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Reliability Analysis of Competing Risks with Masked Failure Causes Based on Progressive Type-II Censoring with Random Removals

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ABSTRACT
This paper considers the reliability analysis of competing risks model based on progressive Type-II censored data with random removals, where the failure causes cannot be fully observed. Assume that the occurrence time of each failure mode follows Pareto distribution, and the number of systems removed at each failure time follows a binomial distribution. Based on the lifetime data containing masked failure causes, the maximum likelihood estimations of the unknown parameters and reliability function are obtained. In addition, the asymptotic confidence intervals of the unknown parameters are also proposed based on normal approximation to the asymptotic distribution of MLEs. In view of the shortcomings for failure cause is completely masked, the maximum likelihood estimation method fails, the Bayesian estimations of parameters and credible interval of the unknown parameters are obtained under the P,Q-symmetric entropy loss function. At last, some analyses of numerical results under different masking levels and removing probabilities are performed by Monte-Carlo simulations for illustrative purposes. The results show that the accuracy of the estimations decreases with increasing the masking level and has nothing to do with removing probability.

Keywords: Pareto distribution; random removals; competing risks with masked failure causes; reliability analysis; maximum likelihood estimation; asymptotic confidence intervals; P,Q-symmetric entropy loss; Bayesian estimation;

1. Introduction

In reliability analysis and lifetime tests, a product is failure may be due to several failure modes, but only the first time and the associated failure mode can be observed. For example, the failure of a bearing assembly may be attributable to bearing failures, shaft failures and so on, but only the first failure time and failure cause can be recorded. That is to say, several failure factors compete for the final failure of the product. It is known as the competing risks model. Recently, a mass of meaningful researches have been achieved by many scholars. Mao et al. [1] discussed the exact inference of competing risks model based on generalized Type-I hybrid censored exponential data. Based on Cox's latent failure time model assumptions, Bhattacharya et al. [2] analyzed the hybrid censored competing risks data. Wu et al. [3] studied the inference for accelerated competing failure models based on Type-I progressive hybrid censored Weibull data. Ahn et al. [4] discussed the problem of group and within-group variable selection for competing risks data. More details can refer to Ahmadi et al. [5], Zhang et al. [6], Delord and Génin [7], and so on.

However, in many situations, the cause of the product failure may not be unavailable to observed because the documentation needed for cause type identification is lost, or the cause type is difficult to determine, or the cause type detection is expensive to do for each subject, etc. This type data is known as masked data. It is also meaningful to study the reliability of the product with masked data. Xu and Tang [8] analyzed the nonparametric Bayesian estimation of competing risks problem with masked data. Hyun et al. [9] studied the proportional hazards model for competing risks data with missing cause of failure. Zheng et al. [10] discussed the problem of competing risks model under accelerated failure time with missing cause of failure. Li and Yu [11] obtained the consistent non-parametric maximum likelihood estimation of the joint distribution function with competing risks data under the dependent masking and right-censoring model. Wang and Yu [12], Wang et al. [13] also did many important work on masked data.

The Pareto distribution is used to model the unequal distribution of personal income and wealth. It has a long heavy tail and has a wide application in economics, business, insurance, reliability, engineering, finance and related areas. Many scholars have discussed the applications of Pareto type distributions in reliability. Abdel-Ghaly et al. [14] studied the estimation of the parameters of Pareto distribution and the reliability function in ALT with censoring. Sarhan and El-Gohary [15] developed the maximum likelihood and Bayes estimators for the parameters in Pareto reliability model with masked data. A bivariate Pareto model was introduced by Sankaran and Kundu [16]. The latest papers can refer to Fernández [17], Bourguignon et al. [18], and so on.

Considering the above mentioned literatures, in this paper, we discuss the reliability of competing risks with masked failure causes based on progressive Type-II censored Pareto data by using maximum likelihood method and Bayesian method. The rest of this paper is organized as follows. In section 2, the model description and assumptions are introduced. In section 3, we derive the maximum likelihood estimators (MLEs) and confidence intervals of unknown parameters and reliability. In section 4, the Bayesian estimators (BEs) and highest posterior density (HPD) credible intervals of unknown parameters and reliability are obtained. In section 5, a simulation study is performed for illustrate purpose. Some conclusions are present in section 6.

