OLD

2015

Part-I 3-Tier

MATHEMATICS

PAPER—II

(Honours)

Full Marks : 90

Time : 4 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Group—A

(Real Analysis)

[Marks : 27]

1. Answer any *one* question : 1×5

(a) (i) Define uniform continuity of a real valued function defined on a subset of R. Let $f:[a, b] \rightarrow \Re$ be continuous on [a, b]. Then show that f is uniformly continuous on [a, b].

(Turn Over)

(ii) Let $f:[a,b]\to\Re$ be monotone on [a, b]. Then show that the set of points of discontinuities of f in [a, b] is a countable set.

(b) (i) Show that every bounded non-empty open subset of R can be expressed as the union of a countable collection of disjoint open intervals.

(ii) Let $\sum u_n$ be a series of positive real numbers and let lim Imf $n\left(\frac{u_n}{u_n+1}-1\right)=r$. Prove that $\sum u_n$ is convergent if $r > 1$. Hence show that the series $1+\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\ldots\ldots+\frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}+\ldots.$ is convergent.

(iii) Let $f:R\to R$ be continuous on R. Prove that the set $S=\{x\in R: f(x)\neq 0\}$ is an open set in R.

2. Answer any *one* question : 1×8

(a) (i) Write down the coefficient of x^7 in the Taylor series expansion of the function

$$f(x) = \log\left(x + \sqrt{1+x^2}\right) \text{ about the origin.}$$

(ii) Show that $\{u_n\}$ is a Cauchy sequence where

$$u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}.$$

(b) (i) Consider the polynomial

$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$, with real coefficients. If p (x) has a real root in the interval [0, 1], then show that

$$\frac{a_0}{1.2} + \frac{a_1}{2.3} + \ldots\ldots + \frac{a_n}{(n+1)(n+2)} = 0.$$

(ii) Evaluate : $\lim_{n \to \alpha} n \,\mathrm{Sin}(2\pi e n!)$

3. Answer any *one* question : 1×4

(a) If f is differentiable on [0, 1], show by Cauchy's Mean Value theorem that the equation $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one solution in (0, 1).

(b) Find the limit points of the set

$$S = \left\{ n + \frac{1}{3m^2} : n, m \in \mathbb{N} \right\}$$

Group B

(Several Variables and Applications)

[Marks : 22]

4. Answer any *two* questions : 2×8

(a) (i) State and Prove Young's theorem.

(ii) Show that f (xy, z − 2x) = 0 satisfies, under suitable conditions, the equation $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x$.

What are these conditions? 6+2

(b) (i) Let u, v be functions of ξ, η, ζ having continuous first order partial derivatives and ξ, η, ζ be functions of x and y having continuous first order partial derivatives. Prove that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(\xi,\eta)}\frac{\partial(\xi,\eta)}{\partial(x,y)} + \frac{\partial(u,v)}{\partial(\eta,\zeta)}\frac{\partial(\eta,\zeta)}{\partial(x,y)} + \frac{\partial(u,v)}{\partial(\xi,\zeta)}\frac{\partial(\xi,\zeta)}{\partial(x,y)}$$

(ii) Prove that the function

$f(x, y) = f(x,y) = \sqrt{|xy|},\ x \neq 0,\ y \neq 0$

$= 0,\ x = 0,\ y = 0.$

is not differentiable at the point (0,0). 5+3

(c) (i) Find the envelope of the family of ellipse

$\frac{x^2}{a^2} + \frac{y^2}{(k-a)^2} = 1$ where k is a constant. 4

(ii) If p_1, p_2 be the radii of curvature at the entremities of any chord of the cardiode $r = a(1+\text{Cos}\theta)$, which passes through the pole, then prove that $p_1^2 + p_2^2 = \frac{16}{9}a^2$. 4

5. Answer any *one* questions : 6×1

(a) By the transformation $\xi = a + \alpha x + \beta y$, $\eta = b - \beta x + \alpha y$ where α, β, a, b are all constants and $\alpha^2 + \beta^2 = 1$, the function u (x, y) is transformed into $U(\xi, \eta)$. Prove that

$$U_{\xi\xi} U\eta\eta - U^2{}_{\xi\eta} = u_{xx} u_{yy} - u^2{}_{xy}.$$

(b) State which type of curve may have asymptotes. Find the equation of the cubic which has the same asymptotes as the curve

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$$

and which touches the axis of y at the origin and goes through the point (3, 2).

Group C

(Integral Calculas)

[Marks : 09]

6. Answer any *one* questions : 1×9

(a) (i) Obtain a reduction formula for $\int \frac{dx}{\left(x^2 + a^2\right)^n}$, where *n* is a positive integer; and hence deduce the value of $\int \frac{dx}{\left(x^2 + a^2\right)^3}$.

(ii) Evaluate : $\iint_{[0,1]\alpha[0,1]} \max\{x, y\}\, dx\, dy$

(b) (i) Find the length of the loop of the curve
$36y^2 = (3x + 7)(3x + 4)^2$.

(ii) Find the volume of the solid generated by revolving the cardioide $r = a(1 - \text{Cos}\theta)$ about the initial line.

Group D

(Differential Equations : Ordinary and Partial)

[Marks : 32]

7. Answer any *one* question : 1×15

(a) (i) Verify y that x = 0 is a regular singular point of the differential equation

$$\left(x^2 - 1\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - xy = 0$$

and find two independent series solutions of the equations near x = 0 and also find the interval where this solution exists.

(ii) Show that of y_1 and y_2 be solutions of the equation $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x alone and $y_2 = y_1 z$ then $z = 1 + ae^{-\int \frac{Q}{y_1} dx}$.

(a is any arbitrary constant)

(b) (i) Solve by Laplace transform $3\frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial t} = 0$ where $y\left(\frac{\pi}{2}, t\right) = 0$, $\left(\frac{\partial y}{\partial x}\right)_{x=0} = 0$ and $y(x, 0) = 30 \operatorname{Cos} 5x$.

(ii) If $Mx - Ny \neq 0$, then show that $\frac{1}{Mx - Ny}$ is an integrating factor of the equation Mdx + Ndy = 0, where M and N (functions of x and y) can be written as $M = yf_1(xy)$, $N = xf_2(xy)$.

(iii) Solve $(p + q)(x + y) = 1$, where the symbols have there usual meaning.

8. Answer any *three* questions : 3×5

(a) Solve : $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = x^{m+1}$

(b) Find the equation of the integral surface of the linear differential equation $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $x^2 + y^2 = 2x$, $z = 0$.

(c) Show that the family of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self-orthogonal where d is a parameter.

(d) Find the eigen values and eigen functions of the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0 \text{ with } y\ (0) \text{ and } y(2\pi) = 0.$$

(e) Solve by the method of variation of Parameters —

$$(2x+1)(x+1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = (2x+1)^2.$$

It is given that $y = x$ and $y = \frac{1}{x+1}$ are two linearly independent solutions of the corresponding homogeneous equation.

9. Answer any *one* question : 1×2

(a) Reduce the equation $\text{Sin}\, y \dfrac{dy}{dx} = \text{Cos}\, x\left(2\text{Cos}\, y - \text{Sin}^2 x\right)$ to a linear equation.

(b) Eliminate the aebitary functions f and φ from y = f (x – at) + φ (x + at) to form a partial differential equation.
