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Double Fuzzy Number and its Application in Risk Analysis in a Production System using Fuzzy Cognitive Map

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ABSTRACT

Any production system consist of different factors and they are interrelated to each other. Corresponding to each factor there also exist some risk in continuation or failure of that production system due to the lack of sufficiency of any factor. since there are interrelationship between the factors a fuzzy cognitive map (FCM)has been considered in this present model. The factors values are taken in linguistic form i.e., fuzzy form. Now the risk is associated with each factor so to relate the risk with the factors a new fuzzy number has been introduced in this paper which is named as "double fuzzy number". Some properties corresponding to this double fuzzy number also has been introduced. To analyze the total risk of the system, the application of this double fuzzy number has been considered. The proposed production system has been analyzed with a numerical example.

Keywords: Risk; Fuzzy Cognitive Map; Production; Linguistic Values; Triangular Fuzzy Number;Double Fuzzy Number.

1. Introduction

Fuzzy set was first introduced by Zadeh [20]. It was established to describe the nondeterministic situation. Real situations are very often uncertain or vague in a number of ways. Due to lack of information, the present and future states of a system is not known completely. So fuzzy set theory has an huge application in real world problems. The fuzzy set theory has encompassed large area of mathematics. Goguen [8, 9] showed the intention of the authors to generalize the classical notion of a set. Zadeh [21] also introduced the concept of linguistic variable and its application to approximate reasoning. Dubois and Prade [6] presented theory and application on fuzzy set theory. Atanassov and Stoeva [1] first introduced intuitonistic fuzzy set.

A production system depends upon various factors. The factors are dependent on each other. As a result the system becomes complex. For complex dynamical systems, conventional methods have limited contribution in modeling and controlling such systems. So a new technique is required for developing and analyzing such system. Some investigators have used fuzzy cognitive Map (FCM) to develop such a complex system.

Fuzzy cognitive maps are symbolic representation for the description and modeling of a system. They consist of concepts which describe different aspects in the behavior of the system.The human experience and the knowledge of the operation of the system are used to develop the fuzzy cognitive map. An FCM illustrates the whole system by a graph showing the cause and effect along concepts and it is a simple way to describe the behavior of a system in a symbolic manner. FCM was first introduced by Kosko [10]. It has been introduced in decision analysis and operation research by Craiger et al [5]. Luo et al. [12] used a TFN-ANP based approach to evaluate virtual research center's comprehensive performance. Kyriakarakos et al. [11] introduced a fuzzy cognitive map petri nets energy management system for autonomous polygeneration microgrids. Papageorgiou et al. [14] presented a fuzzy cognitive based approach for predicting yield in cotton crop production. Elomda et al. [7] introduced an extension of fuzzy decision maps for multi-criteria decision making.

In a production system, there must be some risks in the business of profit/loss in the system. So analyzing risk of a system is very much essential. There are also some fuzzy risk analysis problems in different areas. Various methods are there to investigate fuzzy risk analysis in different areas. Ranking method of fuzzy number is an important techniques to evaluate risk analysis. Chen et al [3] introduced Fuzzy risk analysis based on ranking generalized fuzzy numbers with different left and right heights. Chen and Wang [2] presented the ranking fuzzy number using α cuts, belief feature and signal/noise ratio for risk analysis of a manufacturing system. Chen and Sanguansat [4] introduced a new fuzzy ranking of generalized fuzzy number for risk analysis. Patra and Mondal [15] presented a new ranking method of generalized trapezoidal fuzzy numbers and applied it to evaluate the risk in diabetes problems. Also there are some similarity measure methods to evaluate the risk. Wei and Chen [17] presented a new similarity measure of generalized trapezoidal fuzzy numbers to perform a fuzzy risk analysis using linguistic term values. In 2010 Xu et al. [19] also introduced a new similarity measure using COG point of two linguistic valued trapezoidal fuzzy numbers and a new arithmetic operator of linguistic values trapezoidal fuzzy numbers. Patra and Mondal [16] presented a new similarity measure method of generalized trapezoidal fuzzy numbers and applied it to evaluate the risk in a production system. Wu et al [18] presented risk analysis of corrosion failures of equipment in refining and petrochemical plants using fuzzy set theory. Markowski et al [13] used fuzzy logic to explosion risk assessment.

In spite of the above developments, there are some gaps in the development of fuzzy numbers and risk analysis of a production system using fuzzy cognitive map. In this regard, the present investigation has the following innovative features

• For the first time, double fuzzy number, $\tilde{\tilde{A}} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ where $\tilde{a}_i = (a_{i1}, a_{i2}, a_{i3}), i = 1,2,3$ is defined and used in a production system.

• Here fuzzy cognitive map has been used using the causal relationship among the components of a production system.

• The risk values are connected with each factor of the production system.

In any production system different factors are interrelated. If any of the factors is insufficient to fill up the requirement to run the production process then the system may fail to continue. Therefore there exist a risk in continuation of a production system corresponding to each and every factor connected with this production system. So the evaluation of risk is very important to continuation of a production system.

Again each and every factor associated with a production system have some numerical values. In real life these values are not crisp in nature i.e., they are fuzzy. Now risk is associated with each factor. Corresponding to a particular value of a factor there exist different risk values i.e., they are not crisp i.e., fuzzy in nature. As each factor is fuzzy valued so the concept of double fuzzy number is come to the risk value corresponding to each factor which is introduced in this paper.

In this paper a risk model has been established to analyze the risk of a production system using fuzzy cognitive map and a new type of fuzzy number is proposed by us which has been termed as double fuzzy number. Since a production system is dependent on various factors which are interrelated to each other, fuzzy cognitive map has been introduced to show the causal relationship between them. Using this causal relationship, a steady state of the system has been evaluated by introducing a threshold function for different factors. The risk values for the different factors have been calculated transforming the state value by a fuzzy transformation. As a result, the risk values are converted into the double fuzzy numbers. Considering different weights to the each factors, total risk is calculated.

The rest of the paper organized as follows. In section $\S 2$, some preliminaries ideas about triangular fuzzy number and its operations as well as FCM are discussed. In section $\S 3$, proposed double fuzzy number and its operations are described. In section $\S 4$, risk models have been constructed using FCM and double fuzzy number. In section $\S 5$, a numerical example has been given and illustrated the risk in both models and in section $\S 6$, a conclusion has been followed.