2. Model description and assumptions

2.1. Model description

Suppose n identical systems are put to the test at time $t_0 = 0$, and m failures are going to be observed. At the first observed time point t_1, r_1 of the surviving systems are randomly removed from the $n-1$ working systems. Then, at the second observed time point t_2, r_2 of the surviving systems are randomly removed from the $n-2-r_1$ working systems, and so on. The test terminates at the time when the mth failure is observed at time t_m and the remaining $r_m = n - m - \sum_{i=1}^{m-1} r_i$ survivals are all removed. Then we get the failure data $(t_i, c_i), i = 1, 2, \ldots, m$, where $t_1 \le t_2 \le \cdots \le t_m$ and c_i takes any element in the set of $\{0,1,2,\ldots,k\}$, $c_i = j, j = 1,2,\ldots,k$ indicates the failure is caused by failure mode *j*. Here, $c_i = 0$ denotes that the failure mode of the system cannot be observed.

2.2. Basic assumptions

A1. The failure of a system occurs only due to one of the *k* competing risks causes, the lifetimes of which denoted by $X_1, X_2, \ldots, X_k$ which are independent, and the failure time *T* of the system is the minimum of $X_1, X_2, \ldots, X_k$.

A2. The lifetime of the *j*th competing risks causes denoted by $X_j, j = 1, 2, \ldots, k$, which follows a Pareto distribution $Pa(\tau, \theta_j)$ with scale parameter τ and shape parameter θ_j, whose cumulative distribution function (CDF) and probability density function (PDF) are shown as

$$\begin{aligned} &F_j\left(x;\tau,\theta_j\right) = 1-\left(\tau/x\right)^{\theta_j} \qquad ,x>\tau,\tau>0,\theta_j>0 \\ &f_j\left(x;\tau,\theta_j\right) = \left(\theta_j/\tau\right)\left(\tau/x\right)^{\theta_j+1} \quad ,x>\tau,\tau>0,\theta_j>0 \end{aligned} \tag{1}$$

A3. The random removal numbers $r_i, i = 1, 2, \ldots, m-1$ follows a binominal distribution with parameter *p*, namely, $\left(r_i \mid r_{i-1}, r_{i-2}, \cdots, r_1\right) \sim B(n-m-\sum_{j=0}^{i-1} r_j, p)$. Here, $r_0 = 0$.

A4. The failure time *T* of the system is independent with the random removal numbers.

A5. The failure causes are independent with masking level.

Based on A1-A2, the reliability of system is given by

$$R(t) = P(\min(X_1, X_2, \ldots, X_k) > t) = \prod_{j=1}^{k}[1-F_j(t)]. \tag{2}$$

Theorem 1. Under the assumptions A1-A4, the likelihood function of the unknown parameters when given observed sample $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ can be expressed as

$$L \propto \prod_{i=1}^{m}\prod_{j=1}^{k}\left[h_j(t_i)\right]^{\delta_i(c_j)}\left(\prod_{j=1}^{k}[1-F_j(t_i)]\right)^{r_i+1} p^M(1-p)^N, \tag{3}$$

where $h_j(\cdot) = f_j(\cdot)/[1-F_j(\cdot)]$ is the hazard rate function of *j*th failure cause, $M = \sum_{i=1}^{m-1} r_i$, $N = (m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i$.

Proof. When the *i*th failure time t_i is observed, and the associated failure cause is *j*. Then, r_i of the surviving systems are randomly removed from the test. The likelihood function of the unknown parameters when given t_i can be expressed as

$$L_{i0} \propto \prod_{j=1}^{k}\left[f_j(t_i)\prod_{l=1,l\neq j}^{k}\{1-F_l(t_i)\}\right]^{\delta_i(c_j)}\left(\prod_{j=1}^{k}[1-F_j(t_i)]\right)^{r_i}. \tag{4}$$

