3)$, $td_G(u_2u_3) \neq td_G(u_3u_4)$ and $td_G(u_1u_2) = td_G(u_3u_4)$. Hence, G is neighborly edge totally irregular 3-polar fuzzy graph but G is not strongly edge totally irregular 3-polar fuzzy graph.

Theorem 7.4.4. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of G^* where B is constant. If G is strongly edge irregular m-polar fuzzy graph, then G is an irregular m-polar fuzzy graph.*

Proof. Let $B(uv) = (c_1, c_2, \ldots, c_m)$ for all $uv \in E$ where $c_1, c_2, \ldots, c_m \in [0, 1]$. Since G is strongly edge irregular therefore every pair of edges will have distinct degrees. Let us consider the two adjacent edges u_1v_1 and v_1w_1 having distinct degrees.

This implies that $d_G(u_1v_1) \neq d_G(v_1w_1)$

$\Rightarrow d_G(u_1) + d_G(v_1) - 2B(u_1v_1) \neq d_G(v_1) + d_G(w_1) - 2B(v_1w_1)$

$\Rightarrow d_G(u_1) + d_G(v_1) - 2(c_1, c_2, \ldots, c_m) \neq d_G(v_1) + d_G(w_1) - 2(c_1, c_2, \ldots, c_m)$

$\Rightarrow d_G(u_1) \neq d_G(w_1)$

This shows that the vertex v_1 which is adjacent to the vertices u_1 and w_1 having distinct degrees. Hence, G is irregular. □

Theorem 7.4.5. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of G^* and B is constant. If G is strongly edge totally irregular m-polar fuzzy graph then G is an irregular m-polar fuzzy graph.*

Proof. Similar to the above. □

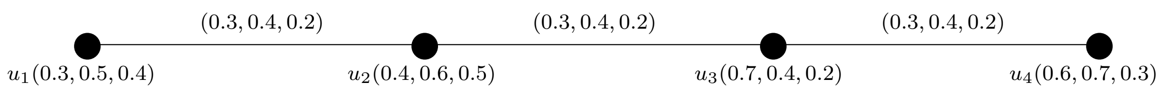

Figure 7.9: Example of 3-polar fuzzy graph which is irregular but neither strongly edge irregular nor strongly edge totally irregular

Remark 7.4.3. *Converse of the Theorems 7.4.4 and 7.4.5 need not be true. For example, consider the 3-polar fuzzy graph of Fig. 7.9. Then we have*

$d_G(u_1) = (0.3, 0.4, 0.2)$, $d_G(u_2) = (0.6, 0.8, 0.4)$,

$d_G(u_3) = (0.6, 0.8, 0.4)$, $d_G(u_4) = (0.3, 0.4, 0.2)$.

So, G is irregular 3-polar fuzzy graph.

Also, $d_G(u_1u_2) = (0.3, 0.4, 0.2)$, $d_G(u_2u_3) = (0.6, 0.8, 0.4)$,

$d_G(u_3u_4) = (0.3, 0.4, 0.2)$, $td_G(u_1u_2) = (0.6, 0.8, 0.4)$,

$td_G(u_2u_3) = (0.9, 1.2, 0.6)$, $td_G(u_3u_4) = (0.6, 0.8, 0.4)$.

Here, G is neither strongly edge irregular nor strongly edge totally irregular.

Theorem 7.4.6. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of G^* and B is constant. If G is strongly edge irregular m-polar fuzzy graph then G is highly irregular m-polar fuzzy graph.*

Proof. Let $B(uv) = (c_1, c_2, \ldots, c_m)$ for all $uv \in E$ where $c_1, c_2, \ldots, c_m \in [0, 1]$. Let u_2 be any vertex adjacent with the vertices u_1, u_3 and u_4. Then u_1u_2, u_2u_3 and u_2u_4 are adjacent edges in G. Let us assume that G is strongly edge irregular m-polar fuzzy graph. Then every pair of edges in G have distinct degrees. So, every pair of adjacent edges in G have distinct degrees.

Hence, $d_G(u_1u_2) \neq d_G(u_2u_3) \neq d_G(u_2u_4)$

$\Rightarrow d_G(u_1) + d_G(u_2) - 2B(u_1u_2) \neq d_G(u_2) + d_G(u_3) - 2B(u_2u_3)$

$\neq d_G(u_2) + d_G(u_4) - 2B(u_2u_4)$

$\Rightarrow d_G(u_1) + d_G(u_2) - 2(c_1, c_2, \ldots, c_m) \neq d_G(u_2) + d_G(u_3) - 2(c_1, c_2, \ldots, c_m)$

$\neq d_G(u_2) + d_G(u_4) - 2(c_1, c_2, \ldots, c_m)$

$\Rightarrow d_G(u_1) \neq d_G(u_2) \neq d_G(u_3)$.

So the vertex u_2 is adjacent to the vertices u_1, u_3 and u_4 with distinct degrees. Hence, G is highly irregular. □

Theorem 7.4.7. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of G^* and B is constant. If G is strongly edge totally irregular m-polar fuzzy graph, then G is highly irregular m-polar fuzzy graph.*

Proof. Similar to the above. □

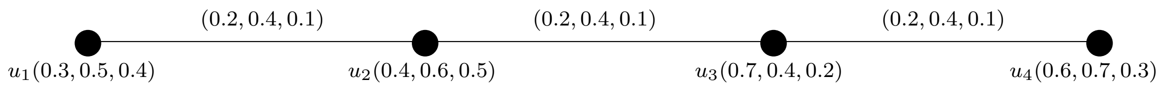

Figure 7.10: G is highly irregular but neither strongly edge irregular nor strongly edge totally irregular

Remark 7.4.4. *Converse of the above Theorems 7.4.6 and 7.4.7 need not be true. For example, consider the 3-polar fuzzy graph G of Fig. 7.10. We have,*

$d_G(u_1) = (0.2, 0.4, 0.1)$, $d_G(u_2) = (0.4, 0.8, 0.2)$,

$d_G(u_3) = (0.4, 0.8, 0.2)$, $d_G(u_4) = (0.2, 0.4, 0.1)$.

Hence, G is highly irregular.

Again, $d_G(u_1u_2) = (0.2, 0.4, 0.1)$, $d_G(u_2u_3) = (0.4, 0.8, 0.2)$, $d_G(u_3u_4) = (0.2, 0.4, 0.1)$.

So, G is not strongly edge irregular.

Also, $td_G(u_1u_2) = (0.4, 0.8, 0.2)$, $td_G(u_2u_3) = (0.6, 1.2, 0.3)$, $td_G(u_3u_4) = (0.4, 0.8, 0.2)$.

So, G is not strongly edge totally irregular also.

Theorem 7.4.8. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of G^* which is a path of $2n$ $(n > 1)$ vertices. If the membership value of the edges $e_1, e_2, \ldots, e_{2n-1}$ are $(a_1^1, a_2^1, \ldots, a_m^1)$, $(a_1^2, a_2^2, \ldots, a_m^2)$, $\ldots$, $(a_1^{2n-1}, a_2^{2n-1}, \ldots, a_m^{2n-1})$ respectively such that $(a_1^1, a_2^1, \ldots, a_m^1) < (a_1^2, a_2^2, \ldots, a_m^2) < \ldots < (a_1^{2n-1}, a_2^{2n-1}, \ldots, a_m^{2n-1})$, then G is both strongly edge irregular and strongly edge totally irregular. $\big($Here, $e_i = v_iv_{i+1}$ for $i = 1, 2, \ldots, (2n-1)\big)$*

Proof. We have

$d_G(v_i) = (a_1^{i-1} + a_1^i, a_2^{i-1} + a_2^i, \ldots, a_m^{i-1} + a_m^i)$ for $i = 2, 3, \ldots, (2n-1)$ and

$d_G(v_1) = (a_1^1, a_2^1, \ldots, a_m^1)$,

$d_G(v_{2n}) = (a_1^{2n-1}, a_2^{2n-1}, \ldots, a_m^{2n-1})$,

$d_G(e_i) = (a_1^{i-1} + a_1^{i+1}, a_2^{i-1} + a_2^{i+1}, \ldots, a_m^{i-1} + a_m^{i+1})$ for $i = 2, 3, \ldots, (2n-2)$,

$d_G(e_1) = (a_1^2, a_2^2, \ldots, a_m^2)$,

$d_G(e_{2n-1}) = (a_1^{2n-2}, a_2^{2n-2}, \ldots, a_m^{2n-2})$.

Hence, G is strongly edge irregular m-polar fuzzy graph.

Again, since

$td_G(e_i) = (a_1^{i-1} + a_1^{i+1} + a_1^i, a_2^{i-1} + a_2^{i+1} + a_2^i, \ldots, a_m^{i-1} + a_m^{i+1} + a_m^i)$

for $i = 2, 3, \ldots, (2n-2)$,

$td_G(e_1) = (a_1^2 + a_1^1, a_2^2 + a_2^1, \ldots, a_m^2 + a_m^1)$,

$td_G(e_{2n-1}) = (a_1^{2n-2} + a_1^{2n-1}, a_2^{2n-2} + a_2^{2n-1}, \ldots, a_m^{2n-2} + a_m^{2n-1})$, therefore G is strongly edge totally irregular m-polar fuzzy graph. □

Theorem 7.4.9. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of G^* which is a cycle of n $(n \geq 4)$ vertices. If the membership value of the edges $e_1, e_2, \ldots, e_n$ are $(a_1^1, a_2^1, \ldots, a_m^1)$, $(a_1^2, a_2^2, \ldots, a_m^2)$, $\ldots$, $(a_1^n, a_2^n, \ldots, a_m^n)$ respectively such that $(a_1^1, a_2^1, \ldots, a_m^1) < (a_1^2, a_2^2, \ldots, a_m^2) < \ldots < (a_1^n, a_2^n, \ldots, a_m^n)$, then G is both strongly edge irregular and strongly edge totally irregular.*

Proof. Let $e_1, e_2, \ldots, e_n$ be the edges of the cycle G^* in that order.

Then we have,

$d_G(v_i) = (a_1^{i-1} + a_1^i, a_2^{i-1} + a_2^i, \ldots, a_m^{i-1} + a_m^i)$ for $i = 2, 3, \ldots, n$,

$d_G(v_1) = (a_1^1 + a_1^n, a_2^1 + a_2^n, \ldots, a_m^1 + a_m^n)$,

$d_G(e_i) = (a_1^{i-1} + a_1^{i+1}, a_2^{i-1} + a_2^{i+1}, \ldots, a_m^{i-1} + a_m^{i+1})$ for $i = 2, 3, \ldots, (n-1)$,

$d_G(e_1) = (a_1^2 + a_1^n, a_2^2 + a_2^n, \ldots, a_m^2 + a_m^n)$,

$d_G(e_n) = (a_1^1 + a_1^{n-1}, a_2^1 + a_2^{n-1}, \ldots, a_m^1 + a_m^{n-1})$.

Hence, G is strongly edge irregular m-polar fuzzy graph.

Again, since

$td_G(e_i) = (a_1^{i-1} + a_1^{i+1} + a_1^i, a_2^{i-1} + a_2^{i+1} + a_2^i, \ldots, a_m^{i-1} + a_m^{i+1} + a_m^i)$ for $i = 2, 3, \ldots, (n-1)$,

$td_G(e_1) = (a_1^2 + a_1^1 + a_1^n, a_2^2 + a_2^1 + a_2^n, \ldots, a_m^2 + a_m^1 + a_m^n)$,

$td_G(e_{2n-1}) = (a_1^1 + a_1^n + a_1^{n-1}, a_2^1 + a_2^n + a_2^{n-1}, \ldots, a_m^1 + a_m^n + a_m^{n-1})$, therefore G is strongly edge totally irregular m-polar fuzzy graph. □

Theorem 7.4.10. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of a graph $G^* = (V, E)$ which is a star $K_{1,n}$. If the membership values of no two edges are same, then G is strongly edge irregular and totally edge regular m-polar fuzzy graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices adjacent to the vertex u_0. Let $e_1, e_2, \ldots, e_n$ be the edges of the star G^* in that order having membership values $(a_1^1, a_2^1, \ldots, a_m^1)$, $(a_1^2, a_2^2, \ldots, a_m^2)$, $\ldots, (a_1^n, a_2^n, \ldots, a_m^n)$ such that $(a_1^1, a_2^1, \ldots, a_m^1) \neq (a_1^2, a_2^2, \ldots, a_m^2) \neq \ldots \neq (a_1^n, a_2^n, \ldots, a_m^n)$. Then

$d_G(e_i = u_0 u_i)$

$= d_G(u_0) + d_G(u_i) - 2B(u_0 u_i)$

$= (a_1^1 + a_1^2 + \ldots + a_1^n, a_2^1 + a_2^2 + \ldots + a_2^n, a_m^1 + a_m^2 + \ldots + a_m^n) + (a_1^i, a_2^i, \ldots, a_m^i)$

$-2(a_1^i, a_2^i, \ldots, a_m^i)$

$= (a_1^1 + a_1^2 + \ldots + a_1^n, a_2^1 + a_2^2 + \ldots + a_2^n, a_m^1 + a_m^2 + \ldots + a_m^n) - (a_1^i, a_2^i, \ldots, a_m^i)$

for $i = 1, 2, \ldots, n$.

