2018

M.Sc.

1st Semester Examination

MATHEMATICS

PAPER—MTM-101

Full Marks : 50

Time : 2 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Real Analysis

1. Answer any *four* questions : 4×2

(a) If E is a measurable subset of $[a, b]$

show that $\int_E f\,dx = 4m(E)$ when $f(x) = 4$.

(b) Is the following a connected subset of $\mathbb{R}^2$

$\{(x, y) \in \mathbb{R}^2 : x^2y^2 = 1\}$? Justify your answer.

(c) Is the following a compact subset of $\mathbb{R}^2$

$\{(x, y) \in \mathbb{R}^2 : x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1\}$? Justify your answer.

(d) Discuss the continuity of a function from a discrete matric space into a metric space.

(e) Evaluate : $\int_{-1}^{1} x^2 \, d(x^2)$.

(f) Define a measurable function.

(g) Define Borel set.

(h) Show that the set of all natural numbers is a null subset of $\mathbb{R}$.

2. Answer any *four* questions : 4×4

(a) Show that the following function is not of bounded variation though continuous

$$f(x) = x \cos\frac{\pi}{x} \quad \text{if } 0 < x \leq 1$$
$$= 0 \quad \text{if } x = 0$$

(b) Let a bounded measurable function $f : E \to R$ satisfy $a < f(x) < b$ for all $x \in E$. Prove that

$$am(E) \leq \int_E f \, dx \leq bm(E).$$

(c) State and prove the Monotone Convergence theorem.

(d) If f is continuous on $[a, b]$ and α is monotonically increasing on $[a, b]$ show that $f \in R(\alpha)$ on $[a, b]$.

(e) Prove that a compact metrix space is separable.

(f) Let $f_n : X \to \mathbb{R}^*$ be measurable for $n = 1, 2, 3, \ldots$ Then show that $\liminf_{n\to\infty} f_n$ and $\inf_{n\to\infty} f_n$ are measurable functions on X.

(g) If $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \ldots,$ and $f(x) = \sum_{n=1}^{\infty} f_n(x),\ x \in X$, then show that

$$\int f\, d\mu = \sum_{n=1}^{\infty} \int f_n\, d\mu .$$

(h) Prove that a continuous image of a connected metric space is connected.

3. Answer any *two* questions : 2×8

(a) Define a function of bounded variation. Prove that a function $f(x)$ is of bounded variation on $[a, b]$ if and only if it can be expressed as difference of two monotone increasing functions.

(b) (i) Let $f(x) = \frac{1}{x^p}$ if $0 < x \leq 1$ and $f(0) = 0$. Find necessary and sufficient condition on p such that $f \in L^1[0, 1]$. Compute $\int_0^1 f(x)\lambda(x)$ in that case.

(ii) Evaluate the following :

$$\int_{-1}^{3} 2\cos x \, d(2x + [x]).$$ 6+2

(c) (i) Show that if a metric space X is compact then it is closed and bounded.

(ii) Show that every path connected metric space is connected. Give an example to show that the converse is not true. 4+4

(d) (i) Let $f : X \to [0, \infty]$ be measurable and $\phi(E) = \int_E f \, d\mu$ for every measurable set E in X. Show that ϕ is a measurable function and $\int g \, d\phi = \int gf \, d\mu$ for every measurable function g on X with range in $[0, \infty]$.

(ii) Show that the Cantor set is a null set. 5+3

[Internal Assessment — 10 Marks]

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