Based on A3, $\left(r_i \mid r_{i-1}, r_{i-2}, \cdots, r_1\right) \sim B(n-m-\sum_{j=0}^{i-1} r_j, p)$, so we have

$$P_i = P\left(R_i = r_i \mid R_{i-1} = r_{i-1}, \cdots, R_1 = r_1\right) = \binom{n-m-\sum_{j=0}^{i-1} r_j}{r_i} p^{r_i}(1-p)^{n-m-\sum_{j=0}^{i} r_j}, \tag{5}$$

where, $0 \le r_i \le n-m-\sum_{v=1}^{i-1} r_v, i = 1, 2, \cdots, m-1$. Then the likelihood function of unknown parameters with t_i and random removals r_i is

$$L_i = L_{i0} P_i \propto \prod_{j=1}^{k} \left[f_j(t_i) \prod_{l=1, l \neq j}^{k} \{1 - F_l(t_i)\} \right]^{\delta_i(c_j)} \left(\prod_{j=1}^{k} [1 - F_j(t_i)] \right)^{r_i} p^{r_i} (1-p)^{n-m-\sum_{j=0}^{i} r_j} . \tag{6}$$

Then the full likelihood function is

$$L = \prod_{i=1}^{m} L_i \propto \prod_{i=1}^{m} \prod_{j=1}^{k} \left[f_j(t_i) \prod_{l=1, l \neq j}^{k} \{1 - F_l(t_i)\} \right]^{\delta_i(c_j)} \left(\prod_{j=1}^{k} [1 - F_j(t_i)] \right)^{r_i} p^{r_i} (1-p)^{n-m-\sum_{j=0}^{i} r_j}$$

$$\propto \prod_{i=1}^{m} \prod_{j=1}^{k} \left[h_j(t_i) \right]^{\delta_i(c_j)} \left(\prod_{j=1}^{k} [1 - F_j(t_i)] \right)^{r_i+1} p^M (1-p)^N .$$

The proof holds.

3. Maximum likelihood estimation

In this section, the MLEs of θ_j and p are derived. Under progressive Type-II censoring scheme, m failures are observed, where $m_j, j = 1, 2, \ldots, k$ failures are caused by jth failure modes and m_0 failure causes are masked. Then the equation (3) can be rewritten as follows

$$L \propto \left[\prod_{i=1}^{m} t_i^{-1} \right] \prod_{j=1}^{k} \theta_j^{m_j} \left(\theta_1 + \theta_2 + \ldots + \theta_k \right)^{m_0} \left[\prod_{i=1}^{m} \left(\tau / t_i \right)^{(\theta_1 + \theta_2 + \ldots + \theta_k)(r_i + 1)} \right] p^M (1-p)^N . \tag{7}$$

3.1. MLEs of $\boldsymbol{\theta_j}, \boldsymbol{p}$ and reliability $\boldsymbol{R}$

Based on equation (7), the log-likelihood function of unknown parameters is

$$\log L = -\sum_{i=1}^{m} \ln t_i + \sum_{j=1}^{k} m_j \log \theta_j + m_0 \log \left(\theta_1 + \theta_2 + \ldots + \theta_k \right)$$

$$+ \left(\theta_1 + \theta_2 + \ldots + \theta_k \right) \sum_{i=1}^{m} \left(r_i + 1 \right) \log \left(\tau / t_i \right) + M \ln p + N \ln \left(1 - p \right).$$

Then, we can get the likelihood equations as follows

$$\begin{cases} \dfrac{\partial \log L}{\partial \theta_j} = \dfrac{m_j}{\theta_j} + \dfrac{m_0}{\theta_1 + \theta_2 + \ldots + \theta_k} + \displaystyle\sum_{i=1}^{m} \left(r_i + 1 \right) \left(\ln \left(\tau / x_i \right) \right) = 0, \\ \dfrac{\partial \log L}{\partial p} = \dfrac{M}{p} - \dfrac{N}{1-p} = 0, \end{cases}$$

Solve the above equation, we can obtain

$$\hat{\theta}_j = \left[-m_j \left(\sum_{i=1}^{m} \left(r_i + 1 \right) \left(\ln \left(\tau / x_i \right) \right) \right)^{-1} \right] \left[\frac{m_0}{m_1 + m_2 + \ldots + m_k} \right], \quad \hat{p} = \frac{M}{M+N}.$$

Based on the invariance of MLEs, the MLE of R can be given by

$$\hat{R}(t) = (\tau / t)^{\hat{\theta}_1 + \hat{\theta}_2 + \ldots + \hat{\theta}_k} .$$