2. Preliminaries

2.1. Triangular fuzzy number and its operations

Triangular Fuzzy Number (TFN): A TFN $\widetilde{A}$ is specified by the triplet (a_1, a_2, a_3) and is defined by its continuous membership function $\mu_{\widetilde{A}}(x): F \to [0,1]$ as follows

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & if \;\; a_1 \le x \le a_2 \\ \dfrac{a_3-x}{a_3-a_2} & if \;\; a_2 \le x \le a_3 \\ 0 & otherwise \end{cases} \tag{1}$$

Some Operations in Fuzzy Numbers: Assume that there are two trapezoidal fuzzy numbers $\widetilde{A} = (a_1, b_1, c_1)$ and $\widetilde{B} = (a_2, b_2, c_2)$ where $a_1 \le b_1 \le c_1$ and $a_2 \le b_2 \le c_2$, then some arithmetic operations between two triangular fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ are defined as follows:

(1) The additional operator denoted by $\oplus$ of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \oplus \tilde{B} = (a_1, b_1, c_1) \oplus (a_2, b_2, c_2)$$
$$= (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

(2) The substraction operator denoted by ! of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} - \tilde{B} = (a_1, b_1, c_1) - (a_2, b_2, c_2)$$
$$= (a_1 - c_2, b_1 - b_2, c_1 - a_2)$$

(3) The multiplication operator denoted by $\otimes$ of $\tilde{A}$ and $\tilde{B}$ is defined by

$$\tilde{A} \otimes \tilde{B} = (a_1, b_1, c_1) \otimes (a_2, b_2, c_2)$$
$$= (a_1 a_2, b_1 b_2, c_1 c_2)$$

(4) The divisional operator denoted by % of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \% \tilde{B} = (a_1, b_1, c_1) \% (a_2, b_2, c_2)$$
$$= (a_1/c_2, b_1/b_2, c_1/a_2)$$

The risk value is always lies between $[01]$. For this reason some operations of triangular fuzzy numbers are denoted as follows. Assume that there are two triangular fuzzy numbers $\tilde{A} = (a_1, b_1, c_1)$ and $\tilde{B} = (a_2, b_2, c_2)$ where $0 \le a_1 \le b_1 \le c_1 \le 1$ and $0 \le a_2 \le b_2 \le c_2 \le 1$, then some arithmetic operations between two triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are defined as follows: [18]

(1) The additional operator denoted by $\oplus$ of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \oplus \tilde{B} = (a_1, b_1, c_1) \oplus (a_2, b_2, c_2)$$
$$= (a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2, c_1 + c_2 - c_1 c_2)$$

(2) The substraction operator denoted by ! of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} ! \tilde{B} = (a_1, b_1, c_1) \oplus (a_2, b_2, c_2)$$
$$= (max\{(a_1 - c_2), 0\}, max\{(b_1 - b_2, 0\}, max\{(c_1 - a_2, 0\})$$

(3) The multiplication operator denoted by $\otimes$ of $\tilde{A}$ and $\tilde{B}$ is defined by

$$\tilde{A} \otimes \tilde{B} = (a_1, b_1, c_1) \otimes (a_2, b_2, c_2)$$
$$= (a_1 a_2, b_1 b_2, c_1 c_2)$$

(4) The divisional operator denoted by % of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \% \tilde{B} = (a_1, b_1, c_1) \% (a_2, b_2, c_2)$$
$$= (a_1/c_2, b_1/b_2, c_1/a_2)$$
$$where\ a/b = \begin{cases} a/b, \ if\ a < b \\ 1, \quad otherwise \end{cases}$$

2.2. Fuzzy cognitive map

A Fuzzy cognitive map (FCM)[5,7,10,11,14] represents a graphical causal relationship of all attributes known as concepts in a system. It was originally introduced by Kosko [10] as an extension of cognitive map model. In this model there are some concepts known as nodes and some causal relations associated with each edge. Each edge (C_i, C_j) is associated with a number known as weight (w_{ij}) which determines the degree of causal relation between the concepts C_i and C_j. The weight for any interconnection is simply a crisp number that ranges over $[-1,1]$. Three possible states arise there such as (i) the weight w_{ij} equal to 0 indicates that there is no relationship between two concepts C_i and C_j, (ii) $w_{ij} > 0$ indicates that the increase (decrease) in C_i leads to increase (decrease) in C_j, (iii) $w_{ij} < 0$ implies that the increase (decrease) in C_i leads to decrease (increase) in C_j. A graphical representation of FCM has been given in Figure 1 where each node indicates the concept and each directed edge indicates it causal relationships.

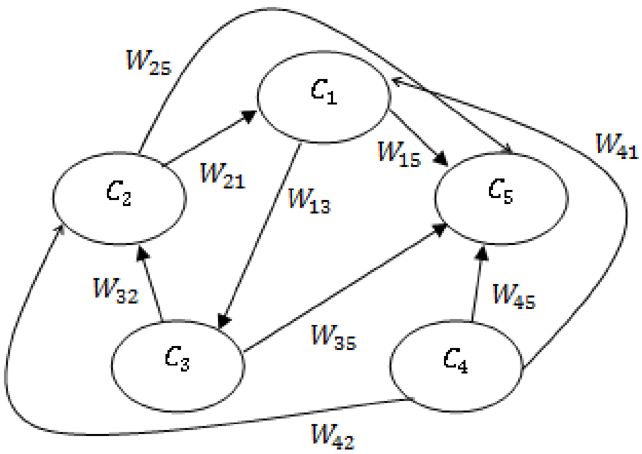

Figure 1: Simple Fuzzy cognitive map

The influence of a specific concept to other concepts can be calculated using the following equation

$$A_i^{(t+1)} = f(k_1 \sum_{j=1, j\neq i}^{n} A_j^{(t)} W_{ji} + k_2 A_i^{(t)})$$

The coefficient k_2 represents the proportion of the contribution of the previous value of the concept in the computation of a new value. k_1 represents the influence of the interconnected concepts in the configuration of the new value of the concept A_i. In this paper, it is considered that the influence of the previous value of each concept is very high and it is supposed that $k_1 = k_2 = 1$. This means that the previous value of each concept has a great influence in the determination of new value. Hence the value of A_i for each concept C_i can be calculated as

$$A_i^{(t+1)} = f(\sum_{j=1, j\neq i}^{n} A_j^{(t)} W_{ji} + A_i^{(t)}) \tag{2}$$

where A_i^{t+1} is the value of the concept C_i at time $(t+1)$ and A_i^t is the value of the concept C_i at time t and W_{ji} is the interconnection from the concept C_j to C_i. f is the threshold transformation function that is used to normalized concept values to a certain binary or bipolar ranges. These threshold function may be different such as

Hard limit function

$$f(x) = \begin{cases} 1 & for\ x \geq 1 \\ 0 & for\ x < 1 \end{cases} \qquad x \in [0,1]$$

Hyperbolic tangent function

$$f(x) = \tanh(x)$$
$$= (1-e^{-x})/(1+e^{-x}) \quad x \in [-1,1]$$

Logistic function

$$f(x) = 1/(1+e^{-x}) \quad x \in [0,1]$$

The threshold function operates until the steady state arises. The simplicity of FCM model becomes apparent from its mathematical representation and operation. Suppose FCM is consisted of n concepts. It is mathematically represented by a $1\times n$ state vector A which gathers the value of n concepts by an $n\times n$ weight matrix W. Then the fuzzy cognitive map operation with compact mathematical model described in equation 1 transform into the following form

$$A^{(t+1)} = f(A^{(t)}W + A^{(t)}) \tag{3}$$

Therefore equation (4) computes a new state vector $A^{(t+1)}$.