We see that all edges have distinct degrees. So, G is strongly edge irregular. Also,

$td_G(e_i = u_0 u_i)$

$= (a_1^1 + a_1^2 + \ldots + a_1^n, a_2^1 + a_2^2 + \ldots + a_2^n, a_m^1 + a_m^2 + \ldots + a_m^n) - (a_1^i, a_2^i, \ldots, a_m^i)$

$+(a_1^i, a_2^i, \ldots, a_m^i)$

$= (a_1^1 + a_1^2 + \ldots + a_1^n, a_2^1 + a_2^2 + \ldots + a_2^n, a_m^1 + a_m^2 + \ldots + a_m^n)$ for $i = 1, 2, \ldots, n$.

Since all edges have the same total degrees therefore G is totally edge regular. □

Theorem 7.4.11. *Let $G = (V, A, B)$ be an m-polar fuzzy graph of a graph $G^* = (V, E)$ which is a Barbell graph $B_{n,p}$. If the membership values of no two edges are same, then G is strongly edge irregular but not strongly edge totally irregular m-polar fuzzy graph.*

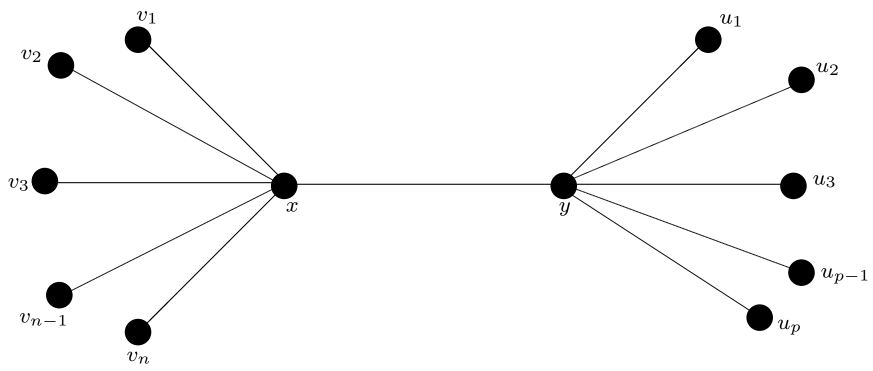

Figure 7.11: The m-polar fuzzy graph G of the Barbell graph $B_{n,p}$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices adjacent to the vertex x and $e_1, e_2, \ldots, e_n$ be the edges incident with the vertex x in that order having membership values $(a_1^1, a_2^1, \ldots, a_m^1)$, $(a_1^2, a_2^2, \ldots, a_m^2)$, $\ldots, (a_1^n, a_2^n, \ldots, a_m^n)$. Again let $u_1, u_2, \ldots, u_p$ be the vertices adjacent to the vertex y and $f_1, f_2, \ldots, f_p$ be the edges incident with the vertex y in that order having membership values such that $(a_1^1, a_2^1, \ldots, a_m^1) < (a_1^2, a_2^2, \ldots, a_m^2)$ $< \ldots, (a_1^n, a_2^n, \ldots, a_m^n) < (b_1^1, b_2^1, \ldots, b_m^1) < (b_1^2, b_2^2, \ldots, b_m^2) < \ldots, (b_1^p, b_2^p, \ldots, b_m^p) <$ $(a_1, a_2, \ldots, a_m)$ where $(a_1, a_2, \ldots, a_m)$ is the membership value of the edge xy (see Fig. 7.11). Then,

$d_G(xy) = \big(a_1^1 + a_1^2 + \ldots + a_1^n, a_2^1 + a_2^2 + \ldots + a_2^n, a_m^1 + a_m^2 + \ldots + a_m^n\big) + (a_1, a_2, \ldots, a_m) +$ $\big(b_1^1 + b_1^2 + \ldots + b_1^p, b_2^1 + b_2^2 + \ldots + b_2^p, b_m^1 + b_m^2 + \ldots + b_m^p\big) + (a_1, a_2, \ldots, a_m) - 2(a_1, a_2, \ldots, a_m)$ $= \big(a_1^1 + a_1^2 + \ldots + a_1^n, a_2^1 + a_2^2 + \ldots + a_2^n, a_m^1 + a_m^2 + \ldots + a_m^n\big) + \big(b_1^1 + b_1^2 + \ldots + b_1^p, b_2^1 +$ $b_2^2 + \ldots + b_2^p, b_m^1 + b_m^2 + \ldots + b_m^p\big)$,

$td_G(xy) = \big(a_1^1 + a_1^2 + \ldots + a_1^n, a_2^1 + a_2^2 + \ldots + a_2^n, a_m^1 + a_m^2 + \ldots + a_m^n\big) + \big(b_1^1 + b_1^2 +$ $\ldots + b_1^p, b_2^1 + b_2^2 + \ldots + b_2^p, b_m^1 + b_m^2 + \ldots + b_m^p\big) + (a_1, a_2, \ldots, a_m)$,

$d_G(e_i) = \big(a_1^1 + a_1^2 + \ldots + a_1^n, a_2^1 + a_2^2 + \ldots + a_2^n, a_m^1 + a_m^2 + \ldots + a_m^n\big) + (a_1, a_2, \ldots, a_m) +$ $(a_1^i, a_2^i, \ldots, a_m^i) - 2(a_1^i, a_2^i, \ldots, a_m^i) = \big(a_1^1 + a_1^2 + \ldots + a_1^n, a_2^1 + a_2^2 + \ldots + a_2^n, a_m^1 + a_m^2 +$ $\ldots + a_m^n\big) + (a_1, a_2, \ldots, a_m) - (a_1^i, a_2^i, \ldots, a_m^i)$ for $i = 1, 2, \ldots, n$.

Similarly, $d_G(f_i) = \big(b_1^1 + b_1^2 + \ldots + b_1^p, b_2^1 + b_2^2 + \ldots + b_2^p, b_m^1 + b_m^2 + \ldots + b_m^p\big) +$ $(a_1, a_2, \ldots, a_m) - (b_1^i, b_2^i, \ldots, b_m^i)$ for $i = 1, 2, \ldots, p$.

We see that every pair of edges have distinct degrees. So, G is strongly edge irregular.

Also, $td_G(e_i) = \big(a_1^1+a_1^2+\ldots+a_1^n, a_2^1+a_2^2+\ldots+a_2^n, a_m^1+a_m^2+\ldots+a_m^n\big)+(a_1, a_2, \ldots, a_m)$ for $i = 1, 2, \ldots, n$ and

$td_G(f_i) = \big(b_1^1 + b_1^2 + \ldots + b_1^p, b_2^1 + b_2^2 + \ldots + b_2^p, b_m^1 + b_m^2 + \ldots + b_m^p\big) + (a_1, a_2, \ldots, a_m)$ for $i = 1, 2, \ldots, p$.

We see that each e_i, $i = 1, 2, \ldots, m$ have same total degrees and each f_i, $i = 1, 2, \ldots, p$ have same total degrees. So, G is not strongly edge totally irregular. $\square$

7.5 Summary

In this chapter, the definition of edge regular, partial edge regular and fully edge regular m-polar fuzzy graphs are given and some properties of them are studied. The condition under which edge regular m-polar fuzzy graph and totally edge regular m-polar fuzzy graphs are equivalent is mentioned. The notion of strongly edge irregular and strongly edge totally irregular m-polar fuzzy graphs. Characterization of strongly edge irregular and strongly edge totally irregular m-polar fuzzy graphs are given. Several important properties of them have been investigated.

Chapter 8

Morphism of m-polar fuzzy graphs

8.1 Introduction

In order to achieve a good correspondence between two graphs, the most used concept is the one of graph isomorphism and a lot of work is dedicated to the search for the best isomorphism between two graphs or subgraphs. However in a number of cases, the bijective condition is too strong and the problem is expressed rather as an inexact graph matching problem. For instance, inexact graph matching appears as an important area of research in the pattern recognition. Several researches use graphs to represent the knowledge and the information extracted for instance from images, where vertices represent the segments or entities of the image and edges show the relationships between them. Examples of areas in which this type of representation is used are cartography, robotics and autonomous agents, character recognition and recognition of brain structures. Graph matching is used when the recognition is based on comparison with a model for instance. One graph represents the model and another one the image where recognition has to be performed. Because of the schematic aspect of the model (atlas or map for instance) and of the difficulty to segment accurately the image into meaningful entities, no isomorphism can be expected between both graphs. Such problems call for inexact graph matching. Similar examples can be found in other fields. In this chapter, we have introduced the notion of m-polar ψ-morphism on m-polar fuzzy graphs. The action of m-polar ψ-morphism on m-polar fuzzy graphs is studied and we established some results on weak and co-weak isomorphism. d_2-degree and total d_2-degree of a vertex in m-polar fuzzy graphs are defined and studied $(2, \overline{k})$-regularity and totally $(2, \overline{k})$-regularity. A real life situation of a company where a group of people decides which product design to manufacture has been modeled as a 4-polar fuzzy graphs.

8.2 Regularity and isomorphism on m-polar fuzzy graphs

Regular graphs are the most widely studied classes. For example, regular fuzzy graphs play a key role in designing reliable communication networks. Here, the notion of m-polar ψ-morphism is introduced in m-polar fuzzy graphs. Also, d_2-degree, total d_2-degree, $(2, \overline{k})$-regularity and totally $(2, \overline{k})$-regularity are defined in m-polar fuzzy graphs and studied some important properties of them.

Definition 8.2.1. *Let $G = (V, A, B)$ be an m-polar fuzzy graph. Then d_2- degree of a vertex u in G is $d_2(u) = (d_2^1(u), d_2^2(u), \ldots, d_2^m(u))$ where $d_2^i(u) = \sum p_i \circ B^2(uv)$ is such that $p_i \circ B^2(uv) = sup\{p_i \circ B(uu_1) \wedge p_i \circ B(u_1v)\}$.*

The minimum d_2-degree of G is denoted as $\delta_2(G) = (\delta_2^1(G), \delta_2^2(G), \ldots, \delta_2^m(G))$ where $\delta_2^i(G) = \wedge\{d_2^i(u) : u \in V\}$. The maximum d_2-degree of G is denoted as $\triangle_2(G) = (\triangle_2^1(G), \triangle_2^2(G), \ldots, \triangle_2^m(G))$ where $\triangle_2^i(G) = \vee\{d_2^i(u) : u \in V\}$.