3.2. Asymptotic confidence intervals

In this subsection, the asymptotic confidence intervals for the unknown parameters are obtained. The asymptotic result can be expressed as follows

$$(\hat{\theta}_1 - \theta_1, \hat{\theta}_2 - \theta_2, \ldots, \hat{\theta}_k - \theta_k, \hat{p} - p) \to N_{k+1}(\mathbf{0}, \mathbf{I}^{-1}(\theta_1, \theta_2, \ldots, \theta_k, p)),$$

where $\mathbf{I}(\theta_1, \theta_2, \ldots, \theta_k, p)$ is the Fisher information matrix for the parameters

$(\theta_1, \theta_2, \ldots, \theta_k, p)$. The elements of matrix $\mathbf{I}$ are as follows

$$I_{jj} = -\frac{\partial^2 \log L}{\partial \theta_j^2} = \frac{m_j}{\theta_j^2} + \frac{m_0}{(\theta_1 + \theta_2 + \ldots + \theta_k)^2},\ j = 1, 2, \ldots, k,$$

$$I_{j+1,j+1} = -\frac{\partial^2 \log L}{\partial p^2} = \frac{M}{p^2} + \frac{N}{(1-p)^2},$$

$$I_{ij} = I_{ji} = 0, (\, i = 1, 2, \ldots, k;\ j = i+1, \ldots, k+1; i \neq j).$$

Denote V as the approximate asymptotic variance-covariance matrix for the MLEs of unknown parameters $\theta_1, \theta_2, \ldots, \theta_k, p$, and $\hat{V}$ as the estimate of V, then

$$\hat{V}\left(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k, \hat{p}\right) = \begin{bmatrix} \hat{V}_{11} & \cdots & \hat{V}_{1k} & \hat{V}_{1,k+1} \\ \vdots & \ddots & \vdots & \vdots \\ \hat{V}_{k1} & \cdots & \hat{V}_{k,k} & \hat{V}_{k,k+1} \\ \hat{V}_{k+1,1} & \cdots & \hat{V}_{k+1,k} & \hat{V}_{k+1,k+1} \end{bmatrix} = \begin{bmatrix} \hat{I}_{11} & \cdots & \hat{I}_{1k} & \hat{I}_{1,k+1} \\ \vdots & \ddots & \vdots & \vdots \\ \hat{I}_{k1} & \cdots & \hat{I}_{k,k} & \hat{I}_{k,k+1} \\ \hat{I}_{k+1,1} & \cdots & \hat{I}_{k+1,k} & \hat{I}_{k+1,k+1} \end{bmatrix}^{-1}.$$

Therefore, the approximate $100(1-\alpha)\%$ confidence intervals for $\theta_1, \theta_2, \ldots, \theta_k, p$ are given by

$$\left[\hat{\theta}_j - z_{\alpha/2}\sqrt{\hat{V}_{jj}}, \hat{\theta}_j + z_{\alpha/2}\sqrt{\hat{V}_{jj}}\right], j = 1, 2, \cdots, k,$$

$$\left[\hat{p} - z_{\alpha/2}\sqrt{\hat{V}_{k+1,k+1}}, \hat{p} + z_{\alpha/2}\sqrt{\hat{V}_{k+1,k+1}}\right],$$

where $z_{\alpha/2}$ is the $\alpha/2$ percentile of the standard normal distribution.

4. Bayesian estimation

In the analysis of section 3, the MLEs of $\theta_1, \theta_2, \ldots, \theta_k, p$ are obtained. However, we cannot obtain the MLEs of the unknown parameters when the failure causes are completely masked. In this situation, Bayesian method is an alternative approach.