3. Double fuzzy number

Definition: Double fuzzy number: A fuzzy number is called double fuzzy number if any point of this fuzzy number is also fuzzy number and it is denoted as $\approx\!\!\!\!\!A$ ($\overset{\approx}{A}$).

For example $\overset{\approx}{A} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ is known as triangular double fuzzy number whose membership function is defined as

$$\mu_{\overset{\approx}{A}}(x) = \begin{cases} \dfrac{x-\tilde{a}_1}{\tilde{a}_2-\tilde{a}_1} & if\ \tilde{a}_1 \leq x \leq \tilde{a}_2 \\ \dfrac{\tilde{a}_3-x}{\tilde{a}_3-\tilde{a}_2} & if\ \tilde{a}_2 < x \leq \tilde{a}_3 \\ 0 & otherwise \end{cases} \tag{4}$$

where $\tilde{a}_1 = (a_{11}, a_{12}, a_{13}), \tilde{a}_2 = (a_{21}, a_{22}, a_{23}), \tilde{a}_3 = (a_{31}, a_{32}, a_{33})$ are also triangular fuzzy number and the membership values of a_i's are defined as

$$\mu_{\tilde{a}_i}(x) = \begin{cases} \dfrac{x - a_{i1}}{a_{i2} - a_{i1}} & if \;\; a_{i1} \le x \le a_{i2} \\ \dfrac{a_{i3} - x}{a_{i3} - a_{i2}} & if \;\; a_{i2} < x \le a_{i3} \\ 0 & otherwise \end{cases} \tag{5}$$

$i = 1,2,3.$

The triangular fuzzy number can be also written as $\approx{A} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ The graphical representation of triangular double fuzzy number is given in the Figure 2

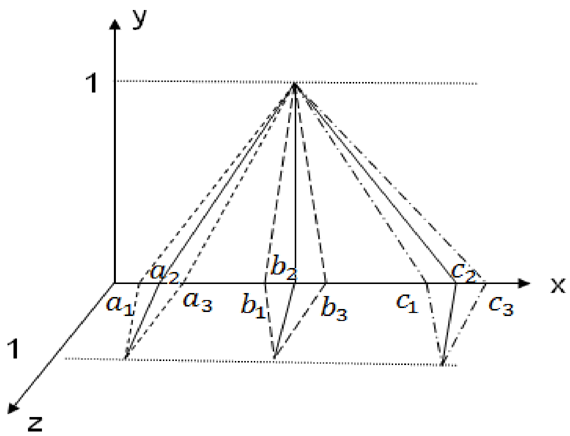

Figure 2: membership graph of triangular double fuzzy number

Property 1. Let $\stackrel{\approx}{A}$ and $\stackrel{\approx}{B}$ are two triangular double fuzzy numbers where

$\stackrel{\approx}{A} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ and

$\stackrel{\approx}{B} = ((b_{11}, b_{12}, b_{13}), (b_{21}, b_{22}, b_{23}), (b_{31}, b_{32}, b_{33}))$

with the conditions $0 \le a_{i1} \le a_{i2} \le a_{i3} \le 1$, $0 \le b_{i1} \le b_{i2} \le b_{i3} \le 1$, $i = 1,2,3$ and $a_{ij} \le a_{i+1\,j}$, $b_{ij} \le b_{i+1\,j}$, $i = 1,2$ & $j = 1,2,3$ then the sum of these two triangular double fuzzy numbers is given

$\stackrel{\approx}{A} \oplus \stackrel{\approx}{B} = ((a_{11} + b_{11} - a_{11}b_{11}, a_{12} + b_{12} - a_{12}b_{12}, a_{13} + b_{13} - a_{13}b_{13}), (a_{21} + b_{21} - a_{21}b_{21}, a_{22} + b_{22}$

$-a_{22}b_{22}, a_{23}+b_{23}-a_{23}b_{23}), (a_{31}+b_{31}-a_{31}b_{31}, a_{32}+b_{32}-a_{32}b_{32}, a_{33}+b_{33}-a_{33}b_{33}))$

Proof: Let $\tilde{\tilde{A}} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ and $\tilde{\tilde{B}} = ((b_{11}, b_{12}, b_{13}), (b_{21}, b_{22}, b_{23}), (b_{31}, b_{32}, b_{33}))$ are two triangular double fuzzy numbers where $0 \le a_{i1} \le a_{i2} \le a_{i3} \le 1$ and $0 \le b_{i1} \le b_{i2} \le b_{i3} \le 1$, $i = 1,2,3$.

Consider that $\tilde{a}_i = (a_{i1}, a_{i2}, a_{i3})$ and $\tilde{b}_i = (b_{i1}, b_{i2}, b_{i3})$, $i = 1,2,3$. Now, $\tilde{a}_i$ and $\tilde{b}_i$ are two triangular fuzzy numbers, so

$$\tilde{a}_i + \tilde{b}_i = (a_{i1}+b_{i1}-a_{i1}b_{i1}, a_{i2}+b_{i2}-a_{i2}b_{i2}, a_{i3}+b_{i3}-a_{i3}b_{i3}), i = 1,2,3$$

Since $a_{ij}, b_{ij} \ge 0$ and $a_{ij} \le 1$, so $1-a_{ij} \ge 0$ for $i, j = 1,2,3$. Hence

$$a_{ij}+b_{ij}-a_{ij}b_{ij} = a_{ij}+b_{ij}(1-a_{ij}) \ge 0$$

Again $a_{ij}, b_{ij} \le 1$ then

$$a_{ij}+b_{ij}-a_{ij}b_{ij} \le (1+1-1.1) = 1$$

i.e., $0 \le a_{ij}+b_{ij}-a_{ij}b_{ij} \le 1$.