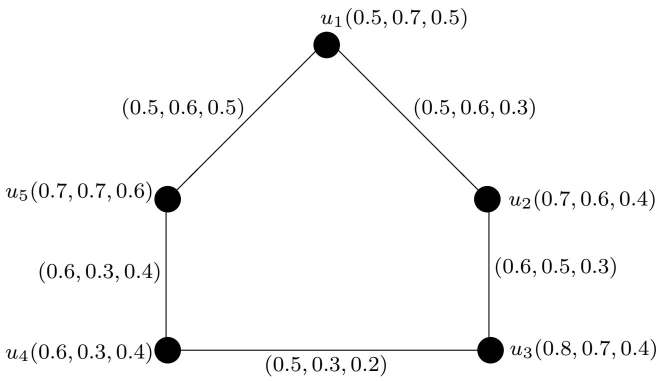

Figure 8.1: 3-polar fuzzy graph G

Example 8.2.1. *Let G be a 3-polar fuzzy graph where $V = \{u_1, u_2, u_3, u_4, u_5\}$ and $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1\}$ (see Fig. 8.1). By routine computations we have,*

$d_2^1(u_1) = \{0.5 \vee 0.6\} + \{0.5 \vee 0.6\} = 1, d_2^2(u_1) = \{0.6 \vee 0.5\} + \{0.6 \vee 0.3\} = 0.8,$

$d_2^3(u_1) = \{0.3 \vee 0.3\} + \{0.5 \vee 0.4\} = 0.7, d_2^1(u_2) = \{0.5 \vee 0.5\} + \{0.5 \vee 0.6\} = 1,$

$d_2^2(u_2) = \{0.6 \vee 0.6\} + \{0.3 \vee 0.5\} = 0.9, d_2^3(u_2) = \{0.5 \vee 0.3\} + \{0.2 \vee 0.3\} = 0.5,$

$d_2^1(u_3) = \{0.5 \vee 0.6\} + \{0.6 \vee 0.5\} = 1, d_2^2(u_3) = \{0.6 \vee 0.5\} + \{0.3 \vee 0.3\} = 0.8,$

$d_2^3(u_3) = \{0.3 \vee 0.3\} + \{0.4 \vee 0.2\} = 0.5.$

Hence, $d_2(u_1) = (1, 0.8, 0.7)$, $d_2(u_2) = (1, 0.9, 0.5)$, $d_2(u_3) = (1, 0.8, 0.5)$.

Definition 8.2.2. *If* $d_2(u) = \overline{k}$ *for all* $u \in V$ *then* g *is said to be* $(2, \overline{k})$*- regular* m*-polar fuzzy graph.*

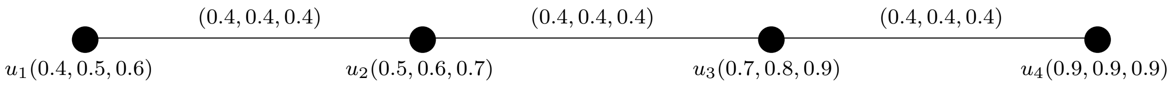

Figure 8.2: $(2, (0.4, 0.4, 0.4))$-regular 3-polar fuzzy graph G

Example 8.2.2. *Consider the 3-polar fuzzy graph as in Fig. 8.2. Here,* $d_2(u_1) = d_2(u_2) = d_2(u_3) = d_2(u_4) = (0.4, 0.4, 0.4)$. *So,* G *is* $(2, (0.4, 0.4, 0.4))$*-regular 3-polar fuzzy graph.*

Definition 8.2.3. *The total* d_2*- degree of a vertex* $u \in V$ *is defined as* $td_2(u) = (td_2^1(u), td_2^2(u), \ldots, td_2^m(u))$, *where* $td_2^i(u) = \sum p_i \circ B^2(uv) + p_i \circ A(u)$, $i = 1, 2, \ldots, m$.

Note 8.2.1. *If each vertex of* G *has the same total* d_2*-degree* $\overline{l}$, *then* G *is said to be totally* $(2, \overline{l})$*-regular* m*-polar fuzzy graph.*

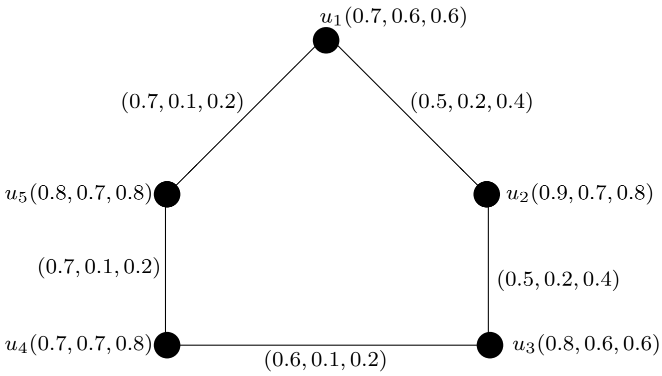

Figure 8.3: Totally $(2, (1.9, 0.9, 1.2))$-regular 3-polar fuzzy graph G

Example 8.2.3. *Consider the 3-polar fuzzy graph* G *with the vertex set* $V = \{u_1, u_2, u_3, u_4, u_5\}$ *and edge set* $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1\}$ *(see Fig. 8.3). We see that*

$d_2(u_1) = (1.2, 0.3, 0.6)$, $d_2(u_2) = (1, 0.2, 0.4)$, $d_2(u_3) = (1.1, 0.3, 0.6)$,

$d_2(u_4) = (1.2, 0.2, 0.4)$, $d_2(u_5) = (1.1, 0.2, 0.4)$ *and*

$td_2(u_1) = td_2(u_2) = td_2(u_3) = td_2(u_4) = td_2(u_5) = (1.9, 0.9, 1.2)$.

Since each vertex has the same total d_2*-degree, therefore* G *is totally* $(2, (1.9, 0.9, 1.2))$*-regular 3-polar fuzzy graph. Although,* G *is not* $(2, \overline{k})$*-regular.*

Theorem 8.2.1. *Let $G = (V, A, B)$ be an m-polar fuzzy graph. Then $A(u) = \overline{c} = (c_1, c_2, \ldots, c_m)$ for all $u \in V$ if and only if the following are equivalent:*

(i) G is a $(2, \overline{k})$-regular m-polar fuzzy graph,

(ii) G is a totally $(2, \overline{k} + \overline{c})$-regular m-polar fuzzy graph.

Proof. Suppose that $A(u) = \overline{c} = (c_1, c_2, \ldots, c_m)$ for all $u \in V$. We will show that the statements (i) and (ii) are equivalent.

$(i) \Rightarrow (ii)$: Let G be a $(2, \overline{k})$-regular m-polar fuzzy graph. Therefore, $d_2(u) = \overline{k}$ for all $u \in V$. Now, $td_2(u) = \overline{k} + \overline{c}$ for all $u \in V$. So, G is totally $(2, \overline{k} + \overline{c})$- regular.

$(ii) \Rightarrow (i)$: Now, suppose that G is totally $(2, \overline{k} + \overline{c})$-regular.

Then $td_2(u) = \overline{k} + \overline{c}$ for all $u \in V$,

i.e. $d_2(u) + A(u) = \overline{k} + \overline{c}$ for all $u \in V$,

i.e. $d_2(u) = \overline{k}$ for all $u \in V$,

i.e. G is $\overline{k}$-regular.

Conversely, let (i) and (ii) are equivalent.

Let G be both totally $(2, \overline{k} + \overline{c})$-regular and $(2, \overline{k})$-regular.

Then we have, $td_2(u) = \overline{k} + \overline{c}$ and $d_2(u) = \overline{k}$ for all $u \in V$,

i.e. $d_2(u) + A(u) = \overline{k} + \overline{c}$ and $d_2(u) = \overline{k}$ for all $u \in V$.

So, $A(u) = \overline{c}$ for all $u \in V$. Hence the result. □

Definition 8.2.4. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two m-polar fuzzy graphs. Then a bijective function $\psi : V_1 \to V_2$ is called an m-polar morphism or m-polar ψ-morphism if there exist positive real numbers k_1, k_2 such that for $i = 1, 2, \ldots, m$*

(i) $p_i \circ A_2(\psi(u)) = k_1 p_i \circ A_1(u)$ for all $u \in V_1$ and

(ii) $p_i \circ B_2(\psi(u)\psi(v)) = k_2 p_i \circ B_1(uv)$ for all $uv \in \widetilde{V_1^2}$.

In this case, ψ is called (k_1, k_2) m-polar ψ-morphism from G_1 onto G_2. If $k_1 = k_2 = k$, then we call ψ an m-polar k-morphism. When $k = 1$, we obtain usual m-polar morphism.

Note 8.2.2. *Let $G_1 = (V_1, A_1, B_1)$, $G_2 = (V_2, A_2, B_2)$ and $G_3 = (V_3, A_3, B_3)$ be three m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$, $G_2^* = (V_2, E_2)$ and $G_3^* = (V_3, E_3)$. Let A_1, A_2 and A_3 denote the membership functions of the vertices in G_1, G_2, G_3 respectively; B_1, B_2, B_3 denote the membership functions of the edges in G_1, G_2, G_3 respectively.*

Theorem 8.2.2. *The relation ψ-morphism is an equivalence relation in the collection of m-polar fuzzy graphs.*

Proof. Let $\mathcal{G}$ be the collection of all m-polar fuzzy graphs. Define a relation '$\sim$' on $\mathcal{G} \times \mathcal{G}$ as follows: for $G_1, G_2 \in \mathcal{G}$, we say $G_1 \sim G_2$ if and only if there exist a (k_1, k_2) m-polar ψ-morphism from G_1 onto G_2 for some non-zero k_1 and k_2.

We show that $\sim$ is an equivalence relation. First, we see that $\sim$ is reflexive by simply taking the identity mapping from G_1 onto itself.

Let $G_1, G_2 \in \mathcal{G}$ and $G_1 \sim G_2$. Then there exists a (k_1, k_2) ψ-morphism from G_1 onto G_2 for some non-zero k_1 and k_2.

Therefore $p_i \circ A_2(\psi(u)) = k_1 p_i \circ A_1(u)$ for all $u \in V_1$ and

$p_i \circ B_2(\psi(u)\psi(v)) = k_2 p_i \circ B_1(uv)$ for all $uv \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$.

Now consider the function $\psi^{-1} : V_2 \to V_1$. Let $x, y \in V_2$.

Since ψ is bijective, therefore there exist $u, v \in V_1$ such that $\psi(u) = x$ and $\psi(v) = y$. Then, $p_i \circ A_1(\psi^{-1}(x)) = p_i \circ A_1(u) = \frac{1}{k_1} p_i \circ A_2(\psi(u)) = \frac{1}{k_1} p_i \circ A_2(x)$ and

$p_i \circ B_1(\psi^{-1}(x)\psi^{-1}(y)) = p_i \circ B_1(uv) = \frac{1}{k_2} p_i \circ B_2(\psi(u)\psi(v)) = \frac{1}{k_2} p_i \circ B_2(xy)$ for $i = 1, 2, \ldots, m$. Thus, ψ^{-1} is a $(\frac{1}{k_1}, \frac{1}{k_2})$ m-polar morphism from G_2 to G_1.

Hence, $G_2 \sim G_1$. So, $\sim$ is symmetric.

Again, let $G_1, G_2, G_3 \in \mathcal{G}$ be such that $G_1 \sim G_2$ and $G_2 \sim G_3$.

Then there exist a (k_1, k_2) m-polar ψ_1 morphism from G_1 onto G_2 and a (k_3, k_4) m-polar ψ_2 morphism from G_2 onto G_3 for some non-zero real numbers k_1, k_2, k_3 and k_4. Then,

$p_i \circ A_2(\psi_1(u)) = k_1 p_i \circ A_1(u)$ for all $u \in V_1$,

$p_i \circ B_2(\psi_1(u)\psi_1(v)) = k_2 p_i \circ B_1(uv)$ for all $uv \in \widetilde{V_1^2}$,

$p_i \circ A_3(\psi_2(u)) = k_3 p_i \circ A_2(u)$ for all $u \in V_2$,

$p_i \circ B_3(\psi_2(u)\psi_2(v)) = k_4 p_i \circ B_2(uv)$ for all $uv \in \widetilde{V_2^2}$, $i = 1, 2, \ldots, m$.

Let $\psi_3 = \psi_2 \circ \psi_1 : V_1 \to V_3$ be a mapping.

Now, $p_i \circ A_3(\psi_3(u))$

$= p_i \circ A_3(\psi_2 \circ \psi_1(u))$

$= p_i \circ A_3(\psi_2(\psi_1(u)))$

$= k_3 p_i \circ A_2(\psi_1(u))$

$= k_3 k_1 p_i \circ A_1(u)$ and

$p_i \circ B_3(\psi_3(u)\psi_3(v))$

$= p_i \circ B_3(\psi_2 \circ \psi_1(u)\psi_2 \circ \psi_1(v))$

$= p_i \circ B_3(\psi_2(\psi_1(u))\psi_2(\psi_1(v)))$

$= k_4 p_i \circ B_2(\psi_1(u)\psi_1(v))$

$= k_4 k_2 p_i \circ B_1(uv)$, $i = 1, 2, \ldots, m$.