4.1. Prior and posterior distribution

Suppose the conjugate prior distribution of θ_j is Gamma distribution $Ga(a_j, b_j)$ and the prior distribution of *p* is an uniform distribution $U(0,1)$, namely,

$$\pi\left(\theta_j \mid a_j, b_j\right) = b_j^{a_j}\left[\Gamma\left(a_j\right)\right]^{-1}\theta_j^{a_j-1}\exp\left\{-b_j\theta_j\right\},\ \theta_j > 0,$$

and

$$\pi(p) = \begin{cases} 1, p \in (0,1), \\ 0, \text{otherwise}. \end{cases}$$

Hence, the joint prior distribution of $\theta_1, \theta_2, \ldots, \theta_k, p$ is

$$\pi(\theta_1, \theta_2, \ldots, \theta_k, p) = \prod_{j=1}^{k} b_j^{a_j}\left[\Gamma\left(a_j\right)\right]^{-1}\theta_j^{a_j-1}\exp\left\{-b_j\theta_j\right\}. \tag{8}$$

Combine equation (7) with equation (8), we can obtain the joint density function of $\theta_1, \theta_2, \ldots, \theta_k, p$ and $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ by using the multiple expansion theorem,

$$f\left(\theta_1,\theta_2,\ldots,\theta_k,p,\mathbf{t}\right)\propto\sum_{m_{0j}\ge 0,\sum_{j=1}^{k}m_{0j}=m_0}\binom{m_0}{m_{01},m_{02},\ldots,m_{0k}}\prod_{j=1}^{k}\left\{\theta_j^{m_{0j}+m_j+a_j-1}\left[e^{-b_j}\prod_{i=1}^{m}(\tau/t_i)^{r_i+1}\right]^{\theta_j}\right\}$$
$$\times\prod_{j=1}^{k}b_j^{a_j}\left[\Gamma(a_j)\right]^{-1}\left[\prod_{i=1}^{m}t_i^{-1}\right]p^M(1-p)^N.$$

Then, the joint posterior density function of $\theta_1,\theta_2,\ldots,\theta_k,p$ is

$$f\left(\theta_1,\theta_2,\ldots,\theta_k,p\,|\,\mathbf{t}\right)=f\left(\theta_1,\theta_2,\ldots,\theta_k,p,\mathbf{t}\right)/\int_0^\infty\cdots\int_0^\infty\int_0^1 f\left(\theta_1,\theta_2,\ldots,\theta_k,p,\mathbf{t}\right)\mathrm{d}p\,\mathrm{d}\theta_1\mathrm{d}\theta_2\cdots\mathrm{d}\theta_k.$$

The posterior density functions of $\theta_1,\theta_2,\ldots,\theta_k,p$ are

$$\pi(\theta_j\,|\,\mathbf{t})=\frac{\sum_{m_{0j}\ge 0,\sum_{j=1}^{k}m_{0j}=m_0}\binom{m_0}{m_{01},m_{02},\ldots,m_{0k}}\theta_j^{A_j-1}B_j^{\theta_j}\prod_{l=1,l\ne j}^{k}C_l\Gamma(A_l)}{\sum_{m_{0j}\ge 0,\sum_{j=1}^{k}m_{0j}=m_0}\binom{m_0}{m_{01},m_{02},\ldots,m_{0k}}\prod_{l=1}^{k}C_l\Gamma(A_l)},\ j=1,2,\ldots,k,$$

$$\pi\left(p\,|\,\mathbf{t}\right)=\left[Be\left(M+1,N+1\right)\right]^{-1}p^M(1-p)^N.$$

where $A_j=m_{0j}+m_j+a_j$, $B_j=e^{-b_j}\prod_{i=1}^{m}(\tau/t_i)^{r_i+1}$, $C_j=\left(-\log B_j\right)^{-A_j}$.

4.2. Bayesian estimation of $\boldsymbol{\theta_j}, \boldsymbol{p}$ and reliability *R*

The P, Q-symmetric entropy loss function is defined as

$$L(\beta,\hat\beta)=(\beta/\hat\beta)^P+(\hat\beta/\beta)^Q-2,$$

where $\hat\beta$ is an estimator of β. Denote the prior and posterior distributions of β are $\pi(\beta)$ and $\pi(\beta\,|\,data)$, respectively. Under the P, Q-symmetric entropy loss function, the Bayesian estimation of any function $h(\beta)$ of β is given by

$$\hat h_B=E[h(\beta)\,|\,data]=\left[\frac{P\int_{\mathfrak{B}}h^P(\beta)\pi(\beta\,|\,data)d\beta}{Q\int_{\mathfrak{B}}h^{-Q}(\beta)\pi(\beta\,|\,data)d\beta}\right]^{\frac{1}{P+Q}},\tag{9}$$

where $\mathfrak{B}$ is the support of β.

