2015

MATHEMATICS

[Honours]

PAPER – I (New)

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

[NEW SYLLABUS]

GROUP – A

(*Classical Algebra*)

[*Marks* : 30]

1. Answer any *one* question : 15 × 1

(*a*) (*i*) If $(1+z)^n=(1-z)^n$ then show that the values of z are $i\ \tan\frac{r\pi}{n}$, where $r=0, 1, 2, \ldots, (n-1)$, but omitting $\frac{n}{2}$, if n is even. 5

(*ii*) The equation $ax^3+bx^2+cx+d=0$ has two equal root β. The show that $(9ad-bc)^2=4\ (b^2-3ac)\ (c^2-3bd)$. Also find the value of β. 4

(*iii*) If $a_1, a_2, \cdots a_n\,; b_1, b_2, \cdots b_n$ be all real numbers, then show that

$$\left(a_1^2+a_2^2+\cdots+a_n^2\right)\left(b_1^2+b_2^2+\cdots+b_n^2\right)$$
$$\geq\left(a_1b_1+a_2b_2+\cdots+a_nb_n\right)^2,$$

the equality occurs when either

(*I*) $a_i=0$ or $b_i=0$ or both $a_i=0$ and $b_i=0$ $(i=1, 2, 3, \cdots n)$. 6

Or

(*II*) $a_i=Kb_i$ for some non-zero real K, $i=1, 2, 3, \cdots n$.

(*b*) (*i*) If the equation $f = 0$ has all its roots real then show that the equation $ff'' - f'^2 = 0$ has all its roots imaginary; where dashes denote derivatives with respect to x. 6

(*ii*) If a, b, c, be positive rational numbers then prove that 5

$$a^a b^b c^c \geq \left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} \left(\frac{b+c}{2}\right)^{\frac{b+c}{2}} \left(\frac{c+a}{2}\right)^{\frac{c+a}{2}}$$

$$\geq \left(\frac{a+b+c}{3}\right)^{a+b+c}$$

(*iii*) Define Log z, where z is a non-zero complex number. Prove that $\text{Log} z_1 + \text{Log} z_2 = \text{Log}(z_1 z_2)$ where z_1, z_2 be two distinct complex numbers such that $z_1 z_2 \neq 0$. Does the above relation hold for $z_1 = z_2$? Justify. 1 + 2 + 1

2. Answer any *one* question : 8 × 1

(*a*) (*i*) State Descartes' rule of signs regarding the number of positive roots of an equation with real coefficients. Apply this rule to show that the equation

$x^4 + 12x - 5 = 0$ has two real roots and two non-real roots. If one of the non-real roots be $1 + 2i$, find all the roots of the equation. 5

(*ii*) If x, y, z are positive numbers and $x + y + z = 1$ then show that 3

$$8xyz \le (1-x)(1-y)(1-z) \le \frac{8}{27}$$

(*b*) (*i*) Form an equation whose roots are the special roots of $x^{15} - 1 = 0$ and hence show that the roots of 5

$$x^4 - x^3 - 4x^2 + 4x + 1 = 0 \text{ are}$$

$$2\cos\frac{2r\pi}{15},\ r = 1, 2, 4, 7.$$

(*ii*) If $2\cos\theta = t$, prove that 3

$$\frac{1+\cos 7\theta}{1+\cos\theta} = \left(t^3 - t^2 - 2t + 1\right)^2$$

3. Answer any *one* question : 4 × 1

(*a*) Prove that the equation $(x+1)^4 = a(x^4 + 1)$ is a reciprocal equation if $a \neq 1$ and solve it when $a = -2$. 4

(*b*) Prove that the least value of $x + 2y + 4z$ is $4\sqrt{3}$, where x, y, z are positive real numbers satisfying the condition $x^2y^3z = 8$. 4

4. Answer any *one* question : 3 × 1

(*a*) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are $\alpha\beta + \beta\gamma$, $\beta\gamma + \gamma\alpha$, $\gamma\alpha + \alpha\beta$. 3

(*b*) Determine k and solve the equation if the roots are in arithmetic progression $8x^3 - 12x^2 - kx + 3 = 0$. 3

GROUP – B

(*Abstract Algebra*)

[*Marks* : 35]

5. Answer any *three* questions : 8 × 3

(*a*) (*i*) Let $f : A \to B$ be an injective mapping from a set A into the set B. If C and D be subsets of A, then prove that

$$f(C \cap D) = f(C) \cap f(D)$$ 3

(*ii*) In a group G, prove that for any two elements $a, b \in G$ the equation $ax = b$ has a unique solution in G. Also show that the subset $A = \{a \in G : ag = ga$ for all $g \in G\}$ is a subgroup of G and also show that A is a normal subgroup of G. 5

