OLD

2015

Part-I 3-Tier

MATHEMATICS

PAPER—I

(Honours)

Full Marks : 90

Time : 4 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Group—A

(Classical Algebra)

[Marks : 27]

1. Answer any *one* question : 1×15

(a) (i) If $x = \text{Cos}\theta + i\text{Sin}\theta$ and $1+\sqrt{1-a^2} = na$,

prove that $1 + a\,\text{Cos}\theta = \frac{a}{2n}(1+nx)\left(1+\frac{n}{x}\right)$

(ii) Prove that every polynomial equation of degree *n* has exactlly *n* roots.

(iii) Prove that the special roots of the equation $x^9 - 1 = 0$ are the roots of the equation $x^6 + x^3 + 1 = 0$ are their values are

$$\text{Cos}\frac{2r\pi}{9} \pm i\text{Sin}\frac{2r\pi}{9},\ r = 1,\ 2,\ 4.$$

(b) (i) If $\tan x = \frac{n \text{ Sin } y}{1 - n \text{ Cos } y}$ $(n < 1)$, show that

$$x = n \text{ Sin } y + \frac{n^2}{2} \text{ Sin } 2y + \frac{n^2}{3} \text{ Sin } 3y + \ldots\ldots$$

(ii) If a_1, a_2, ..., a_n be *n* positive rational numbers, not all equal, then show that

$$a_1^{a_1} a_2^{a_2} \ldots\ldots a_n^{a_n} > \left(\frac{a_1 + a_2 + \ldots\ldots + a_n}{n}\right)^{a_1 + a_2 + \ldots + a_n}$$

(iii) Solve by Ferrari's method, the equation $x^4 + 2x^3 - 5x^2 - 10x - 3 = 0$.

2. Answer any *one* question : 8×1

(a) (i) Find the condition that the roots of $ax^3 + 3bx^2 + 3cx + d = 0$ are in H.P.

(ii) Show that :

$$2^n + 4^n + 6^n + \ldots\ldots + (2m)^n > m(m+1)^n$$

(b) (i) If $\text{Sinh}^{-1}(x+iy)+\text{Sinh}^{-1}(x-iy)=\text{Cosh}^{-1}a$, where a is constant, then show that (x, y) lies on an ellipse or a hyparabola according as $a>1$ or $a<1$.

(ii) Show that 2 Sin 10°, 2 Sin 50° and (– 2 Sin 70°) are the roots of the equation $x^3 - 3x + 1 = 0$.

3. Answer any *one* question : 1×4

(a) Solve the equation :

$$x^3 + x^2 + 3x + 27 = 0,$$

if it has three distinct roots of equal moduli.

(b) Use Strum's method to find the number and position of the real roots of the equation

$$x^4 - 3x^3 - 2x^2 + 7x + 3 = 0.$$

Group B

(Abstract Algebra)

[Marks : 36]

4. Answer any *three* questions : 3×8

(a) (i) Prove that an equivalence relation on a set S determines a partition of S. Also prove that each partition of a set S determines an equivalence relation on S. 6

(ii) In a ring (R, +, •); of $a^2 = a$ for all $a \in R$, then show that charecteristics of a is 2. 2

(b) (i) If (G, o) be a semi-group and for any two elements a, b in G, each of the equations $a_0x = b$ and $y_0a = b$ has a unique solution in G, then show that (G, o) is a group.

(ii) Prove that every sub group of a cyclic group is cyclic.

(c) (i) If G is a group such that $(ab)^n = a^n b^n$, for three consecutive integers n = m, m + 1, m + 2 and for all a, $b \in G$, then show that G is abelian.

(ii) If show that R is a commutative ring if $x^3 = x$ for all $x \in R$.

(d) (i) Prove that left cosets aH, bH of H in G will be identical iff $a^{-1}b \in H$.

(ii) In a group (G, .) if for any two a, $b \in G$,

$a^{-1}.(a.b)^{-1}.a^2.b^2 = b$, show that (G, .) is an abelian group.

(d) (i) Show that the set of matrices $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is a subring of the ring of 2×2 matrices with integral elements.

(ii) Let (G, ·) be a group and H be a non-empty finite subset of G. Then prove that (H, ·) is a subgroup of (G, ·) if and only if $a \in H$, $b \in H$ implies $a \cdot b \in H$.

5. Answer any *three* questions : 3×4

(a) Prove that intesection of two subrings of a ring (R, +, .) is a subring of R.

(b) Let R be a ring and $R_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in R \right\}$,

Prove that $f : R_1 \to R$ defined by

$f\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = a$ for all $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in R_1$ is an isomorphism.

(c) For any two positive integers a, b ($\neq 0$), prove that three exist two unique integers q and r such that $a = bq + r$ where $0 \leq r < |b|$.

(d) If p be prime and not a divisor of a, then prove that $a^{p-1} \equiv 1 \pmod{p}$.

(e) If p be a prime number,

then show that $\phi(p^k) = p^k\left(1 - \frac{1}{p}\right)$,

where k is a positive integer and ϕ is Euler's [illegible]n function.

Group C

(Linear Algebra)

[Marks : 27]

6. Answer any *one* question : 1×15

(a) (i) Find for what values of a and b the following system of equations has (i) a unique solution (ii) no solution (iii) infinite number of solutions over the field of rational numbers

$$2x_1 + x_2 + 4x_3 = 1$$
$$5x_1 + 2x_2 + 7x_3 = a$$
$$10x_1 + 4x_2 + bx_3 = a + 1$$

(ii) If a vector space V is the set of real valued continuous functions over R, then show that the set W of all solutions of $3\frac{d^2y}{dx^2} + 14\frac{dy}{dx} - 5 = 0$ is a subspace of V.

(iii) Prove that :

$$\begin{vmatrix} b^2c^2 + a^2d^2 & bc + ad & 1 \\ c^2a^2 + b^2d^2 & ca + bd & 1 \\ a^2b^2 + c^2d^2 & ab + cd & 1 \end{vmatrix} =$$

$$(b - c)(c - a)(a - b)(a - d)(b - d)(c - d)$$

(b) (i) Let $U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + b = 0 \right\}$ and

$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c + d = 0 \right\}$ be the subspace of $R_{2\times2}$.

Find dim U, dim W, dim $(U \cap W)$ and dim $(U + W)$.

(ii) Find the eigen values and the corresponding eigen vectors of the following matrix

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix}$$

(iii) Prove that the eigen values of a real symmetric matric are real.

7. Answer any *one* question : 1×8

(a) (i) Determine the linear operator $T : R^3 \to R^2$ if the matrix of T relative to the ordered bases (0, 1, 1), (1, 0, 1), (1, 1, 0) of R^3 and (1, 0), (1, 1) of R^2

is $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{bmatrix}$.

(ii) Reduce the quadratic form $5x^2 + 10y^2 + 2z^2 + 12xy + 6yz + 4zx$ to its normal form. Find also the rank and signature.

(b) (i) Prove that every orthogonal set of non-null vectors in an inner product space is linearly independent.

(ii) In an inner product space X, prove that for any

$$x, y \in X (x + y, x + y)^{\frac{1}{2}} \leq (x, x)^{\frac{1}{2}} + (y, y)^{\frac{1}{2}}.$$

8. Answer any *one* question : 4×1

(a) Verify Cayley-Hamilton theorem for the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) Let V (R) be the vector space of polynomials in t over the field of real numbers of degree $\leq n$. Show that the set

$S = \{1, t, t^2, ..., t^n\}$ is a basis of V (R).