Thus, ψ_3 is a (k_3k_1, k_4k_2) m-polar morphism from G_1 onto G_3.

Therefore, $G_1 \sim G_3$ and hence $\sim$ is transitive. So, the relation $\sim$ is an equivalence relation in the collection of m-polar fuzzy graphs. □

Theorem 8.2.3. *Let G_1 and G_2 be two m-polar fuzzy graphs and ψ be a (k_1, k_2) m-polar fuzzy morphism from G_1 onto G_2 for some non-zero k_1 and k_2. Then the image of strong edges in G_1 is also strong edge in G_2 if and only if $k_1 = k_2$.*

Proof. Let u_1v_1 be a strong edge in G_1 and $k_1 = k_2$.

Since ψ is a (k_1, k_2) m-polar fuzzy morphism from G_1 to G_2, therefore we have,

$p_i \circ B_2(\psi(u_1)\psi(v_1))$

$= k_2 p_i \circ B_1(u_1v_1)$

$= k_2\{p_i \circ A_1(u_1) \wedge p_i \circ A_1(v_1)\}$

$= k_2 p_i \circ A_1(u_1) \wedge k_2 p_i \circ A_1(v_1)$

$= k_1 p_i \circ A_1(u_1) \wedge k_1 p_i \circ A_1(v_1)$

$= p_i \circ A_2(u_1) \wedge p_i \circ A_2(v_1)$ for $i = 1, 2, \ldots, m$.

So, the edge $\psi(u_1)\psi(v_1)$ in G_2 is strong.

Conversely, let u_1v_1 be a strong edge in G_1 and its corresponding image $\psi(u_1)\psi(v_1)$ in G_2 is also strong. Then we have,

$k_2 p_i \circ B_1(u_1v_1)$

$= p_i \circ B_2(\psi(u_1)\psi(v_1))$

$= p_i \circ A_2(\psi(u_1)) \wedge p_i \circ A_2(\psi(v_1))$

$= k_1 p_i \circ A_1(u_1) \wedge k_1 p_i \circ A_1(v_1)$

$= k_1 p_i \circ B_1(u_1v_1)$ for each $i = 1, 2, \ldots, m$.

This implies that $k_1 = k_2$. This completes the proof. □

Corollary 8.2.1. *Let G_1 and G_2 be two m-polar fuzzy graphs and G_1 be a (k_1, k_2) m-polar fuzzy morphism to G_2. If G_1 is strong, then G_2 is strong if and only if $k_1 = k_2$.*

Theorem 8.2.4. *If the m-polar fuzzy graph G_1 is co-weak isomorphic to the m-polar fuzzy graph G_2 and G_1 is regular, then G_2 is regular also.*

Proof. Since G_1 is co-weak isomorphic to G_2, therefore there exists a co-weak isomorphism $\phi : V_1 \to V_2$ which is bijective such that

$p_i \circ A_1(u) \leq p_i \circ A_2(\phi(u))$ and

$p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v))$ for all $u, v \in V_1$, $i = 1, 2, \ldots, m$.

Since G_1 is regular, we have $d_{G_1}(u) = (c_1, c_2, \ldots, c_m)$ for all $u \in V_1$.

Now, $d^i_{G_2}(\phi(u))$

$$= \sum_{\substack{\phi(u)\neq\phi(v)\\ \phi(u)\phi(v)\in E_2}} p_i \circ B_2(\phi(u)\phi(v))$$

$$= \sum_{\substack{u\neq v\\ uv\in E_1}} p_i \circ B_1(uv) = c_i \text{ for all } u \in V_1 \text{ and } i = 1, 2, \ldots, m.$$

Hence, G_2 is regular. □

Remark 8.2.1. *If the m-polar fuzzy graph G_1 is co-weak isomorphic to G_2 and G_1 is strong, then G_2 need not be strong.*

Theorem 8.2.5. *Let G_1 and G_2 be two m-polar fuzzy graphs. If G_1 is weak isomorphic to G_2 and G_1 is strong, then G_2 is also strong.*

Proof. Since G_1 is weak isomorphic to G_2, therefore there exists a weak isomorphism $\phi : V_1 \to V_2$ which is bijective such that

$p_i \circ A_1(u) = p_i \circ A_2(\phi(u))$ for all $u \in V_1$ and

$p_i \circ B_1(uv) \leq p_i \circ B_2(\phi(u)\phi(v))$ for all $uv \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$.

As G_1 is strong, $p_i \circ B_1(uv) = min\{p_i \circ A_1(u), p_i \circ A_1(v)\}$ for all $uv \in E_1$, $i = 1, 2, \ldots, m$. Now,

$p_i \circ B_2(\phi(u)\phi(v))$

$\geq p_i \circ B_1(uv)$

$= min\{p_i \circ A_1(u), p_i \circ A_1(v)\}$

$= min\{p_i \circ A_2(\phi(u)), p_i \circ A_2(\phi(v))\}$ and

by definition, $p_i \circ B_2(\phi(u)\phi(v)) \leq min\{p_i \circ A_2(\phi(u)), p_i \circ A_2(\phi(v))\}$ for $\phi(u)\phi(v) \in E_2$, $i = 1, 2, \ldots, m$. Hence, G_2 is strong. □

Corollary 8.2.2. *Let G_1 and G_2 be two m-polar fuzzy graphs. If G_1 is weak isomorphic to G_2 and G_1 is regular, then G_2 need not be regular.*

Theorem 8.2.6. *If the m-polar fuzzy graphs G_1 is co-weak isomorphic with a strong regular m-polar fuzzy graph G_2, then G_1 is strong regular m-polar fuzzy graph.*

Proof. Since G_1 is co-weak isomorphic to G_2 therefore there exists a co-weak isomorphism $\phi : V_1 \to V_2$ which is bijective such that

$p_i \circ A_1(u) \leq p_i \circ A_2(\phi(u))$ for all $u \in V_1$ and

$p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v))$ for all $uv \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$.

Now we have,

$p_i \circ B_1(uv)$

$= p_i \circ B_2(\phi(u)\phi(v))$

$= min\{p_i \circ A_2(\phi(u)), p_i \circ A_2(\phi(v))\}$ (Since G_2 is strong)

$\geq min\{p_i \circ A_1(u), p_i \circ A_1(v)\}$.

But, by definition of m-polar fuzzy graphs,

$p_i \circ B_1(uv) \leq min\{p_i \circ A_1(u), p_i \circ A_1(v)\}$ for all $uv \in \widetilde{V_1^2}$.

So, from the above we have, $p_i \circ B_1(uv) \leq min\{p_i \circ A_1(u), p_i \circ A_1(v)\}$ for all $uv \in E_1$, $i = 1, 2, \ldots, m$. Hence, G_1 is strong.

Also, for $i = 1, 2, \ldots, m$ and $u \in V_1$,

$\sum\limits_{\substack{u \neq v \\ uv \in E_1}} p_i \circ B_1(uv) = \sum\limits_{\substack{\phi(u) \neq \phi(v) \\ \phi(u)\phi(v) \in E_2}} p_i \circ B_2(\phi(u)\phi(v)) =$ constant (Since G_2 is regular).

Hence, G_1 is regular. □

Theorem 8.2.7. *Let G_1 and G_2 be two isomorphic m-polar fuzzy graphs. Then G_1 is strong regular if and only if G_2 is strong regular.*

Proof. As G_1 is isomorphic to G_2, therefore there exists an isomorphism $\phi : V_1 \to V_2$ which is bijective and satisfies $p_i \circ A_1(u) = p_i \circ A_2(\phi(u))$ for all $u \in V_1$ and

$p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v))$ for all $uv \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$.

Now, G_1 is strong

$\Leftrightarrow p_i \circ B_1(uv) = min\{p_i \circ A_1(u), p_i \circ A_1(v)\}$ for all $uv \in E_1$, $i = 1, 2, \ldots, m$

$\Leftrightarrow p_i \circ B_2(\phi(u)\phi(v)) = min\{p_i \circ A_2(\phi(u)), p_i \circ A_2(\phi(v))\}$ for all $\phi(u)\phi(v) \in E_2$, $i = 1, 2, \ldots, m$

$\Leftrightarrow G_2$ is strong.

Again, G_1 is regular

$\Leftrightarrow \sum\limits_{\substack{u \neq v \\ uv \in E_1}} p_i \circ B_1(uv) =$ constant for all $u \in V_1$,

$\Leftrightarrow \sum\limits_{\substack{\phi(u) \neq \phi(v) \\ \phi(u)\phi(v) \in E_2}} p_i \circ B_2(\phi(u)\phi(v)) =$ constant for all $\phi(u) \in V_2$

$\Leftrightarrow G_2$ is regular. □

Theorem 8.2.8. *A strong m-polar fuzzy graph G is strong regular if and only if its complement $\overline{G}$ is strong regular.*

Proof. From Proposition 6.11 of [45], we have if $G = (V, A, B)$ is a strong m-polar fuzzy graph, then $\overline{G} = (V, \overline{A}, \overline{B})$ is also a strong m-polar fuzzy graph where $\overline{A} = A$ and $\overline{B}$ is defined by $p_i \circ \overline{B}(xy) = min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy)$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$.

Now, G is strong regular if and only if $p_i \circ B(xy) = min\{p_i \circ A(x), p_i \circ A(y)\}$ if and only if $p_i \circ \overline{B}(xy) = min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy) = p_i \circ B(xy) - p_i \circ B(xy) = 0$ if and only if $\sum p_i \circ \overline{B}(xy) = 0$ if and only if $\overline{G}$ is strong regular. □

8.3 Modeling of products design in a company as a 4-polar fuzzy graph

Here, we model a real life situation of a company where a group of people decides which product design to manufacture. This type of network is an ideal example of m-polar fuzzy graphs. It is very important for a company to decide which product design to manufacture so that they can make profit as much as possible. A very good product design is gladly acceptable to the peoples if it is also cheap in price. The determination of which product design to manufacture is called the decision making problem. By taking the very good decision (very good product design), one company can spread their product all over the world keeping in mind that the product design is very good, demandable, cheap, easily accessible, etc. Before manufacturing a product design, engineers and manufacturers test several important things in a product. Suppose a company has to decide to manufacture a product design among five products, say D_1, D_2, D_3, D_4 and D_5. A product design is manufactured by a company keeping in mind its market demand, price, time taken to manufacture and accessibility.

We consider the above as a set, say $M = \{$demand, price, time, accessibility$\}$ and the set of product designs as $D = \{D_1, D_2, D_3, D_4, D_5\}$. Since all the above characteristics of a product design according to the different company are uncertain in real life, therefore we consider a 4-polar fuzzy subset A of the set D. This situation can be represented as a 4-polar fuzzy graph by considering the different product design as the nodes and edges between them represent the relationship between two product designs (see Fig. 8.4). The membership value of each node represents the degree of

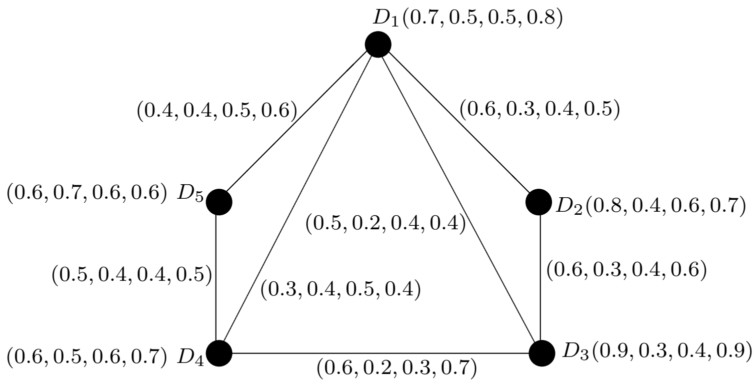

Figure 8.4: Modeling of products design in a company as a 4-polar fuzzy graph G

demand, price, time taken to manufacture and accessibility to people in global market. Edge membership values which represent the relationship between the product design can be calculated by using the relation $p_i \circ B(uv) \leq min\{p_i \circ A(u), p_i \circ A(v)\}$ for all $u, v \in D, i = 1, 2, \ldots, 4$. The edge between two product designs represents the degree of using common power equipments, raw materials, engineer employs and agencies involved for both products.

From the Fig. 8.4, we see that $G = (D, A, B)$ is a 4-polar fuzzy graph and the product design D_3 has maximum demand, minimum price, minimum time to manufacture and has maximum accessibility compared to all others product designs.