(*b*) (*i*) Define a partition on a non empty set. Prove that a partition of a set induces an equivalence relation on that set. 1 + 3

(*ii*) Let $S = \{x \in R : -1 < x < 1\}$ and $f : R \to S$ be a mapping defined by

$$f(x) = \frac{x}{1+|x|}$$

show that f is invertible and find f^{-1}. 3 + 1

(*c*) (*i*) In the field R of real numbers, show that $S = \{a + b\sqrt{3}\ ; a, b \in Q\}$ is a subfield but $T = \{b\sqrt{3}, b \in Q\}$ is not, where Q is the set of relational numbers. 5

(*ii*) Define S_3, the symmetric group of order 3, show that if is not abelian. 3

(*d*) (*i*) In a group $(G, *)$ the elements a, b commute and $O(a)$ and $O(b)$ are prime to each other. Show that $O(a * b) = O(a) \cdot O(b)$. 4

(*ii*) A subgroup H of a group G is normal if and only if $aHa^{-1} = H$ for every $a \in G$. 4

(*e*) (*i*) Prove that the intersection of two subrings is a subring. Cite an example to show that union of two subrings may be a subring. 2 + 1

(*ii*) Find the order of the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix}$$

in S_6. Decompose f as a product of transpositions. Give an example to show that S_6 is not an abelian group. 2 + 1 + 2

6. Answer any *two* questions : 4 × 2

(*a*) $(R, +, \cdot)$ is a field. Another law of composition 'X' is defined in R by taking $a \times b = a.u.b$, where a, b are two elements of R and $u\ (\neq 0)$

is a fixed element of R. Prove that $(R, +, x)$ is a ring. Is $(R, +, x)$ a field ? Justify. 4

(*b*) Show that a cyclic group G with generators of finite order n is isomorphic to the multiplicative group of nth order unity. 4

(*c*) Prove that a finite integral domain is a field. 4

7. Answer any *one* question : 3×1

(*a*) If a is an idempotent element of a ring R, then prove that for any $b \in R$, the product $(1 - a)\, ba$ is nilpotent. 3

(*b*) Let G be a group and $a \in G$. If $O(a) = 24$, find $O(a^4)$, $O(a^7)$ and $O(a^{10})$. 3

GROUP – C

(*Linear Algebra*)

[*Marks* : 25]

8. Answer any *one* question : 15×1

(*a*) (*i*) Prove that the rank of the product of two matrices cannot exceed the rank of either factor. If a matrix of rank r be multiplied by a non-singular matrix, what will be the rank of the product ? 5

(*ii*) Using Jacobi's theorem prove that 5

$$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = (af - be + cd)^2$$

(*iii*) Define eigenvalues and eigenvectors of a matrix. Find the eigenvalues and eigenvectors of 5

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(*b*) (*i*) If the row rank of the matrix

$$\begin{pmatrix} 3 & 4 & -3 & 5 \\ 1 & 2 & -1 & 7 \\ 4 & 1 & 2 & 9 \\ 2 & -1 & 4 & K \end{pmatrix}$$

is 3 then find the value of K. 5

(*ii*) For what values of a the following system of equations is consistent ?

$$x - y + z = 1$$
$$x + 2y + 4z = a$$
$$x + 4y + 6z = a^2$$

Solve the above equations considering that value of 'a'. 5

(*iii*) What is a real quadratic form ? Check whether the form

$$4x^2 + 9y^2 + 2z^2 + 8yz + 6zx + 6xy$$

is positive definite or not. 5

9. Answer any *one* question : 8 × 1

(*a*) (*i*) Apply the Gram-Schmidt process to the vectors (1, 0, 1), (1, 0, – 1) and (1, 3, 4) to obtain an orthonormal basis for R^3 with the standard inner product. 5

(*ii*) Prove that the eigenvectors corresponding to two distinct eigenvalues of a real symmetric matrix are orthogonal. 3

(*b*) (*i*) A is a non singular matrix such that the sum of the elements in each row is K. Prove that the sum of the elements in each row of A^{-1} is K^{-1}. 3

(*ii*) State Schwarz' inequality in Euclidean space. In a Euclidean space V, prove that two vectors α, β are linearly dependent iff $|(\alpha,\beta)| = \|\alpha\| \, \|\beta\|$. 1 + 4

10. Answer any *one* question : 2 × 1

(*a*) For what real value(s) of k the set of vectors $\{(k, 1, 1), (1, k, 1), (1, 1, k)\}$ is linearly dependent in R^3. 2

(*b*) If V be a vector space over a field F and let $\alpha \in V$ then prove that $W = \{c\,\alpha : c \in F\}$ forms a subspace of V. 2