Therefore the sum of $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is

$$\tilde{\tilde{A}} \oplus \tilde{\tilde{B}} = (\tilde{a}_1+\tilde{b}_1, \tilde{a}_2+\tilde{b}_2, \tilde{a}_3+\tilde{b}_3)$$

$$= ((a_{11}+b_{11}-a_{11}b_{11}, a_{12}+b_{12}-a_{12}b_{12}, a_{13}+b_{13}-a_{13}b_{13}), (a_{21}+b_{21}-a_{21}b_{21}, a_{22}+b_{22}$$
$$-a_{22}b_{22}, a_{23}+b_{23}-a_{23}b_{23}), (a_{31}+b_{31}-a_{31}b_{31}, a_{32}+b_{32}-a_{32}b_{32}, a_{33}+b_{33}-a_{33}b_{33}))$$

Property 2. Let $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are two triangular double fuzzy numbers where

$\tilde{\tilde{A}} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ and

$\tilde{\tilde{B}} = ((b_{11}, b_{12}, b_{13}), (b_{21}, b_{22}, b_{23}), (b_{31}, b_{32}, b_{33}))$

with the conditions $0 \le a_{i1} \le a_{i2} \le a_{i3} \le 1$, $0 \le b_{i1} \le b_{i2} \le b_{i3} \le 1$, $i = 1,2,3$ and $a_{ij} \le a_{i+1j}$, $b_{ij} \le b_{i+1j}$, $i = 1,2$ & $j = 1,2,3$ then the substraction of these two triangular double fuzzy numbers is given by

$$\tilde{\tilde{A}} - \tilde{\tilde{B}} = ((max(a_{11}-b_{33},0), max(a_{12}-b_{32},0), max(a_{13}-b_{31},0)), (max(a_{21}-b_{23},0),$$
$$max(a_{22}-b_{22},0), max(a_{23}-b_{21},0)), (max(a_{31}-b_{13},0), max(a_{32}-b_{12},0), max(a_{33}-b_{11},0)))$$

Proof: Let $\tilde{\tilde{A}} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ and $\tilde{\tilde{B}} = ((b_{11}, b_{12}, b_{13}), (b_{21}, b_{22}, b_{23}), (b_{31}, b_{32}, b_{33}))$ are two triangular double fuzzy numbers where $0 \le a_{i1} \le a_{i2} \le a_{i3} \le 1$ and $0 \le b_{i1} \le b_{i2} \le b_{i3} \le 1$, $i = 1,2,3$.

Consider that $\tilde{a}_i = (a_{i1}, a_{i2}, a_{i3})$ and $\tilde{b}_j = (b_{j1}, b_{j2}, b_{j3})$, $i, j = 1,2,3$. Now, $\tilde{a}_i$ and $\tilde{b}_j$ are the triangular fuzzy numbers, so

$\tilde{a}_i - \tilde{b}_j = (max(a_{i1} - b_{j3}, 0), max(a_{i2} - b_{j2}, 0), max(a_{i3} - b_{j1}, 0)), i, j = 1,2,3$

Now $0 \le a_{ij} \le 1$ and $0 \le b_{kl} \le 1$ for $i, j, k, l = 1,2,3$. So

$max(a_{ij} - b_{kl}, 0) = 0$ when $a_{ij} \le b_{kl}$

$max(a_{ij} - b_{kl}, 0) = a_{ij} - b_{kl} > 0$ when $a_{ij} > b_{kl}$

The value of $a_{ij} - b_{kl}$ is maximum when the value of a_{ij} is maximum i.e., 1 and the value of b_{kl} is minimum i.e., 0. Hence the maximum value of $a_{ij} - b_{kl}$ is 1 i.e., $a_{ij} - b_{kl} \le 1$. Therefore, for all cases $0 \le max(a_{ij} - b_{kl}, 0) \le 1$

So the difference between $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is given by

$$\tilde{\tilde{A}} - \tilde{\tilde{B}} = (\tilde{a}_1 - \tilde{b}_3, \tilde{a}_2 - \tilde{b}_2, \tilde{a}_3 - \tilde{b}_1)$$

$$= ((max(a_{11} - b_{33}, 0), max(a_{12} - b_{32}, 0), max(a_{13} - b_{31}, 0)), (max(a_{21} - b_{23}, 0),$$
$$max(a_{22} - b_{22}, 0), max(a_{23} - b_{21}, 0)), (max(a_{31} - b_{13}, 0), max(a_{32} - b_{12}, 0), max(a_{33} - b_{11}, 0)))$$

Property 3. Let $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are two triangular double fuzzy numbers where

$\tilde{\tilde{A}} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ and

$\tilde{\tilde{B}} = ((b_{11}, b_{12}, b_{13}), (b_{21}, b_{22}, b_{23}), (b_{31}, b_{32}, b_{33}))$

with the conditions $0 \le a_{i1} \le a_{i2} \le a_{i3} \le 1$, $0 \le b_{i1} \le b_{i2} \le b_{i3} \le 1$, $i = 1,2,3$ and $a_{ij} \le a_{i+1j}$, $b_{ij} \le b_{i+1j}$, $i = 1,2$ & $j = 1,2,3$ then the multiplication of these two triangular double fuzzy numbers is given by

$$\tilde{\tilde{A}} \otimes \tilde{\tilde{B}} = ((a_{11} * b_{11}, a_{12} * b_{12}, a_{13} * b_{13}), (a_{21} * b_{21}, a_{22} * b_{22}, a_{23} * b_{23}), (a_{31} * b_{31}, a_{32} * b_{32}, a_{33} * b_{33}))$$

Proof: Let $\tilde{\tilde{A}} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ and $\tilde{\tilde{B}} = ((b_{11}, b_{12}, b_{13}), (b_{21}, b_{22}, b_{23}), (b_{31}, b_{32}, b_{33}))$ are two triangular double fuzzy numbers where $0 \le a_{i1} \le a_{i2} \le a_{i3} \le 1$ and $0 \le b_{i1} \le b_{i2} \le b_{i3} \le 1$, $i = 1,2,3$.