8.4 Summary

In this chapter, the notion of m-polar ψ-morphism is introduced on m-polar fuzzy graphs. The action of m-polar ψ-morphism on m-polar fuzzy graphs is studied and we established some results on weak and co-weak isomorphism. d_2-degree and total d_2-degree of a vertex in m-polar fuzzy graphs are defined and studied $(2, \overline{k})$-regularity and totally $(2, \overline{k})$-regularity. Finally, we have modeled a real life situation in terms of 4-polar fuzzy graph as an application.

Chapter 9

Generalized regular bipolar fuzzy graphs and product bipolar fuzzy line graphs*

9.1 Introduction

In 2011, Akram [3, 6] introduced bipolar fuzzy graphs with many properties. In 2012, Akram and Dudek [5] introduced regular bipolar fuzzy graphs. The aim of this chapter is to point out some errors in [5] by counterexamples in Definitions 3.3, 3.5, Propositions 3.9, 3.10 and Theorem 3.17. Finally, we introduced generalized regular bipolar fuzzy graphs. The notion of product bipolar fuzzy line graph is introduced and investigated some of its properties. A necessary and sufficient condition is given for a product bipolar fuzzy graph to be isomorphic to its corresponding product bipolar fuzzy line graph. It is also examined when an isomorphism between two product bipolar fuzzy graphs follows from an isomorphism of their corresponding fuzzy line graphs.

9.2 Counterexamples

Here, we assume that $G^* = (V, E)$ represents a crisp graph and $G = (V, A, B)$ represents a bipolar fuzzy graph of it.

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First, we recall some definitions given in [5]. Also, we recall the Propositions 3.9, 3.10 and Theorem 3.17 of [5].

Definition 9.2.1. *(Definition 3.3 of [5]) Let $G = (V, A, B)$ be a bipolar fuzzy graph on G^*. If all the vertices have the same open neighborhood degree n, then G is called an n-regular bipolar fuzzy graph. The open neighborhood degree of a vertex x in G is defined by $deg(x) = (deg^P(x), deg^N(x))$, where $deg^P(x) = \sum_{x \in V} \mu_A^P(x)$ and $deg^N(x) = \sum_{x \in V} \mu_A^N(x)$.*

Definition 9.2.2. *(Definition 3.4 of [5]) Let $G = (V, A, B)$ be a regular bipolar fuzzy graph. The order of a regular bipolar fuzzy graph G is $O(G) = (\sum_{x \in V} \mu_A^P(x), \sum_{x \in V} \mu_A^N(x))$. The size of a regular bipolar fuzzy graph G is $S(G) = (\sum_{xy \in V} \mu_A^P(xy), \sum_{xy \in E} \mu_A^N(xy))$.*

Definition 9.2.3. *(Definition 3.5 of [5]) Let $G = (V, A, B)$ be a bipolar fuzzy graph. If each vertex of G has the same closed neighborhood degree m, then G is called a totally regular bipolar fuzzy graph. The closed neighborhood degree of a vertex x in G is defined by $deg[x] = (deg^P[x], deg^N[x])$, where $deg^P[x] = deg^P(x) + \mu_A^P(x)$, $deg^N[x] = deg^N(x) + \mu_A^N(x)$.*

Proposition 9.2.1. *(Proposition 3.9 of [5]) The size of a n-regular bipolar fuzzy graph G is $\frac{nk}{2}$, where $|V| = k$.*

Proposition 9.2.2. *(Proposition 3.10 of [5]) If G is a m-totally regular bipolar fuzzy graph, then $2S(G) + O(G) = mk$, where $|V| = k$.*

Theorem 9.2.1. *(Theorem 3.17 of [5]) Let $G = (V, A, B)$ be a bipolar fuzzy graph where crisp graph G^* is an odd cycle. Then G is regular bipolar fuzzy graph if and only if B is a constant function.*

To find out out the flaws of Definition 9.2.1 and Definition 9.2.3, we give counterexamples to Propositions 9.2.1, 9.2.2 and Theorem 9.2.1.

First of all, we point out that the Definition 9.2.1 is itself meaningless. The adjacency between vertices is missing in the definition. So, according to [5], the open neighborhood degree of a vertex x is $deg(x) = (deg^P(x), deg^N(x))$, where $deg^P(x) = \sum_{\substack{y \in V \\ xy \in E}} \mu_A^P(y)$ and $deg^N(x) = \sum_{\substack{y \in V \\ xy \in E}} \mu_A^N(y)$.

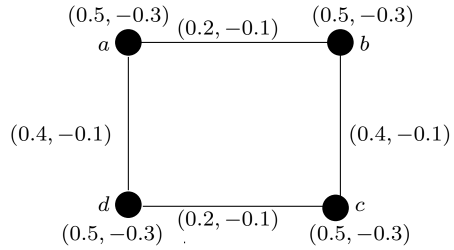

Figure 9.1: Bipolar fuzzy graph G

If we use the above definition, still the Propositions 9.2.1, 9.2.2 and Theorem 9.2.1 do not hold good at all.

Example 9.2.1. *Consider the graph* $G^* = (V, E)$ *where* $V = \{a, b, c, d\}$ *and* $E = \{ab, bc, cd, ad\}$. *Now, let* $G = (V, A, B)$ *be a bipolar fuzzy graph of* G^* *(see Fig. 9.1).*

Then, $deg(a) = deg(b) = deg(c) = deg(d) = (1, -0.6)$.

Hence, G *is* $(1, -0.6)$*-regular. Also,* $S(G) = (1.2, -0.4)$.

Here, $n = (1, -0.6)$ *and* $k = 4$.

So, by Proposition 9.2.1, $S(G) = \frac{nk}{2} = (2, -1.2)$.

But, $S(G) = (1.2, -0.4)$.

Remark 9.2.1. *This example shows that Proposition 9.2.1 is not true.*

Example 9.2.2. *Let* G *be same as of Example 9.2.1. Then,* $deg[a] = deg[b] = deg[c] = deg[d] = (1.5, -0.9)$. *So, the graph* G *of Fig. 9.1 is* $(1.5, -0.9)$*-totally regular.*

Also, $O(G) = (2, -1.2)$. *Here,* $m = (1.5, -0.9)$ *and* $k = 4$.

So, $2S(G) + O(G) = 2(1.2, -0.4) + (2, -1.2) = (4.4, -2)$ *and* $mk = (6, -3.6)$.

We see that, $2S(G) + O(G) = (4.4, -2) \neq (6, -3.6) = mk$.

Remark 9.2.2. *This example shows that Proposition 9.2.2 is not true.*

Example 9.2.3. *Let* $G^* = (V, E)$ *be an odd cycle where* $e_1 = v_0v_1, e_2 = v_1v_2, e_3 = v_2v_3 = v_2v_0$ *be the edges of* G^* *such that* $v_0 = v_3$. *Let* $G = (V, A, B)$ *be a bipolar fuzzy graph of* G^* *where* $A(v_0 = v_3) = A(v_1) = A(v_2) = (0.5, -0.3)$ *and* $B(e_1) = (0.2, -0.1), B(e_2) = (0.4, -0.1), B(e_3) = (0.2, -0.1)$. *Then,* $deg(v_0) = deg(v_1) = deg(v_2) = (1, -0.6)$, *i.e.* G *is* $(1, -0.6)$*-regular but* B *is not a constant.*

Again, consider another bipolar fuzzy graph $G = (V, A, B)$ *of* G^* *where* $A(v_0 = v_3) = (0.5, -0.3), A(v_1) = (0.3, -0.2), A(v_2) = (0.4, -0.3)$ *and* $B(e_1) = B(e_2) = B(e_3) = (0.2, -0.1)$. *In this case,* $deg(v_0) = (0.7, -0.5), deg(v_1) = (0.9, -0.6)$,
$deg(v_2) = (0.8, -0.5)$. *So,* G *is not regular although* B *is constant.*

Remark 9.2.3. *This example shows that Theorem 9.2.1 is not true.*

Remark 9.2.4. *We also point out that in Definition 9.2.2 there is a typing mistake which is corrected in the next section.*

9.3 Main results

In this section, we mainly provide the modified version of Definitions 9.2.1, 9.2.2, 9.2.3 and a proof of Propositions 9.2.1, 9.2.2 and Theorem 9.2.1.

Definition 9.3.1. *(The correction of Definition 9.2.1) The open neighborhood degree of $x \in V$ in G is defined by $deg(x) = (deg^P(x), deg^N(x))$, where $deg^P(x) = \sum\limits_{\substack{x \neq y \\ xy \in E}} \mu_B^P(xy)$ and $deg^N(v) = \sum\limits_{\substack{x \neq y \\ xy \in E}} \mu_B^N(xy)$. If all the vertices of G have same open neighborhood degree (d_1, d_2), then G is said to be (d_1, d_2)-regular. In this case, we also call G as a generalized regular bipolar fuzzy graph.*

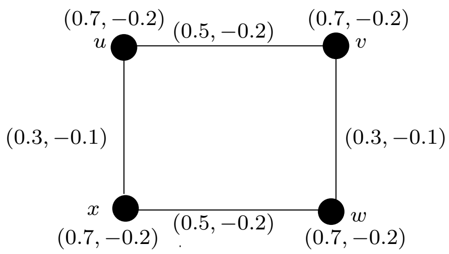

Figure 9.2: $(0.8, -0.3)$-regular and $(1.5, -0.5)$-totally regular bipolar fuzzy graph G

Example 9.3.1. *From Fig. 9.2 we have, $deg(u) = deg(v) = deg(w) = deg(x) = (0.8, -0.3)$. Hence, G is $(0.8, -0.3)$-regular.*

Definition 9.3.2. *(The correction of Definition 9.2.2) The order of G is denoted as $O(G) = (O^P(G), O^N(G))$ where $O^P(G) = \sum\limits_{v \in V} \mu_A^P(v)$ and $O^N(G) = \sum\limits_{v \in V} \mu_A^N(v)$. The size of G is denoted as $S(G) = (S^P(G), S^N(G))$ where $S^P(G) = \sum\limits_{uv \in E} \mu_B^P(uv)$ and $S^N(G) = \sum\limits_{uv \in E} \mu_B^N(uv)$.*

Example 9.3.2. *From Fig. 9.2, we have $O(G) = (2.8, -0.8)$ and $S(G) = (1.6, -0.6)$.*

Definition 9.3.3. *(The correction of Definition 9.2.3) The closed neighborhood degree of $x \in V$ in G is denoted as $deg[x] = (deg^P[x], deg^N[x])$, where $deg^P[x] = deg^P(x) + \mu_A^P(x)$ and $deg^N[x] = deg^N(x) + \mu_A^N(x)$. If each vertex of G has equal* *closed neighborhood degree* (f_1, f_2), *then* G *is said to be* (f_1, f_2)*-totally regular. In this case, we call* G *as a generalized totally regular bipolar fuzzy graph.*

Example 9.3.3. *The bipolar fuzzy graph* G *in Fig. 9.2 is* $(1.5, -0.5)$*-totally regular, since* $deg[u] = deg[v] = deg[w] = deg[x] = (1.5, -0.5)$.

Proposition 9.3.1. *Let* $G = (V, A, B)$ *be a* (d_1, d_2)*-regular bipolar fuzzy graph. Then size of* G *is given by* $S(G) = \frac{n}{2}(d_1, d_2)$ *where* $|V| = n$.

Proof. Immediate from the definition. □

Proposition 9.3.2. *Let* $G = (V, A, B)$ *be a* (f_1, f_2)*-totally regular bipolar fuzzy graph. Then* $2S(G) + O(G) = n(f_1, f_2)$ *where* $|V| = n$.

Proof. Follows from the definition. □

Theorem 9.3.1. *Let* $G = (V, A, B)$ *be a bipolar fuzzy graph of an odd cycle* $G^* = (V, E)$. *Then* G *is regular if and only if* $B = (\mu_B^P, \mu_B^N)$ *is constant.*

Proof. Suppose G is a (d_1, d_2)-regular.

Let $e_1, e_2, \ldots, e_{2n+1}$ be the edges of G^* such that $e_i = v_{i-1}v_i \in E$, $v_0, v_i \in V$, $i = 1, 2, \ldots, 2n + 1$ and $v_0 = v_{2n+1}$.

