2015

MATHEMATICS

[Honours]

PAPER – VII

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

GROUP – A

(*Mathematical Probability*)

[*Marks* : 36]

1. Answer any *one* question : 15 × 1

(*a*) (*i*) Define random variable and its distribution function. Show that distribution function of a random variable is continuous on the right at all points. 1 + 1 + 3

(*ii*) Let $f(x)$ be the probability mass function of a discrete random variable X, for $x = 0, 1, 2, \ldots$. Given that $f(x) = \frac{\lambda}{x} f(x-1)$, for $x = 1, 2, \ldots$. Determine $f(x)$. Find the mean value of the random variable X. 2 + 3

(*iii*) The joint density function of two random variables X and Y is given by

$$f(x, y) = x + y, \; 0 < x < 1, \; 0 < y < 1$$
$$= 0 \quad \text{, else where}$$

Find the distribution of XY. 5

(*b*) (*i*) For a bivariate random variable (X, Y), define regression curves. For a bivariate normal distribution, prove that regression curves are identical with regression lines. 1 + 4 (*ii*) Given that $x = 4y + 5$ and $y = kx + 4$ are the regression lines of x on y and of y on x respectively. Show that $0 < k \leq 0{\cdot}25$. If $k = 0{\cdot}10$, find the means of the variables x and y and also their correlation coefficient. 5

(*iii*) State and prove weak law of large numbers. 5

2. Answer any *two* questions : 8 × 2

(*a*) (*i*) Prove that

$$P(\lim A_n) = \lim P(A_n),$$

clearly stating the restriction on the sequence of events $\{A_n\}$. 5

(*ii*) Let X be a continuous random variable having distribution function $F(x)$. Show that $Y = F(x)$ has uniform distribution over (0, 1). 3

(*b*) (*i*) If X_i $(i = 1, \ldots n)$ are mutually independent $N(\mu_i, \sigma_i)$ variates, then find the distribution of $Z = \sum_{i=1}^{n} a_i X_i$. Hence calculate the mean and standard deviation of Z. 4 + 2

(*ii*) If the correlation coefficient $\rho(X, Y)$ between two random variables X and Y exists, then show that $-1 \leq \rho(X, Y) \leq 1$. 2

(*c*) (*i*) Define Bernoulli sequence of trials. Prove that in n Bernoulli trials with probability of failure q, the probability of at most k success is 1 + 4

$$\int_0^q x^{n-k-1}(1-x)^k dx \Big/ \int_0^1 x^{n-k-1}(1-x)^k dx$$

(*ii*) If X is a $\gamma\left(\frac{n}{2}\right)$ variate, then show that $Y = 2X$ has a χ^2-distribution with n degrees of freedom. 3

(*d*) (*i*) If X is a nonnegative random variable having mean m, prove that

$$P(X \geq \tau m) \leq 1/\tau$$

for any $\tau > 0$. 3

(*ii*) For any distribution, prove that the first absolute moment about the mean cannot exceed the standard deviation. 5

3. Answer any *one* question : 3×1

(*a*) Among n coins, $(n-1)$ are of the usual type, while one has head on both sides. A coin is chosen at random and tossed k times. If the coin falls head each time, what is the probability that it is the unusual coin ? 3

(*b*) The random variables X and Y are connected by the linear relation $aX + bY + c = 0$. Find the correlation coefficient between X and Y. 3

4. Answer any *one* question : 2×1

(*a*) $A_1, A_2, A_3, \ldots A_n$ are n mutually independent events. Prove that the probability of accurance of at least one of the events is $1-(1-p_1)(1-p_2) \ldots (1-p_n)$ where $p(A_i) = p_i$, $i = 1, 2, \ldots, n$. 2

(*b*) If X has a t-distribution with n degrees of freedom, then show that $Y = X^2$ has an $F(1, n)$ distribution. 2

GROUP – B

(*Statistics*)

[*Marks* : 27]

5. Answer any *one* question : 15 × 1

(*a*) (*i*) Define standard error of a statistic and give its significance. Show that for any population with variance σ^2 standard error of a sample mean is $\frac{\sigma}{\sqrt{n}}$, where n is size of the sample. 1 + 2 + 4

(*ii*) Prove that for a normal (m, σ) population, the sampling distribution of the statistic is $\frac{ns^2}{\sigma^2}$ χ^2-distribution with $n - 1$ degree of freedom where S^2 is the sample variance of a random sample of size n drawn from the given population. 8

(*b*) (*i*) What is meant by 'statistical hypothesis'? Explain with example the following terms : 4

Simple Hypothesis and Composite hypothesis.

(*ii*) Show that sample variance is not an unbiased estimator of the population variance. Find the unbiased estimator of the population variance. Examine whether this unbiased estimator is consistent or not. 3 + 1 + 2

(*iii*) A drug is given to 10 patients, and the increments in their blood pressure were recorded to be 3, 6, −2, 4, −3, 4, 6, 0, 02. Is it reasonable to believe that the drug has no effect on change of blood pressure? Test at 5% significance level, assuming the population to be normal. 5

6. Answer any *one* question : 8 × 1

(*a*) (*i*) A die was thrown 1,000 times, and the frequencies of the different faces were observed to be the following :

Face	1	2	3	4	5	6	Total
Frequency	105	143	181	157	198	216	1,000

Test if the die is honest. 6

(*ii*) State Neyman-Pearson theorem. 2

(*b*) Define confidence interval in connection with an unknown population parameter. The population of scores of 10-year children in a test is known to have a standard deviation 5·2. If a random sample of size 20 shows a mean of 16·9, first 95% confidence limits for the mean score of the population assuming that the population is normal. Given 2 + 6

$$\frac{1}{\sqrt{2\pi}}\int_{1\cdot96}^{\infty} e^{-\frac{x^2}{2}}\,dx = 0\cdot025.$$