Consider that $\tilde{a}_i = (a_{i1}, a_{i2}, a_{i3})$ and $\tilde{b}_i = (b_{i1}, b_{i2}, b_{i3})$, $i = 1,2,3$. Now, $\tilde{a}_i$ and $\tilde{b}_i$ are the triangular fuzzy numbers, so

$$\tilde{a}_i * \tilde{b}_i = (a_{i1} * b_{i1}, a_{i2} * b_{i2}, a_{i3} * b_{i3}), i = 1,2,3$$

Now $0 \le a_{ij} \le 1$ and $0 \le b_{ij} \le 1$ for $i, j = 1,2,3$. So $0 \le a_{ij} * b_{ij} \le 1$

Hence the multiplication of $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is

$$\tilde{\tilde{A}} \otimes \tilde{\tilde{B}} = (\tilde{a}_1 * \tilde{b}_1, \tilde{a}_2 * \tilde{b}_2, \tilde{a}_3 * \tilde{b}_3)$$

$$= ((a_{11} * b_{11}, a_{12} * b_{12}, a_{13} * b_{13}), (a_{21} * b_{21}, a_{22} * b_{22}, a_{23} * b_{23}), (a_{31} * b_{31}, a_{32} * b_{32}, a_{33} * b_{33}))$$

Property 4. Let $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are two triangular double fuzzy numbers where

$\tilde{\tilde{A}} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ and
$\tilde{\tilde{B}} = ((b_{11}, b_{12}, b_{13}), (b_{21}, b_{22}, b_{23}), (b_{31}, b_{32}, b_{33}))$ with the conditions
$0 \le a_{i1} \le a_{i2} \le a_{i3} \le 1$, $0 \le b_{i1} \le b_{i2} \le b_{i3} \le 1$, $i = 1,2,3$ and $a_{ij} \le a_{i+1j}$, $b_{ij} \le b_{i+1j}$, $i = 1,2$ & $j = 1,2,3$ then the division of these two triangular double fuzzy numbers is given by

$$\tilde{\tilde{A}} \% \tilde{\tilde{B}} = ((a_{11}/b_{33}, a_{12}/b_{32}, a_{13}/b_{31}), (a_{21}/b_{23}, a_{22}/b_{22}, a_{23}/b_{21}), (a_{31}/b_{13}, a_{32}/b_{12}, a_{33}/b_{11}))$$

$$where \quad a/b = \begin{cases} a/b, if\ a < b \\ 1,\ otherwise \end{cases}$$

Proof: Let
$\tilde{\tilde{A}} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$
and $\tilde{\tilde{B}} = ((b_{11}, b_{12}, b_{13}), (b_{21}, b_{22}, b_{23}), (b_{31}, b_{32}, b_{33}))$ are two triangular double fuzzy numbers where $0 \le a_{i1} \le a_{i2} \le a_{i3} \le 1$ and $0 \le b_{i1} \le b_{i2} \le b_{i3} \le 1$, $i = 1,2,3$.
Consider that $\tilde{a}_i = (a_{i1}, a_{i2}, a_{i3})$ and $\tilde{b}_j = (b_{j1}, b_{j2}, b_{j3})$, $i, j = 1,2,3$. Now, $\tilde{a}_i$ and $\tilde{b}_j$ are the triangular fuzzy numbers, so

$$\tilde{a}_i / \tilde{b}_j = (a_{i1}/b_{j3}, a_{i2}/b_{j2}, a_{i3}/b_{j1}), i, j = 1,2,3$$

Now $0 \le a_{ij} \le 1$ and $0 \le b_{kl} \le 1$ for $i, j, k, l = 1,2,3$. Consider that $a_{ij}/b_{kl} = 1$ when $a_{ij} \ge b_{kl}$. So $0 \le a_{ij}/b_{kl} \le 1$ for all $i, j, k, l = 1,2,3,4$, since $a_{ij}, b_{kl} \ge 0$. Hence the division of $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is

$$\tilde{\tilde{A}} \% \tilde{\tilde{B}} = (\tilde{a}_1/\tilde{b}_3, \tilde{a}_2/\tilde{b}_2, \tilde{a}_3/\tilde{b}_1)$$

$$= ((a_{11}/b_{33}, a_{12}/b_{32}, a_{13}/b_{31}), (a_{21}/b_{23}, a_{22}/b_{22}, a_{23}/b_{21}), (a_{31}/b_{13}, a_{32}/b_{12}, a_{33}/b_{11}))$$

Property 5. Let $\tilde{\tilde{A}} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ be a triangular double fuzzy number then the defuzzyfied value d of this triangular double fuzzy number is given by $\frac{1}{9}(a_{11} + a_{12} + a_{13} + a_{21} + a_{22} + a_{23} + a_{31} + a_{32} + a_{33})$

Proof: Let $\tilde{\tilde{A}} = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ be a triangular double fuzzy number and $\tilde{a}_i = (a_{i1}, a_{i2}, a_{i3}), i = 1,2,3$. Since $\tilde{a}_i$'s are the triangular fuzzy number then the defuzzyfied value of $\tilde{a}_i$'s are $(a_{i1} + a_{i2} + a_{i3})/3, i = 1,2,3$. Hence the defuzzified value (d) of $\tilde{\tilde{A}}$ is calculated as

$$d = \frac{\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3}{3}$$

$$= \frac{(a_{11}+a_{12}+a_{13})/3+(a_{21}+a_{22}+a_{23})/3+(a_{31}+a_{32}+a_{33})/3}{3}$$

$$i.e., d = \frac{1}{9}(a_{11}+a_{12}+a_{13}+a_{21}+a_{22}+a_{23}+a_{31}+a_{32}+a_{33})$$

4. Risk model formulation with the linguistic weights using fuzzy cognitive map in a production system

The study of risk in a production system is very important. In this paper a production system is considered which is connected with different factors such as (i) availability of produced item (C_1), (ii) demand of the produced item (C_2), (iii) availability of raw materials (C_3), (iv) quality of raw materials (C_4), (v) transportation cost associated with the system (C_5), (vi) expertiseness of labours (C_6), (vii) selling price of the item (C_7) and $(viii)$ quality of the produced item (C_8) of the system. All these factors are dependent on each other. Any change in one factor has some effect on other factors i.e., if one factor increases then other factors may increase or decrease or may simultaneously be without any effect. For example

1. If C_3 increase, the cause could be a fairly high level of increase of C_1 and a low level of increase of C_5.
2. If C_1 increase, the cause could be a low level of decrease of C_7.
3. If C_2 increase, the cause could be a low level of increase of C_7.
4. If C_4 increase, the cause could be a high level of increase of C_8.
5. If C_5 increase, the cause could be a very low level of increase of C_7.
6. If C_6 increase, the cause could be a fairly low level of increase of the C_1 and a very high level of increase of C_5.
7. If C_7 increase, the cause could be a very high level of decrease of C_2.
8. If C_8 increase, the cause could be a medium level of increase of the C_2.