Based on the subsection 4.1 and the equation (9), we can get the Bayesian estimators of $\theta_1,\theta_2,\ldots,\theta_k,p$ and reliability *R*,

$$\hat\theta_{jB}=\left[\frac{P}{Q}\frac{\sum_{m_{0j}\ge 0,\sum_{j=1}^{k}m_{0j}=m_0}\binom{m_0}{m_{01},m_{02},\ldots,m_{0k}}D_{j1}\Gamma(A_j+P)\prod_{l=1,l\ne j}^{k}C_l\Gamma(A_l)}{\sum_{m_{0j}\ge 0,\sum_{j=1}^{k}m_{0j}=m_0}\binom{m_0}{m_{01},m_{02},\ldots,m_{0k}}D_{j2}\Gamma(A_j-Q)\prod_{l=1,l\ne j}^{k}C_l\Gamma(A_l)}\right]^{\frac{1}{P+Q}},\ j=1,2,\ldots,k,$$

$$\hat p_B=\left[\frac{P}{Q}\cdot\frac{Be\left(M+P+1,N+1\right)}{Be\left(M-Q+1,N+1\right)}\right]^{\frac{1}{P+Q}},$$

$$\hat{R}_B=\prod_{j=1}^{k}\left[\frac{P}{Q}\frac{\sum_{m_{0j}\ge 0,\sum_{j=1}^{k}m_{0j}=m_0}\binom{m_0}{m_{01},m_{02},\ldots,m_{0k}}E_{j1}\Gamma(A_j)\prod_{l=1,l\ne j}^{k}C_l\Gamma(A_l)}{\sum_{m_{0j}\ge 0,\sum_{j=1}^{k}m_{0j}=m_0}\binom{m_0}{m_{01},m_{02},\ldots,m_{0k}}E_{j2}\Gamma(A_j)\prod_{l=1,l\ne j}^{k}C_l\Gamma(A_l)}\right]^{\frac{1}{P+Q}},$$

where $D_{j1}=(-\log B_j)^{-(A_j+P)}, D_{j2}=(-\log B_j)^{-(A_j-Q)}$,

$E_{j1}=\left[-\log(B_1e^{-b_1}(\tau/t)^P)\right]^{-A_j}, E_{j2}=\left[-\log(B_1e^{-b_1}(\tau/t)^{-Q})\right]^{-A_j}$.

4.3. HPD credible intervals

Given credible level α, the HPD credible interval of parameter β can be obtained by solving the following equation

$$\begin{cases}\int_{-\infty}^{\beta_L}\pi(\beta\mid data)d\beta=\alpha/2\\ \int_{-\infty}^{\beta_U}\pi(\beta\mid data)d\beta=1-\alpha/2\end{cases}\tag{10}$$

Then the HPD credible interval of parameter β is $[\beta_L,\beta_U]$.

Replace the $\pi(\beta\mid data)$ by the posterior density functions of $\theta_1,\theta_2,\ldots,\theta_k,p,R$,respectively, then the HPD credible interval of $\theta_1,\theta_2,\ldots,\theta_k,p,R$ can be obtained, namely,

$[\theta_{1L},\theta_{1U}],[\theta_{2L},\theta_{2U}],\ldots,[\theta_{kL},\theta_{kU}],[p_L,p_U],[R_L,R_U]$.

5. Simulation study

The progressive Type-II censored data are generated by the following steps:

Step1. Generate k columns independent uniformly distributed random numbers from $U(0,1)$, denoted by $y_{ij}(i=1,2,\cdots,n;j=1,2,\cdots,k)$.

Step 2. Substitute t in the equation $F^{-1}(t)=\tau/(1-t)^{1/\theta}$ by y_{ij}, then obtain the lifetime data of each competing risks $t_{ij}=F^{-1}(y_{ij})$, then the lifetime of the system is $t_i=\min_{1\le j\le k}(t_{ij})$.

Step 3. Given random removal probability *p*, generate *m* random removal numbers such that $r_i\sim B\left(n-m-\sum_{j=0}^{i-1}r_j,p\right)$.

