2015

M.Sc. Part–I Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—I

Full Marks : 100

Time : 4 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Write the answer to questions of each group in Separate answer booklet.

Group—A

(*Real Analysis*)

[Marks : 40]

Answer *Q. No. 1* and any *three* from *Q. No. 2* to *Q. No. 6.*

1. Answer any *one* of the following : 1×1

(a) If outer measure of a set A is zero, show that its measure is zero.

(b) Define Lebesgue integral for unbounded functions.

2. (a) Let $f : [a, b] \to |\mathbb{R}|$ be a function of bounded variation on [a,c] and on [c, b] where $c \in (a, b)$. Then show that
(i) f is of bounded variation on [a, b], and
(ii) $V_f[a, c] + V_f[c, b] = V_f[a, b]$.

(b) Let $f(x) = |x|$, $x \in [-1, 2]$. Show that f is a function of bounded variation on [−1, 2]. Calculate the positive variation, the negative variation and the total variation of f on [−1, 2].

(c) Give an example of a function f which is continuous on [0, 1] but is not a function of bounded variation on [0, 1] 6+5+2

3. (a) If f is Riemann-Stieltjes integrable w.r.t. α over [a, b], then show that α is also Riemann–Stielfjes integrable w.r.t. f over [a, b], Also, show that the two integrals are related as

$$\int_a^b f d\alpha + \int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a)$$

(b) Let $f(x) = \begin{cases} 0, & 2 \le x < 3 \\ 1, & x = 3 \\ 0, & 3 < x \le 8. \end{cases}$ and

$\alpha(x) = 2x^2 + 5$

Show that f is Riemann–Stieltjes integrable w.r.t. α on [2, 8] Also, find $\int_2^8 f d\alpha$.

(c) Evaluate : $\int_{-1}^{2} (x^2 + e^x) d([x] + 5)$ 5+5+3

4. If A_1, A_2, A_3....are pairarise disjoint measurable subsets of [a, b], then show that $\bigcup_{n=1}^{\infty} An$ is measurable.

Also, show that $m\left(\bigcup_{n=1}^{\infty} An\right) = \sum_{n=1}^{\infty} m(An)$.

(b) Let f, g, be measurable functions on [a, b]. Then show that eath of the sets :

$\{x \in [a, b] | f(x) > g(x)\}$, $\{x \in [a, b] | f(x) \le g(x)\}$,

$\{x \in [a, b] | f(x) < g(x)\}$, $\{x \in [a, b] | f(x) \ge g(x)\}$, and

$\{x \in [a, b] | f(x) = g(x)\}$ are all measurable subsets of [a, b].

(c) Prove that the measure of the cantor set is zero. 6+4+3

5. (a) Show that a bounded function f on [a, b] is Lebesgne integrable on [a,b] if and only if for every $\epsilon > 0$, there exists a measurable partition P of [a, b] such that $U(P, f) - L(P, f) < t$.

(b) Show that the function f (x) defined by

$f(x) = 3$ for rational x in [−2, 8]

$= 4$ for irrational x in [−2, 8]

is not R integrable over [−2, 8] but L-integrable over [−2, 8].

(c) If A_1, A_2, A_3 are measurable sets of [a, b], then prove that

$$m(A_1 \cup A_2 \cup A_3) = m(A_1) + m(A_2) + m(A_3) - m(A_1 \cap A_2)$$
$$- m(A_1 \cap A_3) - m(A_2 \cap A_3) + m(A_1 \cap A_2 \cap A_3).$$

6. (a) Verify Bounded Convergence Theorem for the sequence of functions $f_n(x) = \frac{5}{\left(2 + \frac{3x}{n}\right)^n}, 0 \le x \le 1,$ $n = 1, 2, 3.....$

(b) State the following theorems :

(i) Lebesgue Dominated Convergence Theorem,

(ii) Fatou's Lemma.

(c) Prove that every bounded measurable function on [a,b] is Lebesgue integrable over [a, b].

5+4+4

Group—B

(Complex Analysis)

[Marks : 30]

Answer all questions.