Since causal relationship among the factors have a loop structure, hence it may be difficult to find the value of any other factor directly when change in one or more of the factors are performed. In such situation after continuous changes among the factors due to its interrelationship, the system comes to a stable state after a certain time. To determine the stable state of the system, FCM has been used. So in this paper, these eight factors in the production system have been treated as different concepts (C_i, $i = 1,2,...,8$). Generally the domain of any concept should be $[0,1]$ in the parlance of FCM, but in the proposed problem the domain of any concept may be $[a,b] \subset R$ where $a,b \geq 0$. So it is necessary to normalize the real state values in $[0,1]$. For this purpose, a transformation function $g(x)$ has been defined as follows

$$g(x) = \frac{x-a}{b-a} \tag{6}$$

where the real value x of any concept lies in the interval $[a\,b]$.
Now, as the risk in the system is dependent on the each factor of the system, then eight risk values corresponding to different concepts are considered here and that are denoted as r_i for the concept C_i, $i = 1,2,...,8$ respectively. The graphical representation of the proposed system is described in Figure 3

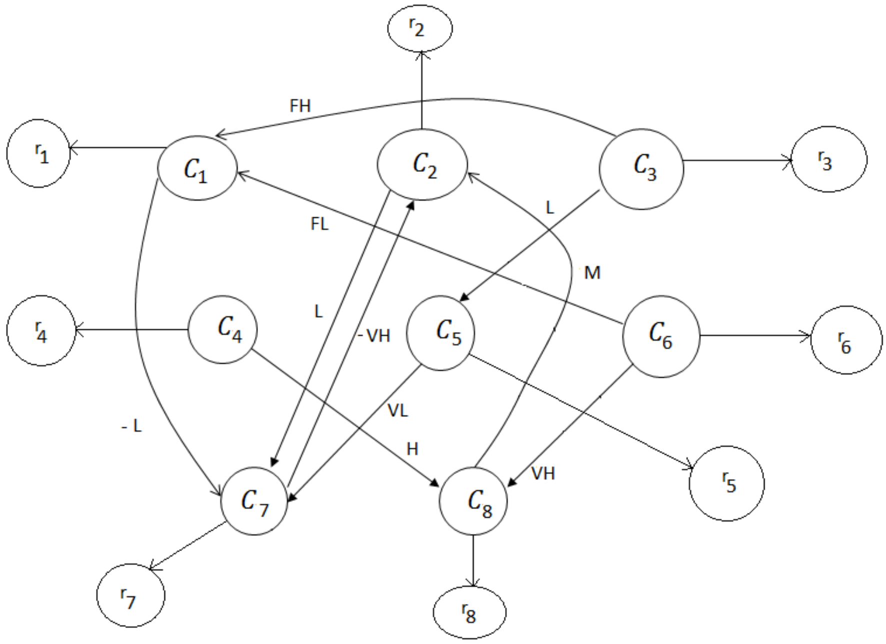

Figure 3: Fuzzy cognitive map of the system with linguistic weights

considering these risk values of all factors, the total risk of the system is calculated as follows.

Step 1: The initial state of the concepts have been considered as

$$\tilde{A}^0 = [\tilde{x}_1\ \tilde{x}_2\ \ldots .\ \tilde{x}_8] \tag{7}$$

where $\tilde{x}_i$'s are triangular fuzzy numbers as $\tilde{x}_i = (x_{i1}, x_{i2}, x_{i3}), i = 1,2,...,8$.
where $0 \le x_{i1} \le x_{i2} \le x_{i3} \le 1$. The value of x_{ij} are obtained after thresholding the actual real value of the concepts using the interface g which transforms the natural measures of the system to their representative concept values in the FCM and vice versa.

Step 2: The causal relationship between any two concepts are taken in the form of linguistic. Here these linguistic values have been considered as triangular fuzzy numbers.

The weights given to each causal relation between any two given concepts in linguistic form can be determined in the following matrix form

$$\tilde{W} = \begin{pmatrix} \tilde{w}_{11} & \tilde{w}_{12} & . & . & \tilde{w}_{18} \\ \tilde{w}_{21} & \tilde{w}_{22} & . & . & \tilde{w}_{28} \\ . & . & . & . & . \\ . & . & . & . & . \\ \tilde{w}_{81} & \tilde{w}_{82} & . & . & \tilde{w}_{88} \end{pmatrix} \tag{8}$$

where $\tilde{w}_{ij}$'s are linguistic in nature determined by the decision maker and denoted by triangular fuzzy number $\tilde{w}_{ij} = (a_{ij}, b_{ij}, c_{ij})$, $a_{ij} \le b_{ij} \le c_{ij}$, when the value of the concept C_j increases with the increase of the concept C_i. Again when the value of the concept C_j increases with the decrease of the concept C_i then $\tilde{w}_{ij} = (-c_{ij}, -b_{ij}, -a_{ij})$.

Step 3: Now any change in concept value has some effects on other concept values. So the state vector $\tilde{A}^{(t+1)}$ at any time $t+1$ can be determined from the previous state vector $\tilde{A}^{(t)}$ at time t as follows

$$\tilde{A}^{(t+1)} = f(\tilde{A}^{(t)}\tilde{W} + \tilde{A}^{(t)}) \tag{9}$$

In this simulation procedure a steady state arise after a certain time. To get the steady state of the system, the following threshold function has been used

$$f(x_{i1}, x_{i2}, x_{i3}) = \left(\frac{1}{1+e^{-x_{i1}}}, \frac{1}{1+e^{-x_{i2}}}, \frac{1}{1+e^{-x_{i3}}}\right) \tag{10}$$

Step 4: At the steady state, concept values are converted to their real measures and that will be a triangular fuzzy number. Now risk is connected to each concept, so the risk interval for each concept C_i is determined as $[(k_{i11}, k_{i12}, k_{i13}), (k_{i21}, k_{i22}, k_{i23})]$, $i = 1,2,...,8$.

Step 5: The risk value corresponding to a fixed concept value may be different i.e., fuzzy. So to convert the concept values to the risk values, two fuzzy transformations $\tilde{\phi} : [\tilde{a}\,\tilde{b}] \to [01]$ and $\tilde{\psi} : [\tilde{a}\,\tilde{b}] \to [01]$ have been defined as follows

$$\tilde{\phi}(x) = \begin{cases} 1 & if\; x < a_1 \\ \dfrac{(\tilde{b} - x)}{\tilde{b} - \tilde{a}} & if\; \tilde{a} \le x \le \tilde{b} \\ 0 & if\; x > b_3 \end{cases} \tag{11}$$

when risk value increase with the decrease of the concept value and

$$\tilde{\psi}(x)=\begin{cases}0 \quad if\; x<a_1\\ \dfrac{(x-\tilde{a})}{\tilde{b}-\tilde{a}} \quad if\; \tilde{a}\le x\le \tilde{b}\\ 1 \quad if\; x>b_3\end{cases} \tag{12}$$

when risk value increase with the increase of the concept value.