Let $\mu_B^P(e_1) = k_1$ and $\mu_B^N(e_1) = k_2$ where $k_1 \in [0, 1]$ and $k_2 \in [-1, 0]$.

G is (d_1, d_2)-regular implies that $deg^P(v_1) = d_1$ and $deg^N(v_1) = d_2$.

This means, $deg^P(v_1) = \mu_B^P(e_1) + \mu_B^P(e_2) = d_1$ and

$deg^N(v_1) = \mu_B^N(e_1) + \mu_B^N(e_2) = d_2$,

i.e. $k_1 + \mu_B^P(e_2) = d_1$ and $k_2 + \mu_B^N(e_2) = d_2$,

i.e. $\mu_B^P(e_2) = d_1 - k_1$ and $\mu_B^N(e_2) = d_2 - k_2$.

Again, $deg^P(v_2) = \mu_B^P(e_2) + \mu_B^P(e_3) = d_1$ and

$deg^N(v_2) = \mu_B^N(e_2) + \mu_B^N(e_3) = d_2$.

This implies, $\mu_B^P(e_3) = d_1 - (d_1 - k_1) = k_1$ and

$\mu_B^N(e_3) = d_2 - (d_2 - k_2) = k_2$ and so on.

Therefore, $\mu_B^P(e_i) = \begin{cases} k_1 & if \quad i \quad is \quad odd \\ (d_1 - k_1) & if \quad i \quad is \quad even \end{cases}$

and $\mu_B^N(e_i) = \begin{cases} k_2 & if \quad i \quad is \quad odd \\ (d_2 - k_2) & if \quad i \quad is \quad even \end{cases}$

Therefore, $\mu_B^P(e_1) = \mu_B^P(e_{2n+1}) = k_1$ and $\mu_B^N(e_1) = \mu_B^N(e_{2n+1}) = k_2$.

Since e_1 and e_{2n+1} are incident on the vertex v_0 and $deg(v_0) = (d_1, d_2)$ therefore,
$\mu_B^P(e_1) + \mu_B^P(e_{2n+1}) = d_1$ and $\mu_B^N(e_1) + \mu_B^N(e_{2n+1}) = d_2$,
i.e. $2k_1 = d_1$ and $2k_2 = d_2$,
i.e. $k_1 = \frac{d_1}{2}$ and $k_2 = \frac{d_2}{2}$.
Therefore, $\mu_B^P(e_i) = \frac{d_1}{2}$ and $\mu_B^N(e_i) = \frac{d_2}{2}$ for $i = 1, 2, \ldots, 2n+1$.
Therefore, B is constant.

Conversely, let us assume that B be a constant. So, let $B(u_1v_1) = (\mu_B^P(u_1v_1), \mu_B^N(u_1v_1))$ $= (k_1, k_2)$ for all $u_1v_1 \in E$ where $k_1 \in [0, 1]$ and $k_2 \in [-1, 0]$.

Then $deg(v) = (deg^P(v), deg^N(v)) = \big(\sum\limits_{\substack{u\neq v\\ uv\in E}} \mu_B^P(uv), \sum\limits_{\substack{u\neq v\\ uv\in E}} \mu_B^P(uv)\big) = (2k_1, 2k_2)$ for all $v \in V$. Consequently, G is $(2k_1, 2k_2)$-regular bipolar fuzzy graph. $\square$

9.4 Product bipolar fuzzy graphs

In this section, we define a new subclasses of bipolar fuzzy graphs, called product bipolar fuzzy graphs.

Definition 9.4.1. *A product bipolar fuzzy graph of a graph $G^* = (V, E)$ is a pair $G = (V, A, B)$ where $A = (\mu_A^P, \mu_A^N)$ is an bipolar fuzzy set in V and $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy relation on $\widetilde{V^2}$ such that $\mu_B^P(xy) \leq \mu_A^P(x) \times \mu_A^P(y)$, $\mu_B^N(xy) \geq -(\mu_A^N(x) \times \mu_A^N(y))$ for all $xy \in \widetilde{V^2}$ and $\mu_B^P(xy) = \mu_B^N(xy) = 0$ for all $xy \in (\widetilde{V^2} - E)$.*

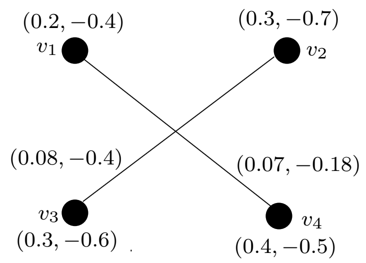

Figure 9.3: Product bipolar fuzzy graph G

Example 9.4.1. *Let us consider the graph $G^* = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_4, v_2v_3\}$. A product bipolar fuzzy graph G of G^* is shown in Fig. 9.3.*

Definition 9.4.2. *A product bipolar fuzzy graph $G = (V, A, B)$ of $G^* = (V, E)$ is said to be strong if $\mu_B^P(xy) = \mu_A^P(x) \times \mu_A^P(y)$ and $\mu_B^N(xy) = -\mu_A^N(x) \times \mu_A^N(y)$ for all $xy \in E$.*

The product bipolar fuzzy graph G in Fig. 9.3 is not strong.

Here after, we assume that G is a product bipolar fuzzy graph of the crisp graph G^*.

9.5 Product bipolar fuzzy line graphs

In this section, first we define a product bipolar fuzzy intersection graph of a product bipolar fuzzy graph. Then we define the product bipolar fuzzy line graphs.

Definition 9.5.1. *Let $P(S) = (S, T)$ be an intersection graph of a simple graph $G^* = (V, E)$. Let $G = (V, A, B)$ be a product bipolar fuzzy graph of G^*. We define a product bipolar fuzzy intersection graph $P(G) = (A_1, B_1)$ of $P(S)$ as follows:*

(i) A_1 *and* B_1 *are bipolar fuzzy subsets of* S *and* T *respectively,*

(ii) $\mu_{A_1}^P(S_i) = \mu_A^P(v_i)$, $\mu_{A_1}^N(S_i) = \mu_A^N(v_i)$,

(iii) $\mu_{B_1}^P(S_iS_j) = \mu_B^P(v_iv_j)$, $\mu_{B_1}^N(S_iS_j) = \mu_B^N(v_iv_j)$ *for all* $S_i, S_j \in S$, $S_iS_j \in T$.

In other words, any product bipolar fuzzy graph of $P(S)$ is called a product bipolar fuzzy intersection graph.

The following proposition is immediate.

Proposition 9.5.1. *Let $G = (V, A, B)$ be a product bipolar fuzzy graph of $G^* = (V, E)$ and $P(G) = (A_1, B_1)$ be a product bipolar intersection graph of $P(S)$. Then the following holds:*

(a) $P(G)$ *is a product bipolar fuzzy graph of* $P(S)$,

(b) $G \cong P(G)$.

Proof. (a) Since G is a product bipolar fuzzy graph, we have by Definition 9.5.1,
$\mu_{B_1}^P(S_iS_j) = \mu_B^P(v_iv_j) \le \mu_A^P(v_i) \times \mu_A^P(v_j) = \mu_{A_1}^P(S_i) \times \mu_{A_1}^P(S_j)$ and
$\mu_{B_1}^N(S_iS_j) = \mu_B^N(v_iv_j) \ge -(\mu_A^N(v_i) \times \mu_A^N(v_j)) = -(\mu_{A_1}^N(S_i) \times \mu_{A_1}^N(S_j))$.

Hence, $P(G)$ is a product bipolar fuzzy graph.

(b) Let us define a mapping $\phi : V \to S$ by $\phi(v_i) = S_i$ for $i = 1, 2, \ldots, n$.

Then clearly ϕ is one to one mapping of V onto S.

Now $v_iv_j \in E$ if and only if $S_iS_j \in T$ and $T = \{\phi(v_i)\phi(v_j) : v_iv_j \in E\}$.

Also, $\mu_A^P(v_i) = \mu_{A_1}^P(S_i) = \mu_{A_1}^P(\phi(v_i))$ and
$\mu_A^N(v_i) = \mu_{A_1}^N(S_i) = \mu_{A_1}^N(\phi(v_i))$ for all $v_i \in V$,
$\mu_B^P(v_iv_j) = \mu_{B_1}^P(S_iS_j) = \mu_B^P(\phi(v_i)\phi(v_j))$ and
$\mu_B^N(v_iv_j) = \mu_{B_1}^N(S_iS_j) = \mu_B^N(\phi(v_i)\phi(v_j))$ for all $v_iv_j \in E$.

Hence, ϕ is an isomorphism of G onto $P(G)$, i.e. $G \cong P(G)$. □

This proposition shows that any product bipolar fuzzy graph is isomorphic to a product bipolar fuzzy intersection graph.

Next, we define the product bipolar fuzzy line graph of a product bipolar fuzzy graph.

Definition 9.5.2. *Let $L(G^*) = (Z, W)$ be a line graph of a simple graph $G^* = (V, E)$. Let $G = (A, B)$ be a product bipolar fuzzy graph of G^*. Then a product bipolar fuzzy line graph $L(G) = (A_1, B_1)$ of G is defined as follows:*

(i) A_1 *and* B_1 *are bipolar fuzzy subsets of* Z *and* W *respectively,*

(ii) $\mu^P_{A_1}(S_x) = \mu^P_B(x) = \mu^P_B(u_x v_x)$,

(iii) $\mu^N_{A_1}(S_x) = \mu^N_B(x) = \mu^N_B(u_x v_x)$,

(iv) $\mu^P_{B_1}(S_x S_y) = \mu^P_B(x) \times \mu^P_B(y) = \mu^P_B(u_x v_x) \times \mu^P_B(u_y v_y)$,

(v) $\mu^N_{B_1}(S_x S_y) = -(\mu^N_B(x) \times \mu^N_B(y)) = -(\mu^N_B(u_x v_x) \times \mu^N_B(u_y v_y))$ *for all* $S_x, S_y \in Z$ *and* $S_x S_y \in W$.

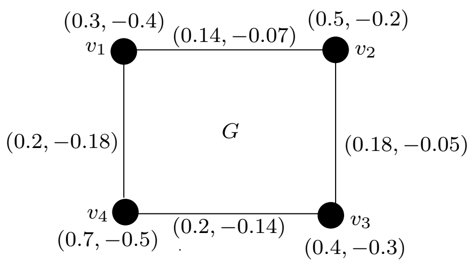

Figure 9.4: G is a product bipolar fuzzy graph

Example 9.5.1. *Let us consider a graph $G^* = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{x_1 = v_1v_2, x_2 = v_2v_3, x_3 = v_3v_4, x_4 = v_4v_1\}$. Let $G = (V, A, B)$ be a product bipolar fuzzy graph of G^* (see Fig. 9.4).*

Now, consider a line graph $L(G^*) = (Z, W)$ such that $Z = \{S_{x_1}, S_{x_2}, S_{x_3}, S_{x_4}\}$ and $W = \{S_{x_1}S_{x_2}, S_{x_2}S_{x_3}, S_{x_3}S_{x_4}, S_{x_4}S_{x_1}\}$. Let A_1 and B_1 be bipolar fuzzy subsets of Z and W respectively. Then by definition of product bipolar fuzzy line graph we have the following:

$\mu^P_{A_1}(S_{x_1}) = \mu^P_B(x_1) = 0.14,\ \mu^P_{A_1}(S_{x_2}) = \mu^P_B(x_2) = 0.18,$

$\mu^P_{A_1}(S_{x_3}) = \mu^P_B(x_3) = 0.2,\ \mu^P_{A_1}(S_{x_4}) = \mu^P_B(x_4) = 0.2,$

$\mu^N_{A_1}(S_{x_1}) = \mu^N_B(x_1) = -0.07,\ \mu^N_{A_1}(S_{x_2}) = \mu^N_B(x_2) = -0.05,$

$\mu^N_{A_1}(S_{x_3}) = \mu^N_B(x_3) = -0.14,\ \mu^N_{A_1}(S_{x_4}) = \mu^N_B(x_4) = -0.18,$

$\mu^P_{B_1}(S_{x_1}S_{x_2}) = \mu^P_B(x_1) \times \mu^P_B(x_2) = 0.14 \times 0.18 = 0.0252,$
$\mu^P_{B_1}(S_{x_2}S_{x_3}) = \mu^P_B(x_2) \times \mu^P_B(x_3) = 0.18 \times 0.2 = 0.036,$
$\mu^P_{B_1}(S_{x_3}S_{x_4}) = \mu^P_B(x_3) \times \mu^P_B(x_4) = 0.2 \times 0.2 = 0.04,$
$\mu^P_{B_1}(S_{x_4}S_{x_1}) = \mu^P_B(x_4) \times \mu^P_B(x_1) = 0.2 \times 0.14 = 0.028,$
$\mu^N_{B_1}(S_{x_1}S_{x_2}) = -(\mu^N_B(x_1) \times \mu^N_B(x_2)) = -(-0.07 \times -0.05) = -0.0035,$
$\mu^N_{B_1}(S_{x_2}S_{x_3}) = -(\mu^N_B(x_2) \times \mu^N_B(x_3)) = -(-0.05 \times -0.14) = -0.007,$
$\mu^N_{B_1}(S_{x_3}S_{x_4}) = -(\mu^N_B(x_3) \times \mu^N_B(x_4)) = -(-0.14 \times -0.18) = -0.0252,$
$\mu^N_{B_1}(S_{x_4}S_{x_1}) = -(\mu^N_B(x_4) \times \mu^N_B(x_1)) = -(-0.18 \times -0.07) = -0.0126.$

Hence, $L(G) = (A_1, B_1)$ is the product bipolar fuzzy line graph of G. It may be noted that, $L(G)$ is neither regular nor totally regular product bipolar fuzzy line graph.