7. Answer any *two* question : 2×2

(a) Find the residue of $\frac{z^2}{z^2 + a^2}$ at $z = ia$. 2

(b) Show that the function $f(z) = e^{\frac{1}{z}}$ has isolated essential singularity at $z = 0$. 2

(c) Find the value of the integral $\int \frac{1}{z} dz$ round a circle whose equation is $|z| = \rho (> 0)$ 2

8. Answer any *four* questions : 4×5

(a) Show by an example that a function $f(z) = u(x, y) + iv(x, y)$ ceases to be differentiable at the point (x_0, y_0) in the domain C even if the Cauchy – Riemann equations are satisfied. 5

(b) Show that an analytic function with constant modulus is constant.

(c) Find the Laurent Expansion of

$$f(z) = \frac{1}{z^2(1-z)}$$ in the region

(i) $0 \langle |z| \langle 1$ (ii) $1 \langle |z| \langle \infty$

(d) Find all the Mobius' transformations which transform the unit circle $|z| \leq 1$ in the z-plane into unit circle $|w| \leq 1$ in the w-plane. 5

(e) Find and discuss the nature of the singularities of

$$f(z) = \frac{z-2}{z^2} \sin \frac{1}{z-1}$$

(f) If f(z) is analytic within and on a closed contour C except at a finite number of poles and is not zero on C, then $\frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z)} dz = N - P$, where N is the number of zeros and P is the number of poles inside C. 5

9. Answer any *one* question : 1×6

(a) Show by the method of contour integration that

$$\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta} = 0.$$ 6

(b) By Calculas residues show that

$$\int_{-\infty}^{\infty} \frac{x \operatorname{Cos} x}{x^2+1} dx = 0$$ 6

Group—C

(Ordinary Differential Equations)

[Marks : 30]

Answer any *two* questions.

10. (a) Using the representation

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} \left\{ \left(z^2 - 1\right)^n \right\},$$

Prove that, $\int_{-1}^{1} P_n(z) P_m(z)\, dz = \begin{cases} 0, \text{ for } n \neq m \\ \frac{2}{2n+1}, \text{ for } n = m \end{cases}$

(b) Find the general solution of the ODE

$2zw''(z) + (1+z)w'(z) - kw(z) = 0$ (Where k is a real constant) in series form. For which value of k there is polynomial solution ? 6

(c) Define INDICIAL equation concerning ODE ? 2

11. (a) If λ_1, λ_2, ...are the positive zeros of the Bessel's function Jn (z), then prove that,

$$\int_0^1 z\, J_n(\lambda_m z)\, J_n(\lambda_p z) = \begin{cases} 0, \text{if } m \neq p \\ \frac{1}{2} J_{n+1}^2(\lambda p), \text{if } m = p \end{cases}$$ 7

(b) Deduce the integral formula for hypergeometic function. 4

(c) If $z > 1$, show that $P_n(z) < P_{n+1}(z)$ 4

12. (a) Prove that if f(z) is continuous and has continuous derivatives in [–1, 1] then f (z) has unique Legendre series expansion and is given by

$$f(z) = \sum_{n=0}^{\infty} C_n P_n(z)$$

Where P_n's are Legendre Polynomials and

$$C_n = \frac{2n+1}{2} \int_{-1}^{1} f(z) P_n(z)\, dz, \ n = 1, 2, 3, \ldots$$ 6

(b) Show that $J_0^2(z) + 2\sum_{n=1}^{\infty} J_n^2(z) = 1$ and prove that for real z, $|J_0(z)| \leq 1$, and $|J_n(z)| < 1/\sqrt{2}$, for all $n \geq 1$. 4

(c) Prove the following :

(i) $\left[J_{1/2}(z)\right]^2 + \left[J_{-1/2}(z)\right]^2 = \left(2/\pi z\right)$

(ii) $J_{3/2}(z) = \sqrt{\frac{2}{\pi z}}\left[\frac{1}{Z}\sin z - \cos z\right]$