Step 6: The risk values $\tilde{\tilde{r}}_i$ obtained in the step 5 are obviously double fuzzy in nature. Let $\tilde{W}_i=(w_{i1},w_{i2},w_{i3})$ is the weight of a concept C_i, then the normalized weight of the concept C_i can be obtained as

$$\tilde{W}_i'=\left(\frac{w_{i1}}{\sum_{i=1}^{8}w_{i3}},\frac{w_{i2}}{\sum_{i=1}^{8}w_{i2}},\frac{w_{i3}}{\sum_{i=1}^{8}w_{i1}}\right) \tag{13}$$

And the total risk $\tilde{\tilde{R}}$ of the system can be determined as

$$\tilde{\tilde{R}}=\sum_{i=1}^{8}\tilde{W}_i'*\tilde{\tilde{r}}_i \tag{14}$$

Step 7: The total risk of the system can be measured by the defuzzyfication of the double fuzzy number $\tilde{\tilde{R}}$ as described in property 5.

5. Numerical illustration

In a production system, it is considered that the system can produce items in the range of $[0 \quad 1000]$ in any instant of time. There is also have a demand of the item in the range of $[0 \quad 900]$ at that time instant. There exists the availability of raw materials in the range of $[500 \quad 1000]$ unit. Now the quality of available raw materials belongs in the range of $[0 \quad 1]$ and the transportation cost in the system is in the range of $[30 \quad 40]$ per unit item. Expertiseness of the labours is in the range of $[0 \quad 1]$. Selling price of an item is in the range of Rs$[500 \quad 600]$. Quality of the produced item is in the range of $[0 \quad 1]$.

Each factor in the system has some risk in profit. Now the risk interval of different factors are given below

Availability of produced item: $[(500,525,550),(800,825,850)]$

Demand of the item: $[(450,475,500),(750,775,800)]$

Availability of raw materials: $[(600,625,650),(900,925,950)]$

Quality of available raw materials: $[(0.5,0.55,0.6),(0.85,0.9,0.95)]$

Total transportation cost: $[(30,31,32),(38,39,40)]$

Expertiseness of the labours: $[(0.6,0.65,0.7),(0.85,0.9,0.95)]$

Quality of the produced item: $[(0.5,0.55,0.6),(0.85,0.9,0.95)]$

Selling price of an item: $[(500,510,520),(580,590,600)]$

Let a production system will start with such causal relationship described in Figure 3. Our aim is to find out the steady state of the system under such type of relationships among the concepts. For this purpose, it is considered that the initial state of the system is

$$\tilde{A}^0 = [(0.85,0.9,0.95)\ (0.75,0.8,0.85)\ (0.55,0.6,0.65)\ (0.75,0.8,0.85)\ (0.15,0.2,0.25)\ (0.8,0.85,0.9)(0.55,0.6,0.65)\ (0.65,0.7,0.75)] \qquad (15)$$

The linguistic values of the weights are represented as triangular fuzzy numbers which are shown in Table-1.

linguistic variable	Triangular fuzzy number
extremely low (EL)	(0,0.1,0.2)
very low (VL)	(0.1,0.2,0.3)
low (L)	(0.2,0.3,0.4)
fairly-low (FL)	(0.3,0.4,0.5)
medium (M)	(0.4,0.5,0.6)
fairly-high (FH)	(0.5,0.6,0.7)
high (H)	(0.6,0.7,0.8)
very-high (VH)	(0.7,0.8,0.9)
extremely high (EH)	(0.8,0.9,1.0)

Table 1**:** Linguistic variables and its domains of triangular fuzzy number

For the described system, using the Table 1, the weight matrix $\tilde{W}$ is found as

$$\tilde{W}=\begin{pmatrix}
(0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (-0.4,-0.3,-0.2) & (0,0,0)\\
(0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0.2,0.3,0.4) & (0,0,0)\\
(0.5,0.6,0.7) & (0,0,0) & (0,0,0) & (0,0,0) & (0.2,0.3,0.4) & (0,0,0) & (0,0,0) & (0,0,0)\\
(0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0.6,0.7,0.8)\\
(0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0.1,0.2,0.3) & (0,0,0)\\
(0.3,0.4,0.5) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0.7,0.8,0.9)\\
(0,0,0) & (-0.9,-0.8,-0.7) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0)\\
(0,0,0) & (0.4,0.5,0.6) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0)
\end{pmatrix} \qquad (16)$$

Using the threshold function the equilibrium state is found as :

C_1	C_2	C_3	C_4
(0.789,0.814,0.836)	(0.592,0.625,0.662)	(0.659,0.659,0.659)	(0.659,0.659,0.659)
C_5	C_6	C_7	C_8
(0.696,0.713,0.730)	(0.659,0.659,0.659)	(0.621,0.684,0.743)	(0.846,0.865,0.881)

Table- 2 : Values of the concepts after simulations

From the above table it is observed that when the weights of the causal relationships of the concepts in the proposed model are triangular fuzzy numbers, then transforming the values of the concepts in Table 2 to real values of the system, following real values of the concepts are obtained: $\tilde{C}_1 = (789,814,836)$, $\tilde{C}_2 = (533,552,596)$, $\tilde{C}_3 = (829.5,829.5,829.5)$, $\tilde{C}_4 = (0.659,0.659,0.659)$, $\tilde{C}_5 = (36.96,37.13,37.3)$, $\tilde{C}_6 = (0.659,0.659,0.659)$, $\tilde{C}_7 = (562.1,568.4,574.3)$, $\tilde{C}_8 = (0.846,0.865,0.881)$. For the above concept values the risk values corresponding to different concept values are determined by the said transformation as

$$\tilde{r}_1 = ((0,0,0.0314),(0,0.0314,0.1133),(0.0314,0.1133,0.252))$$
$$\tilde{r}_2 = ((0.44,0.5967,0.816),(0.5657,0.7433,0.992),(0.62,0.8067,1))$$
$$\tilde{r}_3 = ((0.235,0.3473,0.482),(0.235,0.3473,0.482),(0.235,0.3473,0.482))$$
$$\tilde{r}_4 = ((0.4244,0.6886,1),(0.4244,0.6886,1),(0.4244,0.6886,1))$$
$$\tilde{r}_5 = ((0.496,0.745,1),(0.513,0.766,1),(0.53,0.7875,1))$$
$$\tilde{r}_6 = ((0.5457,0.964,1),(0.5457,0.964,1),(0.5457,0.964,1))$$
$$\tilde{r}_7 = ((0.057,0.1963,0.4283),(0.116,0.27,0.5267),(0.179,0.3488,0.6317))$$
$$\tilde{r}_8 = ((0,0.0543,0.1533),(0,0.1,0.34),(0.0089,0.1529,0.416))$$