Step 4. Based on the characteristic of progressive Type-II censoring scheme, generate *m* failure lifetime data.

Step 5. Given masking level (ML) *q*, obtain the failure causes and $m_0,m_1,\ldots,m_k$.

Suppose $n=30$ identical systems are placed on the life test, each system has two failure modes. The number of failures is $m=15$. Given the values of the parameters $\theta_1=0.8,\theta_2=0.6,\tau=1$, $a_1=6,a_2=5,b_1=7,b_2=8$, $P=1.05,Q=1$. In time $t_0=1.2$, the reliability of system is $R(t_0)=0.7747$. Then the MLEs, Bayesian estimators (BEs), MSEs of two estimators, the confidence intervals (CIs) and HPD credible intervals (HPD-CIs) of $\theta_1,\theta_2,\ldots,\theta_k,p,R$ can be obtained, as well as the 95% credible level's coverage probabilities (CPs) of two intervals.

MLs	Para.	MLEs	MSEs	CIs	CPs
$q=0.2$	θ_1	0.9124	0.1901	[0.7856, 1.0154]	0.8420
	θ_2	0.6690	0.1436	[0.5478, 0.7745]	0.8960
	p	0.1055	0.0019	[0.0946, 0.1231]	0.9420
	R	0.7535	0.0054	[0.6574, 0.8541]	0.9210
$q=0.4$	θ_1	0.9065	0.2195	[0.7541, 1.0210]	0.8510
	θ_2	0.6656	0.1520	[0.5214, 0.7845]	0.8850
	p	0.1073	0.0018	[0.0941, 0.1228]	0.9590
	R	0.7548	0.0054	[0.6411, 0.8452]	0.9170
$q=0.6$	θ_1	0.8771	0.2471	[0.7438, 1.1258]	0.8320
	θ_2	0.6712	0.2046	[0.5102, 0.8420]	0.8740
	p	0.1072	0.0020	[0.0940, 0.1235]	0.9450
	R	0.7574	0.0048	[0.6321, 0.8698]	0.9020
$q=0.8$	θ_1	—	—	—	—
	θ_2	—	—	—	—
	p	0.1081	0.0017	[0.0948. 0.1245]	0.9520
	R	—	—	—	—

Table 1: MLEs, MSEs, CIs and CPs of $\theta_1,\theta_2,\ldots,\theta_k,p,R$ under different MLs when $p=0.1$

MLs	Para.	BEs	MSEs	HPD-CIs	CPs
$q=0.2$	θ_1	0.8013	0.0328	[0.7096, 0.8859]	0.9210
	θ_2	0.5935	0.0295	[0.5124, 0.7011]	0.9100
	p	0.1136	0.0019	[0.0971, 0.1249]	0.9450
	R	0.8165	0.0151	[0.7141, 0.9210]	0.9020
$q=0.4$	θ_1	0.8109	0.0318	[0.6985, 0.8874]	0.9180
	θ_2	0.5842	0.0304	[0.5107, 0.7142]	0.9040
	p	0.1155	0.0018	[0.0960, 0.1265]	0.9540
	R	0.8165	0.0200	[0.7025, 0.9200]	0.8950
$q=0.6$	θ_1	0.8204	0.0297	[0.6952, 0.8812]	0.9120
	θ_2	0.5745	0.0324	[0.5068, 0.7089]	0.9010
	p	0.1153	0.0020	[0.0956, 0.1256]	0.9460
	R	0.8163	0.0221	[0.7005, 0.9187]	0.8950
$q=0.8$	θ_1	0.8313	0.0407	[0.6921, 0.8758]	0.9080
	θ_2	0.5649	0.0363	[0.5024, 0.7002]	0.9050
	p	0.1162	0.0017	[0.0952, 0.1258]	0.9560
	R	0.8163	0.0305	[0.6995, 0.9365]	0.8910

Table 2: BEs, MSEs, HPD-CIs and CPs of $\theta_1,\theta_2,\ldots,\theta_k,p,R$ under different MLs when $p=0.1$

