2015

MATHEMATICS

[Honours]

PAPER – II (New)

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

[NEW SYLLABUS]

GROUP – A

(*Real Analysis*)

[*Marks* : 35]

1. Answer any *one* question : 15×1

(*a*) (*i*) Suppose that $f : [a, b] \to R$ be a continuous function on the closed

(*Turn Over*)

interval $[a, b]$. If $f(a) . f(b)$ is negative then show that there exist points c_1 in the open interval (a, b) such that 5

$$f(c_1) = 0$$

(*ii*) If $f : (a, b) \to R$ is monotone increasing and if $a < c < b$ then show that $f(c-0)$ and $f(c+0)$ exist finitely and 5

$$f(c-0) = \sup_{a<x<c} f(x) \le f(c) \le \inf_{c<x<b} f(x) = f(c+0).$$

(*iii*) Prove that every neighbourhood of a limit point of a set of real numbers contains infinitely many points of that set. Deduce that the set $S = \{2, 3, 4, 5\}$ has no limit point. 1 + 4

(*b*) (*i*) Define Cauchy sequency. Let $D \subseteq R$, and $f : D \to R$ be uniformly continuous on D. Prove that if $\{x_n\}$ be a Cauchy sequence in D, then $\{f(x_n)\}$ is also a Cauchy sequence in R ; where R is the set of real numbers. Is the result true for continuous function f ? Justify.

1 + 4 + 2

(*ii*) State and prove Leibnitz's test for convergence of an alternating series of real numbers. 1 + 3

(*iii*) If the series $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$ converges to a real number s then show that the rearranged series

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\ldots+\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}+\ldots$$

converges to $\frac{s}{2}$. 4

2. Answer any *two* questions : 8×2

(*a*) (*i*) State the least upper bound axiom for the set R of all real numbers. Hence show the a non-empty bounded below subset of R has the greatest lower bound in R. 1 + 3

(*ii*) Prove that every infinite bounded set of real numbers has atleast one accumulation point. 4

(*b*) (*i*) Let Q be the set of irrational numbers and $S \subset Q$, defined by $S = \{x \in Q : x > 0, x^2 < 2\}$. Prove that supremum of S does not belongs to Q. 4

(*ii*) Prove that the set of all irrational numbers is not denumerable. 4

(*c*) (*i*) Obtain Maclaurin's infinite series expansion of $\log(1+x)$, $-1 < x \leq 1$. 3

(*ii*) State and prove Rolle's theorem of differential calculus. 1+4

3. Answer any *one* question : 4×1

(*a*) If

$$\lim_{x \to 0} \frac{ae^x + bc^{-x} + 2\sin x}{\sin x + x\cos x} = 2$$

find the values of a and b. 4

(*b*) If $I_n = \int_0^{\pi/2} x^n \sin x\,dx$ (n is a positive integer), then prove that 4

$$I_n + (n^2 - n)\, I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

GROUP – B

(Several Variables and Applications)

[*Marks* : 20]

4. Answer any *two* questions : 8 × 2

(*a*) (*i*) Let

$$f(x,y) = x\sin\frac{1}{y} + y\sin\frac{1}{x}\ ,\ x, y \neq 0$$
$$0\ ,\ x, y = 0.$$

Examine the existence of double limit and the two repeated limits at the origin. 4

(*ii*) State and prove Euler's theorem on homogeneous function of degree *n* in three independent variables. 4

(*b*) (*i*) Prove that the function $f(x,y) = \sqrt{|xy|}$ is not differentiable at (0, 0). 3

(*ii*) State and prove Young's theorem for the equality of $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ at some point (*a*, *b*) of the domain of $f(x, y)$. 1 + 4 (*c*) (*i*) If ρ_1 and ρ_2 are the radii of curvature at two extremities of any chord of the cardiode $r = a(1 + \cos\theta)$ passing through the pole, prove that

$$\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$$ 4

(*ii*) Find the equation of the cubic which has the same asymptotes as the curve

$$2x^3 + x^2y - 5xy^2 + 2y^3 + 3x + 4y - 7 = 0$$

and which passes through the points (0, 0), (1, −1), (0, 2). 4

5. Answer any *one* question : 4×1

(*a*) Find the pedal equation of

$$r = a + b\cos\theta$$

with respect to the pole. 4

(*b*) Show that origin is a node, cusp or conjugate point of the curve

$$y^2 = ax^2(1 + x)$$

according as *a* is positive, zero or negative. 4

GROUP – C

(*Analytical Geometry of Two Dimensions*)

[*Marks* : 20]

6. Answer any *two* questions : 8×2

(*a*) If

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight lines, prove that the area of the triangle formed by the bisectors of the angles between them and the axis of X is

$$\left(\frac{\sqrt{(a-b)^2 + 4h^2}}{2h}\right) \cdot \left(\frac{ac - g^2}{ab - h^2}\right).$$ 8

(*b*) Reduce the equation

$$x^2 + 4xy + y^2 - 2x + 2y + 6 = 0$$

to its canonical form. Name the conic, find the eccentricity of the conic. $5 + 1 + 2$

(*c*) Let P and Q be two points on the conic $\frac{l}{r} = 1 - e\cos\theta$ with $(\alpha - \beta)$ and $(\alpha + \beta)$ as vectorial angles. Prove that the locus of the foot of the perpendicular from the pole on the straight line PQ is

$$\left(e^2 - \sec^2\beta\right)r^2 + 2ler\cos\theta + l^2 = 0. \qquad 8$$

7. Answer any *one* question : 4×1

(*a*) Find the locus of the poles of the normal chords of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad 4$$

(*b*) Show that the equation of the auxiliary circle of the conic $\frac{l}{r} = 1 + e\cos\theta$ is

$$l^2 - 2ler\cos\theta + r^2(e^2 - 1) = 0. \qquad 4$$

GROUP – D

(*Differential Equation* - I)

[*Marks* : 15]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) Show that

$$(2x^3 + 3xy + c)^2 - 4(x^2 + y)^3 = 0,$$

where c is a constant, is the general solution of the differential equation

$$y = 2px + p^2, \; p = \frac{dy}{dx}.$$

Can you conclude about the singular solution of the equation from c-discriminant relation and p-discriminant relation ? Justify your answer. 7

(*ii*) Suppose that a, b are two constants and u is a function of x. Prove that, on the interval $(-\infty, \infty)$, $y = u\,e^{-ax/2}$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

if and only if u satisfies the differential equation

$$\frac{d^2u}{dx^2}+\left(\frac{4b-a^2}{4}\right)u=0.$$ 4

(*iii*) Find the orthogonal trajectories of the family of co-axial circles

$$x^2+y^2+2gx+c=0,$$

where g is the parameter and c is constant. 4

(*b*) (*i*) Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=e^{-2x}\sec x$$ 5

(*ii*) Solve the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0,$$

given that $\left(x+\frac{1}{x}\right)$ is a solution. 5

(*iii*) *Find the eigenvalues* of eigen functions of the differential equation

$$\frac{d^2 y}{dx^2} + \lambda y = 0, \text{ (λ being a real numbers)},$$

which satisfies the boundary conditions

$$y(0) = 0 \text{ and } y(\pi) = 0.$$ 5
