2017

M.Sc. Part–I Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—II

Full Marks : 100

Time : 4 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

The symbols have their usual meanings.

Write the answer to questions of each group in Separate answer booklet.

Group—A

(*Algebra*)

[Marks : 50]

Answer *Q. No. 1* and any *three* from the rest.

1. Answer any *one* question : 1×2

(a) When an integral domain R is called unique factorisation domain? 2

(b) Define 'radius' and 'diameter' of a connected graph. 2

2. (a) State and prove Canchy's theorem for a finite group G. 7

(b) Define Euler graph. State and prove the necessary and sufficient condition for Euler graph. 5

(c) If F is a field, then prove that $F(x)$, the polynomial in x over F, is an Euclidean domain. 4

3. (a) Let R be a commulative ring with unity. Show that an ideal M of R is maximal if and only if R/M is a field. 5

(b) Define lattice with respect to Poset and also with respect to algebra. Show that the two definitions are equivalent. 5

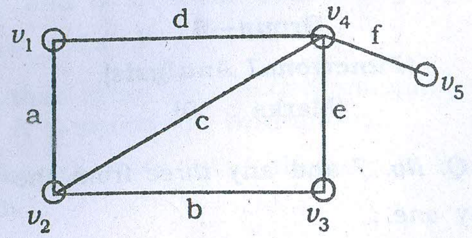
(c) Define chromatic number of a graph. Prove that a graph with n vertices is a tree if and only if its chromatic polynomial $P_n(\lambda) = \lambda(1-\lambda)^{n-1}$. 6

4. (a) Define distributive lattice, Show that $b = c$ in a distributive lattice $(L, \wedge, \vee)$, where $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ for all $a \in L$. 5

(b) State and prove the fundamental theorem on homomorphism of rings. 6

(c) Define kernel and image of a group homomorphism f from the group G to the group G'. Show that Ker(f) is a normal subgroup of G. 5

5. (a) Find the Incidence, Adjacency and circuit matrices of the following graph : 6



(b) Let H be a normal subgroup of a group G. Prove that

(i) If G is abelian then G/H is also abelian.

(ii) If G is cyclic then also G/H. 6

(c) Show that a group of order 75 is not simple.

6. (a) Define component of a graph. Show that a graph G is disconnected if and only if its vetex set V can be partitioned into two non-empty disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2. 6

(b) Let $S = \{z \in C : z^7 = 1\}$ and $G = (S, .)$, and $G' = (Z_7, +)$ be two groups, Define a mapping $f : G \to G'$ such that G and G' are isomorphic. 5

(c) Define duality of a lattice. Show that the dual of a lattice is a lattice. 5

Group—B

(*Functional Analysis*)

[Marks : 50]

Answer *Q. No. 7* and any *three* from the rest.

7. Answer any *one* : 2×1

(a) Define a first category set with an example.

(b) Show that every inner product space is uniformly convex.

8. (a) State and prove Cantor's Intersection theorem. 8

(b) Show that every sequentially compact metric space is compact. 4

(c) Show that given any open cover of $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ (considered as a subset of $\mathbb{R}$) we can find a finite number of elements in the cover such that their union contains A.

Can we assert such an analogus result for the set $B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$? 4

9. (a) Use Banach fixed point theorem, to show that the volterra integral equation

$$f(s) = x(s) - \mu \int_a^s k(s,t)x(t)dt$$

has a unique solution if $f(t)$, $y(t)$ are continuous on $[a, b]$ and $k(s, t)$ is continuous on the region $a \le t \le s,\ a \le s \le b$. 8

(b) Show that every contraction mapping on a metric space X is uniformly continuous. 3

(c) Use Banach fixed point theorem to show that the system.

$$x = \frac{1}{3}x - \frac{1}{4}y + \frac{1}{4}z - 1$$

$$y = -\frac{1}{2}x + \frac{1}{5}y + \frac{1}{4}z + 2$$

$$z = \frac{1}{5}x - \frac{1}{3}y + \frac{1}{4}z - 2$$

has a unique solution. 5

10. (a) Establish a necessary and sufficient condition for a normed space X to be complete. 7

(b) Let $X = e([a, b])$ with the sup norm and $k(s, t)$ be a continuous function on $[a, b] \times [a, b]$.

Define $F : X \to X$ by $F(x)(s) = \int_a^b k(s,t)x(t)dt$,

$x \in X,\ s \in [a, b]$.

Then show that $F \in B(X)$ and

$$\|F\| = \sup\left\{\int_a^b |k(s,t)|\,dt : s \in [a,b]\right\} \qquad 6$$

(c) Define bounded linear transformation. Give an example of a linear mapping which is not bounded. 3

11. (a) Let X, Y be Banach spaces and $T \in B(X,Y)$.

Then show that the following are equivalent.

(i) T is bijective,

(ii) There exists $S \in B(Y,X)$ such that $ST = id_X$, $TS = id_Y$. 5

(b) Let $T : l^2 \to l^2$ be given by

$$T(x_1, x_2, x_3 \ldots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right),$$

where $x = (x_1, x_2, x_3 \ldots) \in l^2$

Show the following :

(i) Ran (T) is not closed.

(ii) Graph of T^{-1} is closed and

(iii) T^{-1} is not continuous. 7

(c) Let H be a Hilbert space and $S \in B(H)$.

Prove that for all $x, y \in H$,

$$< Sx, y > = \frac{1}{4}\sum_{n=0}^{3} i^n < S(x + i^n y), (x + i^n y) > \qquad 4$$

12. (a) Establish Bessel's inequality in an inner product space X. 5

(b) Let $T \in B(H)$ be such that $<Tx, x> = 0$ for all $x \in H$, H being a Hilbert space. Then show that $T = 0$. Give an example to show that the above result is not true in case of real Hilbert space. 6

(c) Let $<, >$ be an inner product on a linear space X and $T : X \to X$ be a linear one-to-one map.

Let $\langle x, y \rangle_T = \langle T(x), T(y) \rangle \quad x, y \in X$.

Then show that $<, >_T$ is an inner product on X. 5