The weights of the different concepts are taken as linguistic variables such as $\tilde{W}_1$ = High, $\tilde{W}_2$ = Very High, $\tilde{W}_3$ = High, $\tilde{W}_4$ = Medium, $\tilde{W}_5$ = Low, $\tilde{W}_6$ = Fairly High, $\tilde{W}_7$ = Medium, and $\tilde{W}_8$ = High. Considering the triangular fuzzy value of the linguistic variables, the normalized weights of the concepts are calculated as

$$\tilde{W}_1' = (0.1071,0.1458,0.2)$$
$$\tilde{W}_2' = (0.125,0.1667,0.225)$$
$$\tilde{W}_3' = (0.1071,0.1458,0.2)$$
$$\tilde{W}_4' = (0.0714,0.1042,0.15)$$

$$\tilde{W}'_5 = (0.0357, 0.0625, 0.1)$$
$$\tilde{W}'_6 = (0.0893, 0.125, 0.175)$$
$$\tilde{W}'_7 = (0.0714, 0.1042, 0.15)$$
$$\tilde{W}'_8 = (0.1071, 0.1458, 0.2)$$

Now the total risk in the double fuzzy form has been obtain as

$$\tilde{\tilde{R}} = ((0.1687, 0.3533, 0.5803), (0.1865, 0.3837, 0.6283), (0.2, 0.409, 0.6517))$$

Now defuzzyfing the above risk value, the total risk in the system is found as $R = 0.3957$.

Therefore the risk of the proposed model is 0.3957 when such type of relationship among the factors exists as described in the proposed production system.

6. Conclusion

The idea of double fuzzy number has been introduced in this paper with some operations on it. A production system has been considered to analyze the risk associated with the system. The risk of any production system depends on various factors involved in the system. In a production system, the factors such as availability of produced item, demand of the produced item, availability of raw materials, quality of raw materials, transportation cost associated with the system, expertiseness of labours, selling price of the item, quality of the produced item are interconnected with one another. So the fuzzy cognitive map has been considered in this paper. The causal relationship among the factors i.e., the concepts have been considered as linguistic. Here a model has been considered and solved by fuzzy cognitive maps. A numerical example has been presented to illustrate the analysis of the risk of the system using the proposed double fuzzy number.

REFERENCES

1. K.T.Atanassov and S.Stoeva, Intuitonistic fuzzy sets. In Albrycht and Wiesniewski, (1983) 23-26.
2. S.M.Chen and C.H.Wang, Fuzzy risk analysis based on ranking fuzzy numbers using α cuts, belief features and signal/noise ratios, Expert Systems with Applications, 36 (2009) 5576-5581.
3. S.M.Chen, A.Munif, G.S.Chen, H.C.Liu and B.C.Kuo, Fuzzy risk analysis based on ranking generalized fuzzy numbers with different left heights and right heights, Expert Systems with Applications, 39 (2012) 6320.
4. S.M.Chen and K.Sanguansat, Analyzing fuzzy risk based on a new fuzzy ranking method between generaized fuzzy numbers, Expert Systems with Applications,38 (2011) 2163- .
5. J.P.Craiger, D.F.Goodman, R.J.Weiss and A.Butler, Modeling Organization Behavior with Fuzzy Cognitive Maps, International Journal of Intelligence and Organisations, 1 (1996) 120-123.
6. D.Dubois and H.Prade, Fuzzy Sets and Systems, Theory and Applications, Academic Press. Inc., New York, (1980).
7. B.M.Elomda, H.A.Hefny and H.A.Hassan, An extension of fuzzy decision maps for

multi-criteria decision-making. Egyptian Informatics Journal.. 14 (2013) 147-155.
8. J.A.Goguen, L-Fuzzy sets, JMAA, 18 (1967) 145-174.
9. J.A.Goguen, The logic of inexact concepts. Synthese, 19 (1969) 325-373.
10. B.Kosko, Fuzzy cognitive maps. International Journal of Man-Machine Studies, 24(1) (1986) 65-75.
11. G.Kyriakarakos, A.I.Dounis, K.G.Arvanitis and G.Papadakis, A fuzzy cognitive maps petri nets energy management system for autonomous polygeneration microgrids, Applied Soft Computing, 12(12) (2012) 3587-3795.
12. Z.Luo, J.Zhou, L.P.Zeng, L.Mo and Y.He, A TFN-ANP based approach to evaluate virtual research center comprehensive performence. Expert systems with application, 37(12) (2010) 8379-8386.
13. A.S.Markowski, M.S.Mannan, A.Kotynia and H.Pawlak, Application of fuzzy logic to explosion risk assessment, J Loss Prevent Proc, 24(6) (2011) 780-790.
14. E.I.Papageorgiou, A.T.Markinos and T.A.Gemtos, Fuzzy cognitive map based approch for predicting yield in cotton crop production as a basis for decision support system in precision agriculture application, Applied Soft Computing, 11 (2011).3643-3657.
15. K.Patra and S.K.Mondal, Risk analysis in diabetes prediction based on a new approach of ranking of generalized trapezoidal fuzzy numbers, Cybernetics and Systems: An International Journal, 43(8) (2012) 623-650.
16. K.Patra and S.K.Mondal, Fuzzy risk analysis using area and height based similarity measure on generalized trapezoidal fuzzy numbers and its application, Applied Soft Computing, 28 (2015) 276-284.
17. S.H.Wei and S.M.Chen, A new approch for fuzzy risk analysis based on similarity measures of generalized fuzzy numbers, Expert system with applications, 36(1) (2009) 589-598.
18. W.Wu, G.Cheng, H.Hu and Q.Zhou, Risk analysis of corrosion failures of equipment in refining and petrochemical plants based on fuzzy set theory, Engineering Failure Analysis, 32 (2013) 23-34.
19. Z.Xu, S.Shang, W.Quin and W.Shu, A method for fuzzy risk analysis based on the new similarity of trapezoidal fuzzy numbers, Expert system with applications, 37 (2010) 1920-1927.
20. L.A.Zadeh, Fuzzy Sets, Information and Control, 8 (1965) 338-356.
21. L.A.Zadeh, The concept of linguistic variable and its application to approximate reasoning, I,II,III, Information Sciences, 8 (1975) 199-249, 9 (1976) 43-58.