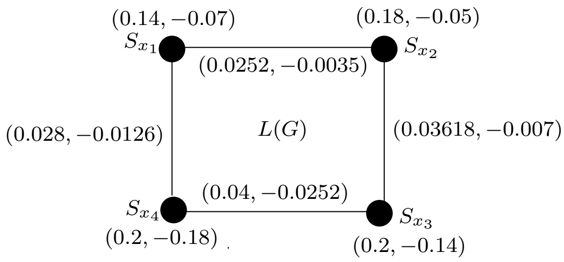

Figure 9.5: The line graph $L(G)$ of G

Proposition 9.5.2. *A product bipolar fuzzy line graph is a strong product bipolar fuzzy graph.*

Proof. Follows from the definition of product bipolar fuzzy line graph. □

Proposition 9.5.3. *If $L(G)$ is a product bipolar fuzzy line graph of the product bipolar fuzzy graph G, then $L(G^*)$ is the line graph of G^*.*

Proof. Since $G = (V, A, B)$ is a product bipolar fuzzy graph and $L(G) = (A_1, B_1)$ is a product bipolar fuzzy line graph, therefore $\mu^P_{A_1}(S_x) = \mu^P_B(x)$ and $\mu^N_{A_1}(S_x) = \mu^N_B(x)$ for all $x \in E$ and so $S_x \in Z \Leftrightarrow x \in E$.
Also, $\mu^P_{B_1}(S_xS_y) = \mu^P_B(x) \times \mu^P_B(y)$ and $\mu^N_{B_1}(S_xS_y) = -(\mu^N_B(x) \times \mu^N_B(y))$ for all $S_x, S_y \in Z$, and so $W = \{S_xS_y : S_x \cap S_y \neq \emptyset, x, y \in E, x \neq y\}$. This completes the proof. □

Proposition 9.5.4. *$L(G) = (A_1, B_1)$ is a product bipolar fuzzy line graph of some product bipolar fuzzy graph $G = (V, A, B)$ if and only if $\mu^P_{B_1}(S_xS_y) = \mu^P_{A_1}(S_x) \times \mu^P_{A_1}(S_y))$ and $\mu^N_{B_1}(S_xS_y) = -(\mu^N_{A_1}(S_x) \times \mu^N_{A_1}(S_y))$ for all $S_xS_y \in W$.*

Proof. Suppose that $\mu^P_{B_1}(S_xS_y) = \mu^P_{A_1}(S_x) \times \mu^P_{A_1}(S_y))$ and

$\mu^N_{B_1}(S_xS_y) = -(\mu^N_{A_1}(S_x) \times \mu^N_{A_1}(S_y))$ for all $S_xS_y \in W$.

Let us now define $\mu^P_A(x) = \mu^P_{A_1}(S_x)$ and $\mu^N_A(x) = \mu^N_{A_1}(S_x)$ for all $x \in E$.

Then, $\mu^P_{B_1}(S_xS_y) = \mu^P_{A_1}(S_x) \times \mu^P_{A_1}(S_y)) = \mu^P_A(x) \times \mu^P_A(y))$ and

$\mu^N_{B_1}(S_xS_y) = -(\mu^N_{A_1}(S_x) \times \mu^N_{A_1}(S_y)) = -(\mu^N_A(x) \times \mu^N_A(y))$.

A bipolar fuzzy set $A = (\mu^P_A, \mu^N_A)$ that yields that the property $\mu^P_B(xy) \le \mu^P_A(x) \times \mu^P_A(y)$ and $\mu^N_B(xy) \ge -(\mu^N_A(x) \times \mu^N_A(y))$ will suffice.

The converse part follows from the Definition 9.5.2. □

Another characterization of product bipolar fuzzy line graphs of product bipolar fuzzy graph is given in the following proposition.

Proposition 9.5.5. *$L(G) = (A_1, B_1)$ is a product bipolar fuzzy line graph of some product bipolar fuzzy graph if and only if $L(G^*) = (Z, W)$ is a line graph satisfying $\mu^P_{B_1}(uv) = \mu^P_{A_1}(u) \times \mu^P_{A_1}(v)$ and $\mu^N_{B_1}(uv) = -(\mu^N_{A_1}(u) \times \mu^N_{A_1}(v))$ for all $uv \in W$.*

Proof. Follows from the Propositions 9.5.3 and 9.5.4. □

Definition 9.5.3. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two product bipolar fuzzy graphs of the graphs $G^*_1 = (V_1, E_1)$ and $G^*_2 = (V_2, E_2)$ respectively. A homomorphism between G_1 and G_2 is a mapping $\phi : V_1 \to V_2$ such that*

(i) $\mu^P_{A_1}(x) \le \mu^P_{A_2}(\phi(x))$ and $\mu^N_{A_1}(x) \ge \mu^N_{A_2}(\phi(x))$ for all $x \in V_1$,

(ii) $\mu^P_{B_1}(xy) \le \mu^P_{B_2}(\phi(x)\phi(y))$ and $\mu^N_{B_1}(xy) \ge \mu^N_{B_2}(\phi(x)\phi(y))$ for all $xy \in \widetilde{V^2_1}$.

A bijective homomorphism with the property that $\mu^P_{A_1}(x) = \mu^P_{A_2}(\phi(x))$ and $\mu^N_{A_1}(x) = \mu^N_{A_2}(\phi(x))$ for all $x \in V_1$ is called a (weak) vertex-isomorphism.

A bijective homomorphism with the property that $\mu^P_{B_1}(xy) \le \mu^P_{B_2}(\phi(x)\phi(y))$ and $\mu^N_{B_1}(xy) \ge \mu^N_{B_2}(\phi(x)\phi(y))$ for all $xy \in \widetilde{V^2_1}$, is called a (weak) line-isomorphism.

If ϕ is both (weak) vertex isomorphism and (weak) line-isomorphism, then ϕ is called a (weak) isomorphism of G_1 onto G_2. If G_1 is isomorphic to G_2, then we write $G_1 \cong G_2$.

Proposition 9.5.6. *$G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two product bipolar fuzzy graphs of the graphs $G^*_1 = (V_1, E_1)$ and $G^*_2 = (V_2, E_2)$ respectively. If ϕ is a weak isomorphism G_1 onto G_2, then ϕ is an isomorphism of G^*_1 onto G^*_2.*

Proof. Obvious. □

Proposition 9.5.7. *Let $L(G) = (A_1, B_1)$ be the product bipolar fuzzy line graph corresponding to the product bipolar fuzzy graph $G = (V, A, B)$ of $G^* = (V, E)$. Suppose that G^* is connected. Then the following hold:*

(i) There exists a weak isomorphism of G onto $L(G)$ if and only if G^* is a cycle and for all $v \in V$, $x \in E$, $\mu_A^P(v) = \mu_B^P(x)$, $\mu_A^N(v) = \mu_B^N(x)$, i.e. $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ are constant functions on V and E, respectively, taking on the same value.

(ii) If ϕ is a weak isomorphism of G onto $L(G)$, then ϕ is an isomorphism.

Proof. Suppose that ϕ is a weak isomorphism of G onto $L(G)$. By Proposition 9.5.6, ϕ is an isomorphism of G^* onto $L(G^*)$. By Proposition 9.5.3, G^* is a cycle ([55], Theorem 8.2).

Let $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{x_1 = v_1v_2, x_2 = v_2v_3, \ldots, x_n = v_nv_1\}$, where $v_1v_2 \ldots v_nv_1$ is a cycle.

Let us define the bipolar fuzzy sets $\mu_A^P(v_i) = s_i$, $\mu_A^N(v_i) = \acute{s_i}$ and

$\mu_B^P(v_iv_{i+1}) = t_i$, $\mu_B^P(v_iv_{i+1}) = \acute{t_i}$ for $i = 1, 2, \ldots, n$ where $v_{n+1} = v_1$, $s_i, t_i \in [0, 1]$, $\acute{s_i}, \acute{t_i} \in [-1, 0]$.

Then for $s_{n+1} = s_1$, $\acute{s}_{n+1} = \acute{s_1}$,

$$(a)\begin{cases} t_i \leq s_i \times s_{i+1} \\ \acute{t_i} \geq -\acute{s_i} \times \acute{s}_{i+1}, i = 1, 2, \ldots, n. \end{cases}$$

Now, $Z = \{S_{x_1}, S_{x_2} \ldots, S_{x_n}\}$ and $W = \{S_{x_1}S_{x_2}\, S_{x_2}S_{x_3}, \ldots, S_{x_n}S_{x_1}\}$.

Also for $t_{n+1} = t_1$ and $\acute{t}_{n+1} = \acute{t_1}$,

$\mu_{A_1}^P(S_{x_i}) = \mu_B^P(x_i) = \mu_B^P(v_iv_{i+1}) = t_i$,

$\mu_{A_1}^N(S_{x_i}) = \mu_B^N(x_i) = \mu_B^N(v_iv_{i+1}) = \acute{t_i}$ and

$\mu_{B_1}^P(S_{x_i}S_{x_{i+1}}) = \mu_B^P(x_i) \times \mu_B^P(x_{i+1}) = t_i \times t_{i+1}$,

$\mu_{A_1}^N(S_{x_i}S_{x_{i+1}}) = -\mu_B^N(x_i) \times \mu_B^N(x_{i+1}) = -\acute{t_i} \times \acute{t}_{i+1}$, $i = 1, 2 \ldots, n$, where $v_{n+1} = v_1, v_{n+2} = v_2$.

Since ϕ is an isomorphism of G^* onto $L(G^*)$, ϕ maps V one-to-one onto Z. Also ϕ preserves adjacency. Hence, ϕ induces a permutation π of $\{1, 2, \ldots, n\}$ such that $\phi(v_i) = S_{x_{\pi(i)}} = S_{v_{\pi(i)}v_{\pi(i+1)}}$ and $x_i = v_iv_{i+1} \to \phi(v_i)\phi(v_{i+1}) = S_{v_{\pi(i)}v_{\pi(i+1)}}S_{v_{\pi(i+1)}v_{\pi(i+2)}}$ for $i = 1, 2, \ldots, (n-1)$.

