2016

MATHEMATICS

[Honours]

PAPER – I

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

[NEW SYLLABUS]

GROUP – A

(*Classical Algebra*)

[*Marks* : 30]

1. Answer any *one* question : 15×1

(a) (i) If $\alpha, \beta, \gamma, \cdots$ be the roots of the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0$. then prove that

$$(1+\alpha^2)(1+\beta^2)(1+\gamma^2)\cdots = (1 - p_2 + p_4 \cdots)^2 + (p_1 - p_3 + p_5 \cdots)^2. \quad 5$$

(ii) The roots of the equation $x^3 + px^2 + qx + r = 0$ are α, β, γ. Find the equation whose roots are $\alpha\beta - \gamma^2$, $\beta\gamma - \alpha^2$, $\gamma\alpha - \beta^2$. Deduce the condition that the roots of the given equation may be in geometric progression. 5

(iii) If $x_1, x_2, \ldots, x_n$ be real numbers satisfying $0 < x_1 < x_2 < \cdots < x_n < \pi/2$, prove that

$$\tan x_1 < \frac{\sin x_1 + \sin x_2 + \cdots + \sin x_n}{\cos x_1 + \cos x_2 + \cdots + \cos x_n} < \tan x_n. \quad 5$$

(*b*) (*i*) If a, b, c, d are four positive real numbers prove that

$$\frac{a^2+b^2+c^2}{a+b+c}+\frac{b^2+c^2+d^2}{b+c+d}+\frac{c^2+d^2+a^2}{c+d+a}$$
$$+\frac{d^2+a^2+b^2}{d+a+b}\geq(a+b+c+d).$$ 5

(*ii*) Prove that

$$x^n-1=(x-1)\prod_{k=1}^{(n-1)/2}\left[x^2-2x\cos\frac{2k\pi}{n}+1\right],$$

if n be an even positive integer. Deduce that

$$\sin\frac{\pi}{25}\sin\frac{2\pi}{25}\sin\frac{3\pi}{25}\cdots\sin\frac{12\pi}{25}=\frac{5}{2^{12}}.$$ 5

(*iii*) Prove that the roots of the equation

$$(2x+3)(2x+1)(x-1)(4x-7)$$
$$+(x+1)(2x-1)(2x-3)=0$$

are all real and different. Separate the intervals in which the roots lie. 5

2. Answer any *one* question : 8×1

(*a*) (*i*) If the equation

$x^4 + px^3 + qx^2 + rx + s = 0$ has roots of the form $\alpha \pm i\alpha, \beta \pm i\beta$ where α, β are real, prove that $p^2 - 2q = 0$, and $r^2 - 2qs = 0$. Hence solve the equation $x^4 + 4x^3 + 8x^2 - 24x + 36 = 0$. 5

(*ii*) If n be an integer and z be a complex number, prove that

$$(1 + \cosh 2z + \sinh 2z)^n = 2^n \cosh^n (\cosh nz + \sinh nz).$$ 3

(*b*) (*i*) If n be a positive integer > 1, then prove that

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} > \frac{1}{2\sqrt{n}}.$$ 3

(*ii*) Solve the equation by Ferrari's method $x^4 - 6x^2 + 16x - 15 = 0$. 5

3. Answer any *one* question : 4 × 1

(*a*) If n be a prime number prove that the special roots of the equation $x^{2n}-1=0$ are the non-real roots of the equation $x^n+1=0$. 4

(*b*) Prove that Gregory' series

$$\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\cdots\cdots$$

to infinity. State the validity of the series and hence prove that

$$\pi=4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right).$$ 4

4. Answer any *one* question : 3 × 1

(*a*) The α, β, γ be the roots of the equation $x^3-px^2+qx-r=0$, then form the equation whose roots are $\beta\gamma+\frac{1}{\alpha}, \gamma\alpha+\frac{1}{\beta}, \alpha\beta+\frac{1}{\gamma}$. 3

(*b*) If $a+b+c=0$, $a^2+b^2+c^2=42$, $a^3+b^3+c^3=105$, show that

$$(a-b)(b-c)(c-a)=\pm 63.$$ 3

GROUP – B

(*Abstract Algebra*)

[*Marks* : 35]

5. Answer any *three* questions : 8 × 3

(*a*) (*i*) If $a, b, \in Z$, not both zero, then show that there exists integers m and n such that $am + bn = gcd(a, b)$. 4

(*ii*) Let S_4 be the set of all permutations on the set {1, 2, 3, 4 }. Prove that there exists at least one element f in S_4 such that f cannot be expressed as g^4 for any $g \in S_4$. 4

(*b*) (*i*) Prove that a non-trivial finite ring having no divisor of zero is a ring with unity. 4

(*ii*) Prove that every group of order less than 6 is commutative. 4

(*c*) (*i*) Let H be a subgroup of a group G and $a \in G$. Then prove that the subset $aHa^{-1} = \{aha^{-1} : h \in H\}$ is a subgroup of G. 4

