2014

M.Sc. Part–I Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—I

Full Marks : 100

Time : 4 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Write the answer to questions of each group in Separate answer booklet.

Group—A

(*Real Analysis*)

[Marks : 40]

Answer *Q. No. 1* and any *three* from *Q. No. 2* to *Q. No. 6.*

1. Answer any *one* of the following : 1×1

(a) Define Riemann-Stieltjes integral for a bounded function f over [a, b].

(b) Define inner measure for any bounded subset of [a, b].

2. (a) Show that the function $f : [a, b] \to \mathbb{R}$ is a function of bounded variation on [a, b] if and only if f can be expressed as the difference of two monotone increasing functions on [a, b]. 6

(b) Let $x_1, x_2 \ldots, x_n$ be an enumeration of all rational points in [0, 1] and let $f : [0, 1] \to \mathbb{R}$ be defined by :

$$f(x_n) = \frac{1}{n^2},\ n = 1, 2, 3, \ldots$$
$$= 0, \text{ elsewhere.}$$

Prove that f is a function of bounded variation on [0, 1]. 4

(c) Show that every function $f : [a, b] \to \mathbb{R}$ of bounded variation is bounded on [a, b]. 3

3. (a) Show that f is Riemann-Stieltjes integral on [a, b]. w.r.t. the monotonic increasing function α if and only if for every $\in > 0$ there exists a partition P of [a, b] such that $U(P,f,\alpha) - L(P,f,\alpha) < \in$. 6

(b) Evaluate : $\int_2^4 (x - [x])\,d(x^3)$. 4

(c) Show that any given finite sum can be expressed as Riemann-Stieltjes integral. 3

4. (a) If $A \subset [a, B]$, then show that $m^*(A) + m_*(A) = (b - a)$, where $A^C = [a, b] - A$. and $m^*(A)$ and $m_*(A)$ denotes respectively the outer measure and inner measure of A. 5

(b) Define a real-valued measurable function f on [a, b]. If $\{f_n\}$ is a sequence of measurable functions on [a, b] such that the sequence $\{f_n(x)\}$ is bounded for every $x \in [a, b]$, then show that :

$\sup_{n \leftarrow \mathbb{N}} f_n(x)$, $\inf_{n \leftarrow \mathbb{N}} f_n(x)$ are measurable. 1+4

(c) Let E = (1, 2) and $E_n = \left(\frac{1}{2^n} + n,\ \frac{1}{2^{n-1}} + n\right)$. Give reasons to justify that $\bigcup_{n=1}^{\infty} E_n$ is measureable. 3

5. (a) Show that every bounded Riemann integrable function is Lebesgue integrable and the two integrals are equal. 7

(b) (i) If f(x) is bounded and Lebesgue integrable on [a, b] and if f(x) = 0 i.e. on [a, b], then show that $\int_a^b f(x)dx = 0$. 4

(ii) Define Lebesgue integral for unbounded function. 2

6. (a) (i) State the following theorems : 4

Lusin's theorem, Egoroff's theorem.

(ii) Show that the set of even integers has measure zero. 4

(b) Let f(x) be defined on [0, 1] as follows :

$f(x) = \frac{1}{2x}$, if $0<x \le 1$

$= 8$ if $x = 0$.

Show that f(x) is not Lebesgue integrable on [0, 1]. 5

Group—B

(*Complex Analysis*)

[Marks : 30]

Answer any *two* questions.

7. (a) If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$, then show that at z, $u_x = v_y$ and $u_y = -v_x$. 5

(b) Determine a function such that the function $f = u + iv$ is analytic on C, the complex plane, where :

$u(x, y) = x(1 - y)$. 5

(c) Use Rouches theorem to find the number of zeros of :

$z^{10} + a_1z^4 + a_2z^3 + a_3z^2 + a_4z + a_5 = 0$

in $|z| = 1$, if $|a_1| > |a_2| + |a_3| + |a_4| + |a_5| + 1$. 5

8. (a) Show that, under suitable conditions, to be stated by you,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a},$$

where C is a closed contour surrounding the point $z = a$. 1+4

(b) (i) Investigate the nature of singularities of the function :

$$f(z) = \frac{z-2}{(z^2-1)^4}\cot\left(\frac{1}{1-z}\right).$$

(ii) Prove that the poles of an analytic function are isolated. 3+2

(c) If $f(z) = \frac{3z-3}{(2z-1)(z-2)}$, find a Laurent series of f(z) about $z = 1$ convergent for $\frac{1}{2} < |z-1| < 1$. 5

9. (a) Find all the Möbius Transformation which transform the half plane $I(z) \geq 0$ onto the unit circular disc $|W| \leq 1$. 5

(b) Evaluate any *two* of the following by the method of contour integration : 5+5

(i) $\int_0^{2\pi} \frac{d\theta}{7+5\cos\theta}$; (ii) $\int_0^{\infty} \frac{x \cos x dx}{x^2+1}$;

(iii) $\int_{-\infty}^{\infty} \frac{\cos x dx}{a^2+x^2}$; (iv) $\int_0^{\infty} \frac{dx}{(1+x^2)^2}$.

Group—C

(Ordinary Differential Equations)

[Marks : 30]

Answer any *two* questions.

10. (a) Find the general solution of the equation :

$2z(1-z)\,\omega''(z) + \omega'(z) + 4\omega(z) = 0$

by Frobenius method about $z = 0$ and show that the equation has a solution which is polynomial in z. 6

(b) Show that when n is a positive integer, $J_n(z)$ is the co-efficient of t^n in the expansion of $e^{\frac{z}{2}(t-yt)}$ in ascending and descending power of t. Also show that $J_{-n}(z)$ is the coefficient of t^{-n} multiplied by $(-1)^n$ in the expansion of the above expression.

Use it or otherwise, prove the following :

$Z\, J_n'(z) = z\, J_{n-1}(z) - n\, J_n(z)$. 6

(c) Under a suitable transformation to be considered by you, prove that hypergeometric function can be reduced to confluent hypergeometric function. 3

11. (a) Find the Legendre polynomial $P_n(z)$ by solving the following differential equation :

$$\left(1-z^2\right)\frac{d^2\omega}{dz^2} - 2z\frac{d\omega}{dz} + n(n+1)\omega = 0 \,.$$ 6

(b) What do you mean by Bessel's functions of order n ? State for what values of n the solutions are independent of Bessel's equation of order n. 3

(c) What are meant by ordinary point and regular singular point of the differential equation :

$$a_0(z)\frac{d^2w}{dz^2} + a_1(z)\frac{dw}{dz} + a_2(z)w = 0 \,.$$ 3

(d) Show that : $\sqrt{\frac{\pi z}{2}}\, J_{\frac{3}{2}}(z) = \frac{1}{z}\sin z - \cos z$. 3

12. (a) Show that $\left(1-2zt+t^2\right)^{-\frac{1}{2}}$, $|z|<1$, $|t|<1$ is the generating function of the Legendre's polynomial. Hence show that :

$$(2n+1)z\,P_n(z)=(n+1)\,P_{n+1}(z)+n\,P_{n-1}(z).$$ 7

(b) Locate and classify the singular points of the following differential equation :

$$z^2\left(z^2-1\right)^2\omega''-z(1-z)\omega'+2\omega=0.$$ 3

(c) Establish Rodrignes formula for Legendre's polynomial. 5