Now, $s_i = \mu_A^P(v_i) \leq \mu_{A_1}^P(\phi(v_i)) = \mu_{A_1}^P(S_{v_{\pi(i)}v_{\pi(i+1)}}) = t_{\pi(i)}$,

$\acute{s_i} = \mu_A^N(v_i) \geq \mu_{A_1}^N(\phi(v_i)) = \mu_{A_1}^N(S_{v_{\pi(i)}v_{\pi(i+1)}}) = \acute{t}_{\pi(i)}$,

$$\begin{aligned} t_i = \mu_B^P(v_iv_{i+1}) &\leq \mu_{B_1}^P(\phi(v_i)\phi(v_{i+1})) \\ &= \mu_{B_1}^P(S_{v_{\pi(i)}v_{\pi(i+1)}}S_{v_{\pi(i+1)}v_{\pi(i+2)}}) \\ &= \mu_{B_1}^P(S_{v_{\pi(i)}v_{\pi(i+1)}}) \times \mu_{B_1}^P(S_{v_{\pi(i+1)}v_{\pi(i+2)}}) \\ &= t_{\pi(i)} \times t_{\pi(i+1)}, \end{aligned}$$

$$\begin{aligned} \acute{t_i} = \mu_B^N(v_iv_{i+1}) &\leq \mu_{B_1}^N(\phi(v_i)\phi(v_{i+1})) \\ &= \mu_{B_1}^N(S_{v_{\pi(i)}v_{\pi(i+1)}}S_{v_{\pi(i+1)}v_{\pi(i+2)}}) \\ &= -\mu_{B_1}^N(S_{v_{\pi(i)}v_{\pi(i+1)}}) \times \mu_{B_1}^N(S_{v_{\pi(i+1)}v_{\pi(i+2)}}) \\ &= -\acute{t}_{\pi(i)} \times \acute{t}_{\pi(i+1)} \text{ for } i = 1, 2, \ldots, n. \end{aligned}$$

That is, $s_i \leq t_{\pi(i)}$, $\acute{s_i} \geq \acute{t}_{\pi(i)}$ and

$$(b) \begin{cases} t_i \leq t_{\pi(i)} \times t_{\pi(i+1)} \\ \acute{t_i} \geq -\acute{t}_{\pi(i)} \times \acute{t}_{\pi(i+1)}, i = 1, 2, \ldots, n. \end{cases}$$

By (b), we have $t_i \leq t_{\pi(i)}$, $\acute{t_i} \geq \acute{t}_{\pi(i)}$ for $i = 1, 2, \ldots, n$ and so $t_{\pi(i)} \leq t_{\pi(\pi(i))}$, $\acute{t}_{\pi(i)} \geq \acute{t}_{\pi(\pi(i))}$ for $i = 1, 2, \ldots, n$. Continuing, we have

$t_i \leq t_{\pi(i)} \leq \ldots \leq t_{\pi^j(i)} \leq t_i$,

$\acute{t_i} \geq \acute{t}_{\pi(i)} \geq \ldots \geq \acute{t}_{\pi^j(i)} \geq \acute{t_i}$ and so $t_i = t_{\pi(i)}$, $\acute{t_i} = \acute{t}_{\pi(i)}$, $i = 1, 2, \ldots, n$, where π^{j+1} is the identity map. Again by (b), we have

$t_i \leq t_{\pi(i)} = t_{i+1}$,

$\acute{t_i} \geq \acute{t}_{\pi(i+1)} = \acute{t}_{(i+1)}$, $i = 1, 2, \ldots, n$ where $t_{n+1} = t_n$, $\acute{t}_{n+1} = \acute{t}_n$.

Hence by (a) and (b), we have $t_1 = \ldots = t_n = s_1 = \ldots s_n$,

$\acute{t_1} = \ldots = \acute{t_n} = \acute{s_1} = \ldots = \acute{s_n}$.

Thus we have not only proved the conclusion about A and B being constant functions, but also we have shown that (ii) holds.

Conversely, suppose that G^* is a cycle and for all $v \in V$, $x \in E$, $\mu_A^P(v) = \mu_B^P(x)$, $\mu_A^N(v) = \mu_B^N(x)$. By Proposition 9.5.3, $L(G^*)$ is the line graph of G^*. Since G^* is a cycle, $G^* \cong L(G^*)$ by ([55], Theorem 8.2). This isomorphism induces an isomorphism of G onto $L(G)$ since $\mu_A^P(v) = \mu_B^P(x)$, $\mu_A^N(v) = \mu_B^N(x)$ for all $v \in V$, $x \in E$ and so $A = B = A_1 = B_1$ on their respective domains. □

Proposition 9.5.8. *Let G_1 and G_2 be two product bipolar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively, such that G_1^* and G_2^* is connected. Let $L(G_1)$ and $L(G_2)$ be the product bipolar fuzzy line graphs corresponding to G_1 and G_2*

respectively. Suppose that it is not the case that one of G_1^* *and* G_2^* *is complete graph* K_3 *and other is bipartite complete graph* $K_{1,3}$. *If* $L(G_1) \cong L(G_2)$, *then* G_1 *and* G_2 *are line isomorphic.*

Proof. Since $L(G_1) \cong L(G_2)$, therefore by Proposition **??**, $L(G_1^*) \cong L(G_2^*)$. Since $L(G_1^*)$ and $L(G_2^*)$ are the line graphs of G_1^* and G_2^*, respectively, by Proposition 9.5.3, we have that $G_1^* \cong G_2^*$ by ([55], Theorem 8.3).

Let ψ be the isomorphism of $L(G_1)$ onto $L(G_2)$ and ϕ be the isomorphism of G_1^* onto G_2^*. Then $\mu_{A_3}^P(S_x) = \mu_{A_4}^P(\psi(S_x)) = \mu_{A_4}^P(S_{\phi(x)})$, $\mu_{A_3}^N(S_x) = \mu_{A_4}^N(\psi(S_x)) = \mu_{A_4}^N(S_{\phi(x)})$, where the latter equalities holds by the proof of ([55], Theorem 8.3) and so $\mu_{B_1}^P(x) = \mu_{B_2}^P(\phi(x))$, $\mu_{B_1}^N(x) = \mu_{B_2}^N(\phi(x))$. Hence G_1 and G_2 are line isomorphic. □

9.6 Summary

In this chapter, we redefined open neighborhood degree and closed neighborhood degree of a vertex in bipolar fuzzy graphs. Finally, we introduced generalized regular bipolar fuzzy graphs and proved some results of it. We introduced a new subclasses of bipolar fuzzy graphs namely product bipolar fuzzy graphs. Then, product bipolar fuzzy line graphs are defined and studied several important results of it.

Chapter 10

Conclusion

Nowadays, uncertainty and impreciseness present in almost all systems. An m-polar fuzzy graph can be used to represent the real world problems which involve multi-case of information and uncertainty. An m-polar fuzzy graph is a generalized structure of a bipolar fuzzy graph which gives more precision, flexibility and compatibility to a system when more than one agreements are to be dealt with. Thus, m-polar fuzzy graphs are the most important research area for the researchers. Application of m-polar fuzzy graphs can be found in image capturing, image segmentation, image shrinking, data mining, communication, planning, scheduling, etc.

The first chapter is the introductory chapter of the thesis.

In Chapter 2, generalized m-polar fuzzy graphs is introduced. Several operations have been defined on m-polar fuzzy graphs. Some useful properties of strong m-polar fuzzy graphs, self-complementary m-polar fuzzy graphs and self-complementary strong m-polar fuzzy graphs are discussed. We are now working to find many more useful results of m-polar fuzzy graphs as an extension of this study.

In Chapter 3, three new operations, viz. direct product, semi-strong product and strong product are defined on m-polar fuzzy graphs. A subclass of m-polar fuzzy graphs called product m-polar fuzzy graph is defined and many properties of them are discussed here. The degree of a vertex in m-polar fuzzy graphs are introduced from two given m-polar fuzzy graphs G_1 and G_2 using the operations of Cartesian product, composition, direct product, semi-strong product and strong product. At the end, an application of 3-polar fuzzy influence graph is given. An algorithm can be designed to find the degree of vertices of an m-polar fuzzy graph.

In Chapter 4, the notions of density of an m-polar fuzzy graphs and balanced m-polar fuzzy graphs are defined. Some results of balanced m-polar fuzzy graphs are discussed here. Here also, an algorithm can be designed to find the density of an m-polar fuzzy graph and to check whether the m-polar fuzzy graph is balanced or not.

In Chapter 5, our study describes the m-polar fuzzy multigraphs, m-polar fuzzy planar graphs, and a very important consequence of m-polar fuzzy planar graphs known as m-polar fuzzy dual graphs. The new parameter "degree of planarity" used in this chapter characterizes an m-polar fuzzy graph in many ways. Several properties can be investigated on regular m-polar fuzzy planar graphs, irregular m-polar fuzzy planar graphs. The graphs such as m-polar fuzzy multigraph, m-polar fuzzy planar graph, and m-polar fuzzy dual graph are also defined. In crisp planar graph, no edge intersects each other. But, the edges of any m-polar fuzzy graph may be m-polar fuzzy weak or m-polar fuzzy strong. Using the concept of m-polar fuzzy weak edge, we define m-polar fuzzy planar graph in such a way that an edge may intersect other edges. But, this facility violates the definition of planarity of graph. Since the role of m-polar fuzzy weak edge is insignificant, the intersection between an m-polar fuzzy fuzzy weak edge with any edge is less important. Motivating from this idea, we allow the intersection of edges in m-polar fuzzy planar graph. It is well known that if the membership values of all edges become one, the graph becomes crisp graph. Keeping this idea in mind, we define a new term called degree of planarity of an m-polar fuzzy graph. If the degree of planarity of an m-polar fuzzy graph is $\mathbf{1} = (1, 1, \ldots, 1)$, then no edge crosses other. This leads to the crisp planar graph. Thus, the planarity value measures the degree of planarity of an m-polar fuzzy graph. This is a very interesting concept of m-polar fuzzy graph theory. Strong m-polar fuzzy planar graph has been exemplified. Another important term of planar graph is 'face' which is redefined in m-polar fuzzy planar graph. In this chapter, new theories have been investigated for m-polar fuzzy planar graph. The m-polar fuzzy dual graph is defined for the m-polar fuzzy planar graph whose degree of planarity is $\mathbf{1} = (1, 1, \ldots, 1)$. These theories will be helpful to improve algorithms in different fields including computer vision, image segmentation, etc. This idea can be extended to the other types of fuzzy graphs such as m-polar fuzzy soft planar graphs, m-polar fuzzy rough planar graphs, etc.

In Chapter 6, the notion of weak self complement m-polar fuzzy graphs, order, size, busy vertices and free vertices of an m-polar fuzzy graphs are defined. Self complement m-polar fuzzy graphs have many important role in the theory of m-polar fuzzy graphs. If an m-polar fuzzy graph is not self complement, then also we can say that it is self complement in some weaker sense. We can establish some useful results with this graph. This motivates to define weak self complement m-polar fuzzy graphs in this chapter. A necessary condition is mentioned for an m-polar fuzzy graph to be weak self complement. Several properties of them are discussed. A relative study of complement and operations on m-polar fuzzy graphs have been made. Some real life problems have been modeled using the concepts of m-polar fuzzy graphs. Many more weak notions can be introduced on m-polar fuzzy graphs to achieve important results.

Chapter 7 deals with the concept of edge regular, strongly regular, biregular, partially edge regular and fully edge regular m-polar fuzzy graphs. Some properties of them are studied. Finally, we introduced the notion of strongly edge irregular and strongly edge totally irregular m-polar fuzzy graphs. Some properties of them are also studied to characterize strongly edge irregular and strongly edge totally irregular m-polar fuzzy graphs.

In Chapter 8, we mainly generalized the usual concept of isomorphism in m-polar fuzzy graphs which we call as m-polar ψ-morphism. The action of m-polar ψ-morphism on m-polar fuzzy graphs are discussed. Then, d_2 degree, total d_2 degree of a vertex, $(2, \overline{k})$-regularity and totally $(2, \overline{l})$-regularity are defined in m-polar fuzzy graphs. A real life situation of a company has been modeled in terms of 4-polar fuzzy graphs as an application.

In Chapter 9, generalized regular bipolar fuzzy graphs are introduced. A subclass of bipolar fuzzy graphs namely product bipolar fuzzy graph is defined. Then the notion of product bipolar fuzzy line graph is introduced and investigated some of its properties. A necessary and sufficient condition is given for a product bipolar fuzzy graph to be isomorphic to its corresponding product bipolar fuzzy line graph. It is also examined when an isomorphism between two product bipolar fuzzy graphs follows from an isomorphism of their corresponding fuzzy line graphs.

The natural extension of these work are

(i) m-polar fuzzy soft graphs,

(ii) m-polar fuzzy soft planar graphs,

(iii) m-polar fuzzy soft hypergraphs,

(iv) m-polar fuzzy soft competition graphs,

(v) m-polar fuzzy rough graphs,

(vi) Applications of m-polar fuzzy soft graphs on decision making problems, etc.

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