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A SIMPLE MEASURE OF DURATION OF DEFAULT RISK-FREE COUPON BONDS

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Abstract

Investments in fixed-income default risk-free coupon bonds are not free from systematic risks arising out of fluctuations in the rate of interest and inflation. Possible fluctuation in the rate of return arising from fluctuations in the rate of interest is known as interest rate risk. Interest rate risk has two components viz., price risk and reinvestment rate risk. When price falls due to increase in interest rate, there is gain from reinvestment of intermediate cash inflows and vice versa. So, change in the interest rate has opposite effects on price and reinvestment of intermediate cash inflows. So, there may exist a point of time at which these effects may off-set each other. This break-even point of time is known as duration. This paper finds a measure of duration which is very easy to calculate yet gives an accurate result.

Key words : Duration, Interest rate risk, Price risk, Reinvestment rate risk, Default risk-free coupon bonds, zero-coupon bonds

Introduction

Investors have various alternatives with respect to investment in fixed-income securities like fixed deposits in post offices and banks, coupon bonds/debentures, zero-coupon bonds etc. Investments in these instruments are associated with risks. Investments in fixed-income securities issued by entities other than Government have systematic as well as unsystematic risks. Generally, investments in the securities issued by the Government do not have any default risk. So, throughout this paper, we have assumed that investments in the bonds issued by the Government are free from any default risk. But investments in fixed-income securities issued by Government entail systematic risk arising from unexpected shocks in interest rates and inflation.

Unexpected change in the rates of interest creates systematic risk for investments in bonds which are issued by the Government and are traded in the secondary market. The

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possibility that actual rate of return would be lower than the expected rate of return due to unexpected change in the rates of interest is known as interest rate risk which has two components viz., price risk and reinvestment rate risk. If there is an unexpected increase in the rates of interest, the price of the bond is likely to fall immediately and vice versa. The possibility that actual rate of return would be lower than the expected rate of return due to fall in price is known as price risk. But when rates of interest increase, investors gain from the reinvestment of intermediate cash inflows and vice versa. The possibility that the actual rate of return would be lower than the expected rate return due to loss from reinvestment of intermediate cash inflows is known as reinvestment rate risk. Thus, unexpected change in the rates of interest in one-direction is likely to have opposite effects on the price of the security and the reinvestment of intermediate cash inflows.

Bonds issued by the Government can be broadly divided into two types: Coupon bonds and Zero-coupon bonds. Zero-coupon bonds are issued at a discount and are redeemed at face values and so there are no interest payments and hence there is no reinvestment rate risk although price risk exists before maturity. In case of coupon bonds, there are reinvestment rate risks since interests are paid periodically and price risk before maturity.

Since unexpected change in the rates of interest is likely to have opposite effects on the price and reinvestment of intermediate cash inflows, it is possible to find out a break-even holding period for such bonds. The objective of this paper is to find a simple measure of that break-even holding period for the default risk-free coupon bonds.

Let us first see how the fair values of default risk-free fixed coupon bonds are determined. In this context, it is pertinent to note that at equilibrium, fair value equals the price of such bonds.

Valuation of default risk-free fixed-coupon bonds

The default risk-free fixed-coupon bonds are valued as follows (assume that interests are paid annually at the end of each year) –

$$B_0 = I\,(1 + i)^{-1} + I\,(1+ i)^{-2} + \ldots + I\,(1 + i)^{-n+1} + I\,(1 + i)^{-n} + F\,(1 + i)^{-n} \qquad \text{— (1)}$$

Where,

B_0 = Current price of the default risk-free bond issued by the Government.

I = Amount of annual fixed interest.

n = Term-to-maturity (in years) of the bond.

i = The expected risk-free rate of return per rupee required by the investor over the life of the bond.

Multiplying both sides with (1+i) we get –

$$B_0 (1+i) = I + I (1 + i)^{-1} + I (1 + i)^{-2} + \ldots + I (1 + i)^{-n+1} + F (1 + i)^{-n+1} \quad \text{— (1a)}$$

