2016

M.Sc. Part-I Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—II

Full Marks : 100

Time : 4 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

The symbols have their usual meanings.

Write the answer to questions of each group in Separate answer booklet.

Group—A

(*Algebra*)

[Marks : 50]

Answer Q. No. 1 and any *three* from the rest.

1. Answer any *one* question : 1×2

(a) Let Z(G) be a center of a group G. If a ∈ Z(G), show that $cl(a) = \{a\}$. 2

(b) Define the following terms : Complete matching, Maximum matching. 1+1

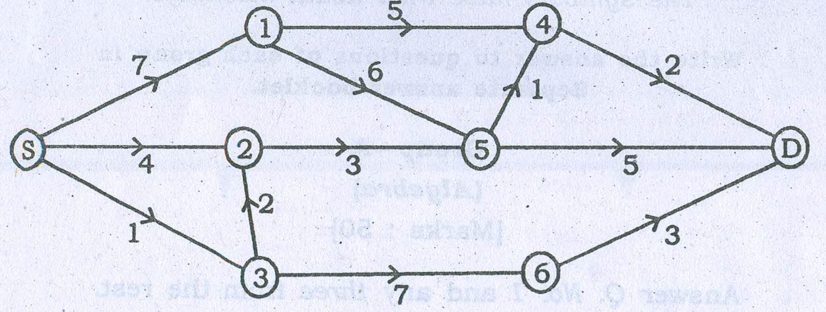
2. (a) Define Poset. Show that $[P(S), \cup, \cap]$ is a lattice, where P(S) ia power set of S. 1+4

(b) Let R be a principal ideal domain. Show that for every $a \in R$, which is not a unit, can be expressed as a product of irreducible elements. 6

(c) If $O(G) = p^2$, where p is prime number, show that G is an abelian group. 5

3. (a) The inner automorphisms of any group G form a normal subgroup of the group of all automorphisms of G. 5

(b) Write an algorithm to find the shortest path from a source vertex (S) to a destination vertex (D) of a directed weighted graph. Demonstrate the algorithm for the following graph : 3+3



(c) Show that a commutative ring with unity has no proper ideals if and only if it is a field. 5

4. (a) Show that any finite cyclic group of order n is isomorphic to Zn. 5

(b) State and prove the Sylow's first theorem for a finite group. 6

(c) Define vertex and edge connectivity of a graph. Prove that vertex connectivity of any graph G can never exceed the edge connectivity of G. 2+3

5. (a) Show that G is a direct product of subgroups H and K iff,

(i) Every $x \in G$ can be uniquely expressed as $x = hk$ for $h \in H$ and $k \in K$.

(ii) $hk = kh$ for $h \in H$ and $k \in K$. 6

(b) Let R be an Euclidian domain. Show that every ideal of R is of the form $I = Ra$ for some $a \in R$. 5

(c) Show that any connected graph with n vertices and (n–1) edges is a tree. 5

6. (a) Show that the mapping $f : C \to M_2(R)$ is defined by $f(x + iy) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ is a homomorphism of rings. Find Ker(f). 4+1

(b) Show that $e \leq 3n - 6$ holds for a simple connected planar graph with n vertices and e edges. 5

(c) State and prove the first isomorphism theorem for groups. 6

Group—B

(Functional Analysis)

[Marks : 50]

Answer *Q. No. 7* and any *three* from the rest.

7. Answer any *one* : 2×1

(a) Define Contraction mapping with an example.

(b) State with justification, whether the following statement is true or false :
Let Y be a Proper dense subspace of a Banach space X. Then Y is not a Banach space with respect to induced norm.

8. (a) Let $x = C^3$. For $x = (x(1), x(2), x(3)) \in X$,

$$\text{let } \|x\| = \left[\left(|x(1)|^2 + |x(2)|^2\right)^{\frac{3}{2}} + |x(3)|^3\right]^{\frac{1}{3}}.$$

Show that $\|\cdot\|$ is a norm onx. 6

(b) Let V, W and U be normed spaces. If $T \in B(V, W)$ and $S \in B(W, U)$, prove that $ST \in B(V, U)$ and $\|ST\| \le \|S\| \; \|T\|$. 6

(c) Check whether $C^1[0, 1]$ with the supremum norm is a Banach space or not. 4

9. (a) Define Hausdorff space.
Show that every normed space is a Hausdorff space. 6

(b) Show that a subset of a metric space is open if and only if it is the union of open balls. 4

(c) Show that for $1 \le p < \infty$,

the set $l^p = \left\{x = \{x_n\}_{n\ge1} : \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}$

is a complete metric space w.r.t. the metric 'd' defined

by $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$ for $x, y \in l^p$. 6

10. (a) Prove the Picard's theorem using Banach fixed point theorem. 8

(b) Estimate the error of the nth approximation in the Banach fixed point theorem. 5

(c) Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$T(x) = \frac{7x+9}{5}, \; x \in \mathbb{R}.$$

Find the fixed point of T using Banach fixed point theorem as a limit of the iterative sequence. 3

11. (a) Show that a real Banach space is a Hilbert space if and only if the Parallelogram law holds. 8

(b) Let F and G be subspaces of a Hilbert space H. Show that $(F+G)^{\perp} = F^{\perp} \cap G^{\perp}$. 3

(c) Let $M : l^1 \to l^1$ be the linear map defined by

$$(Mx)(i) = \sum_{j=1}^{\infty} k_{ij}\, x(j)$$

where $x = (x(1), x(2), \ldots, \ldots) \in l^1$.

Suppose $\sup\left\{\sum_{j=1}^{\infty} |k_{ij}| : i \in \mathbb{N}\right\} < \infty$.

Show that M is bounded. 5

12. (a) Let H be a Hilbert space and fix $y \in H$. Define $T(x) = \langle x, y \rangle$, $x \in H$. Find $\|T\|$. 4

(b) Let H be a Hilbert space and $T \in BL(H)$. Then show that

(i) $\|T\| = \|T^*\|$;

(ii) $\|T^*T\| = \|T\|^2$. 6

(c) Show that $T \in BL(H)$ is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$. 6