2015

M.Sc. Part-I Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—II

Full Marks : 100

Time : 4 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Write the answer to questions of each group in Separate answer booklet.

Group—A

(*Algebra*)

[Marks : 50]

Answer *Q. No. 1* and any *three* from the rest.

1. (a) Find all 2-Sylow subgroups of the symmetric group S_3. 2

(Turn Over)

(b) Define chromatic number of a graph. Find the chromatic number of an odd cycle. 1+1

(c) Show that the polynomial $x^2 - 3$ is irreducible over the field of rational numbers. 2

2. (a) Let L be a complemented distributive lattice. Show that

(i) $(a \vee b)' = a' \wedge b'$ (ii) $(a \wedge b)' = a' \vee b'$

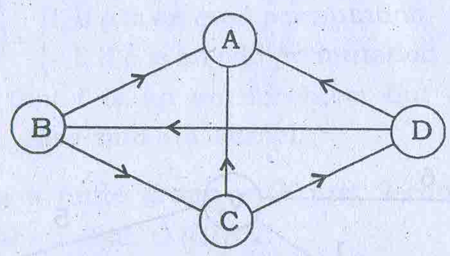
for all $a, b, \in L$. 5

(b) Let $f : R \rightarrow S$ be the homomorphism of a ring R into the ring S and let K be the kernel of homomorphism f, then prove that f (R) is homorphic with the quotient ring R/K. 6

(c) Give an example of a graph which is Euler graph but not Hamiltonion graph and an example for reverse case with proper justifications. 5

3. (a) Define sylow p-subgroup. Show that a finite group G has a unique sylow p-subgroup H if and only if H is normal in G. 1+5

(b) Define digraph. Find the incidence matrix and adjacency matrix of the following digraph : 1+2+2



(c) If G/Z (G) is cyclic then show that G is abelian. 5

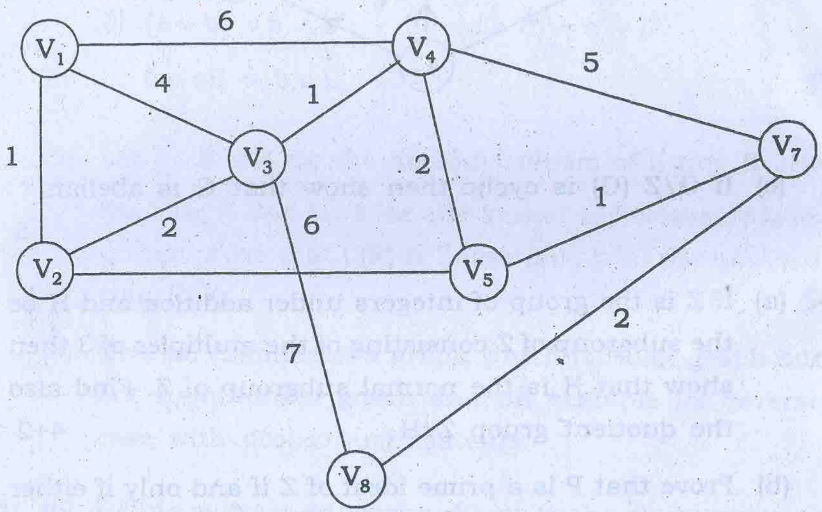
4. (a) If Z is the group of integers under addition and H be the subgroup of Z consisting of the multiples of 3 then show that H is the normal subgroup of Z. Find also the quotient gruop Z/H. 4+2

(b) Prove that P is a prime ideal of Z if and only if either P = 0 or P = pz for some prime p. 5

(c) Prove that a connected graph G is an Euler graph if and only if it can be docomposed into edge disjoint circuits. 5

5. (a) Define manimal ideal. If R is a commutative ring with unity and H is an ideal, then show that R/H is the field if and only if H is manimal ideal. 1+4

(b) Write and demonstrate the Kruskal's algorithm to find a minimal spanning tree for the following weighted graph : 3+3



(c) Show that any subgroup H of order p^{n-1} in a group G of order of order p^n, is normal in G. 5