(*ii*) Prove that every group of prime order is cyclic. 4

(*d*) (*i*) Let (G, o) be a group and (H, o) be a subgroup of (G, o). Let $x, y \in G$ and a relation e is defined on G by "$x\, e\, y$ if and only if $x \text{ o } y^{-1} \in H$". Prove that e is an equivalence relation on G. 5

(*ii*) If S and T be two subrings of a ring R, then show that $S \cap T$ is a subring of R. 3

(*e*) (*i*) D is an integral domain and $a, b \in D$. If $a^m = b^m$ and $a^n = a^n$, where m, n are positive integers relatively prime, then prove that $a = b$. 5

(*ii*) Let G be a group in which $(ab)^3 = a^3 b^3$ for all $a, b, \in G$. Prove that

$$H = \{x^3 : x \in G\}$$

is a normal subgroup of G. 3

6. Answer any *two* questions : 4×2

(*a*) Prove that the number of positive primes is infinite. 4

(*b*) Prove that the order of a permutation on a finite set is the least common multiple of the lengths of its disjoint cycles. 4

(*c*) Prove that if R is a division ring, prove that $Z(R)$ defined by $Z(R) = \{x \in R : xr = rx$ for all $r \in R\}$, is a field. 4

7. Answer any *one* question : 3×1

(*a*) Define the Euler-ϕ-function. Using this function determine the number of generators of the cyclic group G of order 8 and 12 respectively. 3

(*b*) Describe the left cosets and the right cosets of H in G and find $[G : H]$ where G and H are defined as follows : G = Klein's 4-group with elements e, a, b, c and $H = \langle a \rangle$. 3

GROUP – C

(*Linear Algebra*)

[*Marks* : 25]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) Prove that

$$\begin{vmatrix} a\alpha & b\beta & c\gamma & 0 \\ b\beta & c\gamma & 0 & a\alpha \\ c\gamma & 0 & a\alpha & b\beta \\ 0 & c\gamma & b\beta & a\alpha \end{vmatrix} = \begin{vmatrix} \alpha^2 & b^2 & c^2 & 0 \\ \beta^2 & a^2 & 0 & c^2 \\ \gamma^2 & 0 & a^2 & b^2 \\ 0 & \gamma^2 & \beta^2 & \alpha^2 \end{vmatrix}$$

$$= 16s(s - a\alpha)(s - b\beta)(s - c\gamma)$$

where $2s = \alpha + \beta + \gamma$. 5

(*ii*) Prove that a linearly independent set of vectors in a finite dimensional vector space V over a field F is either a basis of V, or it can be extended to a basis of V. 5

(*iii*) Define eigenvalue of eigenvector of a matrix. Prove that it A and P be both $n \times n$ matrices P be, non-singular, then A and $P^{-1} AP$ have the same eigenvalues. 5

(*b*) (*i*) When a set of vectors in a Euclidean space is said to be orthogonal ? Prove that a orthogonal set of non-null vectors in a Euclidean space is linearly independent. 1 + 4

(*ii*) Reduce the following quadratic form to the normal form. Find the rank and signature of each

$2x^2 + 5y^2 + 10z^2 + 4xy + 12yz + 6zx.$ 5

(*iii*) Prove that a matrix is non-singular if and only if it can be expressed as the product of a finite number of elementary matrices. 5

9. Answer any *one* question : 8×1

(*a*) (*i*) S and T are subspaces of the vector space $\mathbb{R}^4$ given by

$S = \{ (x, y, z, w) \in \mathbb{R}^4 : 2x + y + 3z + w = 0\}$

$T = \{ (x, y, z, w) \in \mathbb{R}^4 : x + 2y + z + 3w = 0\}$

Find dim $(S \cap T)$. 4

(*ii*) Prove that the eigenvalues of a real symmetric matrix are all real. 4

(*b*) (*i*) Determine the conditions for which the system of equations has

(*a*) only one solution (*b*) no solution

(*c*) many solutions

$$x + y + z = b$$
$$2x + y + 3z = b + 1$$
$$5x + 2y + 9z = b^2.$$ 3

(*ii*) If S be a real skew-symmetric matrix of order n prove that

(*a*) The matrix $(I_n - S)$ is non-singular (*b*) The matrix $(I_n - S)^{-1}(I_n + S)$ is orthogonal.

(*c*) If X be an eigenvector of S with eigenvalue λ then X is also an eigenvector of $(I_n - S)^{-1}(I_n + S)$ with eigenvalue $\left(\frac{1+\lambda}{1-\lambda}\right)$. 1 + 2 + 2

10. Answer any *one* question : 2 × 1

(*a*) Define basis and dimension of a vector space. 2

(*b*) If A and B are two square matrices of order n, prove that trace (AB) = trace (BA). 2
