2016

MATHEMATICS

[Honours]

PAPER – V

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

[OLD SYLLABUS]

GROUP – A

(*Real Analysis* - II)

[*Marks* : 64]

1. Answer any *two* questions : 15×2

(*a*) (*i*) If f is bounded and Riemann integrable

(*Turn Over*)

on $[a, b]$, then show that $|f|$ is also bounded and Riemann integrable on $[a, b]$. Moreover,

$$\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx.$$

Does the converse of the above statement true ? Support your answer.

4 + 2 + 2

(*ii*) Prove with the help of an example that the equation

$$\int_a^b f'(x)dx = f(b) - f(a)$$

is not always valid. 3

(*iii*) Show that

$$\left|\int_p^q \frac{\sin x}{x}\,dx\right| \le \frac{2}{p}, \text{ if } q > p > 0.$$ 4

(*b*) (*i*) Show that the improper integral

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

is not absolutely convergent. 5

(*ii*) Prove that the function f where

$$f(x,y)=x^2-2xy+y^2+x^3-y^3+x^5$$

has neither a maximum nor a minimum at the origin. 5

(*iii*) If

$$f(x)=-\frac{1}{4}\pi \text{ when } -\pi<x<0,$$

$$=\frac{1}{4}\pi \text{ when } 0<x<\pi;$$

$f(-\pi)=f(0)=0$ and $f(x+2\pi)=f(x)$ for all x, then show that

$$f(x)=\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+.....$$

for all x.

Hence deduce that

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.....$$ 4+1

(*c*) (*i*) Prove that a series $\sum f_n$ converges uniformly in $[a, b]$ if there exists a convergent series $\sum M_n$ of positive numbers s.t. for all n and for all $x \in [a, b]$, $|f_n(x)| \leq M_n$. 5

(*ii*) Prove that

$$\sum_{n=1}^{\infty} a_n \cos nx$$

is uniformly convergent on R if $\sum_{n=1}^{\infty} |a_n|$ converges. 4

(*iii*) If a series $\sum f_n$ converges uniformly to f on $[a, b]$ and each term f_n is integrable on $[a, b]$, then show that f is integrable on $[a, b]$. 6

2. Answer any *two* questions : 8×2

(*a*) (*i*) Show that the sequence $\{f_n\}$, where $f_n = nxe^{-nx^2}$, is pointwise convergent but not uniformly convergent in $[0, \infty)$. 4

(*ii*) Show that

$$\int_0^\infty \frac{x\,dx}{1+x^4 \sin^2 x}$$

is divergent. 4

(*b*) (*i*) Show that

$$\int_0^\infty x^{n-1}e^{-x}dx$$

is convergent if and only if $n > 0$. 4

(*ii*) Evaluate

$$f(y) = \int_0^\infty \frac{\cos xy}{1+x^2}dx.$$ 4

(*c*) (*i*) If $r > 0$ be the radius of convergence of

$$\sum_{n=0}^{\infty} a_n x^n,$$

prove that this power series is uniformly convergent on $[a, b] \subset (-r, r)$. When the series will be uniformly convergent on $[-r, r]$? 3 + 1 (*ii*) Determine the sum function of

$$\sum_{n=0}^{\infty} x^n.$$

Hence prove that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$ 1+3

3. Answer any *three* questions : 4×3

(*a*) (*i*) State first mean value theorem on integral calculus.

(*ii*) Test whether the series

$$\sum_{n=0}^{\infty} \cos nx$$

is a Fourier series. Justify your answer. 2+2

(*b*) Prove using double integral

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)},\ m,n > 0.$$ 4

(*c*) Use Lagrange's method to determine the point(s) on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

so that the tangent line to the point (or points) form with coordinate axes a triangle of smallest area. 4

(*d*) Show that

$$\int_2^\infty \frac{\cos x}{\log x}dx$$

is conditionally convergent. 4

(*e*) Using the method of differentiation under the sign of integration, prove that

$$\int_0^\infty e^{-x^2}\cos 2xt\,dx=\frac{\sqrt{\pi}}{2}e^{-t^2} \quad \text{for all } t\in R.$$ 4

4. Answer any *three* questions : 2×3

(*a*) State Dirichlet's conditions for convergence of Fourier series. 2

(*b*) If for a double series

$$S_{m,n} = (-1)^m \frac{m^2 n^3}{m^3 + n^6},$$

examine whether the series is convergent or not. 2

(*c*) Write the Cauchy's criterian for uniform convergent of a sequence of functions $\{f_n\}$ defined in an interval. 2

(*d*) What is the Weierstrass's *M*-test for uniform convergence ? 2

(*e*) If

$$\log_e x = \int_1^x \frac{dt}{t}, x > 0;$$

then show that $\log_e x$ is strictly increasing on $(0, \infty)$. 2

GROUP – B

(*Metric Space*)

[*Marks* : 14]

5. Answer any *one* question : 8 x 1

(*a*) (*i*) If (X, d) is a metric space then show that

$$\left(X, \frac{d}{1+d}\right)$$

is also a metric space. 4

(*ii*) Prove that in a metric space (X, d), every convergent sequence is a Cauchy sequence. Is the converse true in (X, d) ? Justify your answer. 1+3

(*b*) (*i*) State and prove Cantor inter-section theorem. 1+4

(*ii*) For any subset A of a metric space (X, d), prove that $X \setminus \overline{A} = (X \setminus A)^{\circ}$, where A° and $\overline{A}$ are interior and closure of A respectively. 3

6. Answer any *two* questions : 3 x 2

(*a*) Define interior points of a metric space. Prove, by an example, that

$$\text{int}\,(A \cup B) \neq \text{int}\, A \cup \text{int}\, B$$

for any two subsets A and B of a metric space. 1 + 2

(*b*) Let C be the set of all complex numbers. Show that the function '*d*' defined by $d(z_1, z_2) = |z_1 - z_2|, \forall z_1, z_2 \in C$, is a metric on C. 3

(*c*) When is a metric space said to be complete ? Give an example of an incomplete metric space. 1 + 2

GROUP – C

(*Complex Analysis*)

[*Marks* : 12]

7. Answer any *one* question : 8 × 1

(*a*) (*i*) Let $f(z) = u(x, y) + iv(x, y), z = x + iy$ and $z_0 = x_0 + iy_0$. Let the function f be defined in a domain D except possibly at the point z_0 in D. Then prove that

$$\lim_{z \to z_0} f(z) = w_0 = u_0 + iv_0$$

if and only if $\lim\limits_{z \to z_0} u(x, y) = u_0$ and $\lim\limits_{z \to z_0} v(x, y) = v_0$. 4

(ii) Show that the function $e^x(\cos y + i \sin y)$ is holomorphic and find its derivative. 4

(b) Prove that the function $f(z) = u(x, y) + iv(x, y)$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} (z \neq 0), f(0) = 0$$

is continuous and that Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ does not exist these equation. 8

8. Answer any *one* question : 4×1

(a) Show that if $f(z) = u(x, y) + iv(x, y)$ be analytic function of $z = x + iy$, then the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system. 4

(*b*) Prove that the function

$$u = \frac{1}{2}\log(x^2 + y^2)$$

is harmonic and find its harmonic Conjugate and find the corresponding analytic function $f(z)$ in terms of z. 4