RPs	Para.	MLEs	SEs	CIs	CPs
$p=0.2$	θ_1	0.8810	0.1603	[0.7168, 0.8959]	0.8950
	θ_2	0.6520	0.1299	[0.4985, 0.6981]	0.9010
	p	0.2138	0.0049	[0.1965, 0.2257]	0.9510
	R	0.7595	0.0047	[0.7054, 0.8954]	0.8990
$p=0.4$	θ_1	0.8963	0.1785	[0.7258, 0.9012]	0.9040
	θ_2	0.6502	0.1219	[0.5014, 0.7012]	0.8890
	p	0.4264	0.0115	[0.3978, 0.4241]	0.9450
	R	0.7579	0.0051	[0.7141, 0.9085]	0.9060
$p=0.6$	θ_1	0.8676	0.1613	[0.7104, 0.8898]	0.8840
	θ_2	0.6780	0.1417	[0.5124, 0.7087]	0.8900
	p	0.6246	0.0154	[0.5925, 0.6214]	0.9620
	R	0.7580	0.0049	[0.7089, 0.8969]	0.9210
$p=0.8$	θ_1	0.9057	0.1982	[0.7321, 0.9251]	0.8950
	θ_2	0.6752	0.1499	[0.5098, 0.6984]	0.9000
	p	0.8208	0.0117	[0.7944, 0.8248]	0.9670
	R	0.7537	0.0056	[0.7259, 0.9163]	0.9180

Table 3: MLEs, MSEs, CIs and CPs of $\theta_1, \theta_2, \ldots, \theta_k, p, R$ under different RPs when $q = 0.1$

RPs	Para.	BEs	MSEs	HPD-CIs	CPs
$p=0.2$	θ_1	0.7995	0.0258	[0.7396, 0.8759]	0.9350
	θ_2	0.5926	0.0275	[0.5012, 0.6898]	0.9260
	p	0.2228	0.0048	[0.1987, 0.2248]	0.9680
	R	0.8169	0.0181	[0.7144, 0.8836]	0.9110
$p=0.4$	θ_1	0.8002	0.0285	[0.7368, 0.8996]	0.9260
	θ_2	0.5924	0.0312	[0.5211, 0.6985]	0.9180
	p	0.4196	0.0101	[0.3952, 0.4358]	0.9560
	R	0.8169	0.0209	[0.7414, 0.9055]	0.9020
$p=0.6$	θ_1	0.7984	0.0247	[0.7250, 0.8900]	0.9180
	θ_2	0.5944	0.0295	[0.5266, 0.7102]	0.9330
	p	0.6123	0.0123	[0.5910, 0.6198]	0.9610
	R	0.8168	0.0194	[0.7154, 0.8896]	0.9270
$p=0.8$	θ_1	0.8004	0.0304	[0.7412, 0.9233]	0.9300
	θ_2	0.5940	0.0275	[0.5214, 0.6928]	0.9140
	p	0.7945	0.0086	[0.7926, 0.8150]	0.9690
	R	0.8166	0.0211	[0.7358, 0.9025]	0.9140

Table 4: BEs, MSEs, HPD-CIs and CPs of $\theta_1, \theta_2, \ldots, \theta_k, p, R$ under different RPs when $q = 0.1$.

Compare Table 1 with Table 2, we can find the BEs are better than the MLEs under the same RP. The MSEs of two methods become larger as the increasing of MLs. When the MLs are large enough, the MLEs method cannot obtained the results, but the Bayesian method is still effective. Compared Table 3 with Table 4, when the RP becomes larger, the MLEs and BEs have no significant fluctuation under the same MLs. In general, the 95% CPs of BEs are larger than the MLEs.

6. Conclusions

In this paper, we consider the estimation of the unknown parameters and reliability of the masked risks model with the progressive Type-II censored Pareto data. The lifetimes of failure modes follow to Pareto distributions with a same scale parameter but different shape parameters. Meanwhile, some of the failure causes are masked. The MLEs, Bayesian estimators, confidence intervals and HPD credible intervals are obtained. The simulation study shows that the Bayesian method is better than MLE method in small samples. As the masking level turns to be large, the MLE method is out of effect, but the Bayesian method is still effective. As the random removed probability becomes larger, the MLEs and Bayesian estimators have no significant changing. In the future work, the dependent competing risks with masked failure causes may be considered by using copula function, Marshall-Olkin type distributions, and other methods.

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