6. (a) Let $S = \{1, -1\}$ and $G = (S, \cdot)$ be a multiplicative group. Define a mapping $f : S_3 \to G$ (S_3 is the symmetric group of the symbol $\{1, 2, 3\}$,

by $f(\rho) = \begin{cases} 1, \text{ if } \rho \text{ is an even permutation} \\ -1, \text{ if } \rho \text{ is an odd permutation} \end{cases}$

Show that f is an epimorphism but not monomorphism. Also find its kernel. 4+1

(b) If G is a finite group with just 2-conjugate classes then show that $O(G) = 2$. 5

(c) Define 'lattice' and 'distributive lattice'. Show that $b = c$ in distributive lattice $(L, \wedge, \vee)$, where $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ for all $a \in L$. 1+1+4

Group—B

(Functional Analysis)

[Marks : 50]

Answer *Q. No. 7* and any *three* from the rest.

7. Answer any *one* : 2×1

(a) Define continuity of a function f between two metric spaces.

(b) Define complete orthogonal set in an inner product space X.

8. (a) Define seperable metric space with example. Show that the space ℓ^p, $1 \le p < \infty$ is seperable.

(b) Let (X, d_1) and (Y, d_2) be metric spaces. If $f : X \to Y$ be continuous and X be compact, then show that f is uniformly continuous.

(c) Show that every conpact metric space is seperable.

7+6+3

9. (a) State and Prove Banach fixed point theorem.

(b) A mapping $T : [a, b] \to [a, b]$ is said to satisfy a Lipschitz condition with a Lipschitz constant k on [a, b] if there is a constant k such that for all $x, y \in [a, b]$, $|Tx - Ty| \le k|x - y|$.

(i) Is T a contraction ?

(ii) If T is continuously differentiable, show that T satisfies a lipschitz condition.

(iii) Does the converse of (b) hold ?

(c) Consider the integral operator $T : C[0, 1] \to C[0, 1]$ be defined by $Tx(t) = \int_0^1 k(t, s)\, x(s)\, ds$ where $k(t, s)$ is a given continuous function on the closed square $[0, 1] \times [0, 1]$. Show that T is linear and bounded.

7+5+4

10. (a) Let X = C [0, 1] with the supremum norm. Consider the sequence $x_n(t) = \frac{t^n}{n^2}$, $t \in [0, 1]$. Check whether the series $\sum_{x=1}^{\infty} x_n$ is summable in X.

(b) Suppose $X = C^1$ [0, 1,] i.e. the set of all functions $f : [0, 1] \to \mathbb{R}$ such that f' exists and is continuous. Let Y = c [0, 1] and let X and Y be equipped with Supremum norm. Define $A : X \to Y$ by $Af = f'$. Show that the graph of A is closed.

(c) Let X be a normed and Y be a Banach Space. Then show that B(X, Y) is a Banach Space. 4+6+6

11. (a) State and prove uniform Boundedness theorem.

(b) If $\{e_1, e_2, \ldots e_n\}$ is a finite orthonormal set in an inner product space X and $x \in X$, then show that

$\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \le \|x\|^2$ and $x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i$ is orthogonal to e_j for all j = 1, 2,...n.

(c) Let X be a real normed linear space and suppose f (x) = 0, for all $f \in X^*$. Show that $x = 0$. 6+6+4

12. (a) Let M be a closed subspace of a Hilbert space H and $x \in H$. Then how that there exists unique $y \in M$ and $Z \in M^{\perp}$ such that $x = y + z$.

(b) Let X be an inner product space and $A, B \subset X$.

Then show that (i) $A \subseteq B \Rightarrow B^{\perp} \subseteq A^{\perp}$,

(ii) $A \subseteq A^{\perp\perp}$,

(iii) $A^{\perp} = A^{\perp\perp\perp}$.

(c) If $\| x + \lambda y \| = \| x - \lambda y \|$ is true for all scalar λ, then show that $x \perp y$. Is the converse true ?

7+6+3