7. Answer any *one* question : 4 × 1

(*a*) Show that sample correlation coefficient is independent if change of origin and magnitude of the scale of variables. 4

(*b*) Find the sampling distribution of the mean of Gamma population. 4

GROUP – C

Optional Paper – I

(*Discrete Mathematics*)

[*Marks* : 27]

8. Answer any *one* question : 3 × 1

(*a*) Define cardinality of a set and order set with examples. 3

(*b*) Solve the recurrence relation $s_n - 4s_{n-1} + 4s_{n-2} = 0$, $s_0 = 1$ and $s_1 = 6$ by using generating function where $n \geq 2$. 3

9. Answer any *two* questions : 12 × 2

(*a*) (*i*) If G be a connected planner graph having n vertices, e edges and r regions, then prove that $n - e + r = 2$. 4

(*ii*) Prove that the set D of all factors of 12 under divisiblity forms a lattice. 3

(*iii*) By principle of induction, prove that $3^{4n+2} + 5^{2n+1}$ is a multiple of 14, for all positive intgral value of n including zero. 5

(*b*) (*i*) Prove that the number of vertices of odd degree in a graph is always even. 5

(*ii*) Let I is the set of all integers : $X \equiv y$ (mole 7) is an equivalence relation in I. Find what is the partition of I. 3

(*iii*) If $(A, \leq)$ and $(B, \leq)$ be two partially order sets then prove that $(A \times B, \leq)$ is partially order set with partial order $\leq$ defined by $(a, b) \leq (a', b')$ if $a < a'$ in A and $b < b'$ in B. 4

(c) (i) If a graph has exactly two vertices of odd degree, then prove that there must be a path joining these two vertices. 5

(ii) Find time complexity of linear search algorithm. 3

(iii) Let $\{A, B, C, S)$ be the set of non-terminals, with S being the starting symbol. Let $\{a, b, c\}$ be the set of terminals. Describe the language specified by the following set of production in set theoretic notation : 4

$$\{S \to aS, S \to bA, A \to aA, A \to a\}$$

Optional Paper – II

(*Mathematical Modelling*)

[*Marks* : 27]

8. Answer any *one* question : 15 × 1

(a) A prey-predator model satisfies the differential equation

$$\frac{dx}{dt} = x(a - by)$$

$$\frac{dy}{dt} = -y(p - qx)$$

with $x(0) = x_0$, $y(0) = y_0$, where a, b, p, q are positive constants and $x(t), y(t)$ are the population of prey-predator at t.

(*i*) Find the equilibrium position of these equations. 4

(*ii*) Discuss the stability of these equilirium points 7

(*iii*) Discuss the limitations of this prey -predator model. 4

(*b*) (*i*) Growth of a single species satisfies the differential equation

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{k}\right) \text{ with } x(0) = x_0,\ \alpha > 0,\ k > 0.$$

Find the population at time t. Sketch the graph of $x(t)$ against t and discuss the behaviour of the solution with respect to the initial population. 4 + 4 (*ii*) Show that the mathematical model represented by

$$\frac{dx}{dt} = x(15 - 5x - 3y)$$

$$\frac{dy}{dt} = y(4 - x - y), x \geq 0, y \geq 0$$

has a position of equilibrium. Show also that this position is stable and two species can co-exist. 7

9. Answer any *one* question : 8 × 1

(*a*) Investigate the qualitative behaviour of the solution of the system 8

$$\frac{dx}{dt} = x\left(1 - \frac{x}{30}\right) - \frac{xy}{x+10}$$

$$\frac{dy}{dt} = y\left(\frac{x}{x+10} - 1/3\right)$$

(*b*) Derive the relation between carvelocity (u) and traffic density (p) on a highway. Find speed limitation (u max) on highway. 8

10. Answer any *one* question : 4 × 1

(*a*) Define critical point of the system of differential equation

$$\frac{dx}{dt} = f(x, y) \text{ and } \frac{dy}{dt} = \psi(x, y).$$

When is this critical point said to be stable and asymptotically stable ? 4

(*b*) In a population of birds, the proporttionate birth rate and the proportionate death rate are constants being 0·45 per year and 0·65 per year respectively. Formulate a model of the population and discuss its long term behaviour. 4

Optional Paper — III

(*Application of Mathematics in Finance and Insurance*)

[*Marks* : 27]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) Discuss profit maximisation objectives of a firm. How far is it different from wealth maximisation objective ?

(*ii*) Find the present value of the following cash flows using 5% discounting rate :

Year	Cash Flows Rs.
1-4	2,000
5	3,000
6	5,000
7-8	3,000
9	5,000
10	7,000

6 + 4 + 5

(*b*) (*i*) What do you mean by 'Stand-alone risk' and 'portfolio risk' in corporate security market ?

(*ii*) What do you mean by futures and forwards in respect to financial derivatives ?

(*iii*) Write a note on proximate cause of insurance. 4 + 7 + 4

9. Answer any *one* question : 8 × 1

(*a*) Briefly describe Markowitz model. 8

(*b*) State the concept of compounding and discounting in time value of money. 8

10. Answer any *one* question : 4×1

(*a*) Discuss the benefits of insurance to the society. 4

(*b*) Write a note on internal rate of return (IRR). 4