Subtracting (1) from (1a) we get –

$$B_0 (1 + i - 1) = I - I (1 + i)^{-n} - F (1 + i)^{-n} + F (1 + i)^{-n+1}$$

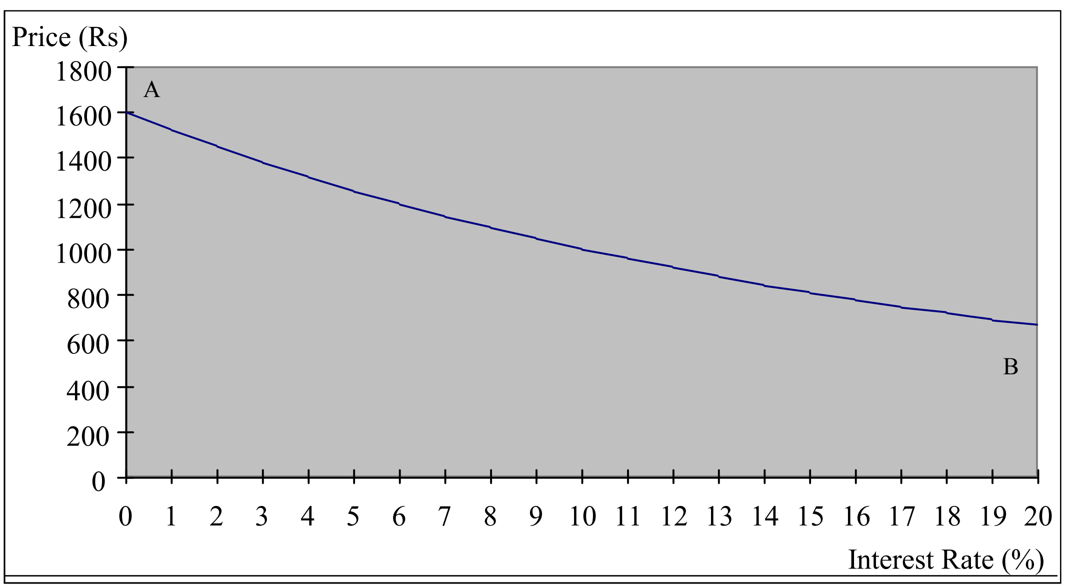
$$\text{Or, } B_0 = I \{1 - (1 + i)^{-n}\} i^{-1} - F (1 + i)^{-n} (1-1-i) i^{-1}$$

$$\text{Or, } B_0 = I i^{-1} \{1 - (1 + i)^{-n}\} + F (1 + i)^{-n} \quad \text{— (1b)}$$

Let us take an example of a default risk-free coupon-bond which has a term to maturity of 6 years at the end of which it will be redeemed at Rs 1,000. Annual interest of Rs 100 is payable at the end of each year. It is assumed that the interest rate may change but after the change the term structure of interest rates will remain flat (i.e., interest rates applicable to bonds having different maturities will be the same). We have computed the price of the bond at various interest rates. The relationship between the price of the bond and the interest rate is shown in Figure – 1.

Figure – 1 shows that as interest rate increases price of the bond falls and so the relationship is represented by a downward slopping curve, AB. As the rate of interest increases the slope of the curve AB also falls implying that change in price falls as interest rate increases. Thus, volatility in the price of the bond depends on the level of interest rate and change in interest rate. So, the slope of the curve and volatility are different at different rate of interest.

Figure - 1: Price and Interest Rate



Now, multiplying the right-hand side of equation (1) with $(1+i)^n / (1+i)^n$ we get,

$$B_0 = [\, I\,(1+i)^{n-1} + I\,(1+i)^{n-2} + \ldots + I\,(1+i) + I + F \,]\,(1+i)^{-n} \qquad \text{— (2)}$$

But the rate of interest prevailing at the time when the intermediate cash inflows are generated can be different. Moreover, the rate of interest applicable to investments in Government bonds having different maturities can also be different. Thus, we can rewrite the equation (2) as follows –

$$B_0 = [\, I\,(1+i_1)^{n-1} + I\,(1+i_2)^{n-2} + \ldots + I\,(1+i_{n-1}) + I + F \,]\,(1+i)^{-n} \qquad \text{— (3)}$$

Here, i_t stands for the expected risk-free rate of interest per rupee prevailing at time t and applicable to bonds having maturity of n-t.

Valuation of default risk-free zero-coupon bonds

The default risk-free zero-coupon bonds are valued as follows: $Z_0 = F\,(1+i)^{-n}$ ---- (4)
Here Z_0 stands for current price of the default risk-free bond issued by the Government. Now let us see the consequences of changes in the rates of interest.

Interest rate risks

When there is an increase in the rates of interest, the existing bonds become less lucrative as compared to the new bonds which offer higher rate of interest. As a result, the prices of the existing bonds decline in order to match the yield-to-maturity on such bonds with the yield-to-maturity of the new bonds. Let us see the effect of increase in the rates of interest on the valuation of coupon bonds. Using equation (3), we get –

$$P_0 = [\, I\,(1+r_1)^{n-1} + I\,(1+r_2)^{n-2} + \ldots + I\,(1+r_{n-1}) + I + F \,]\,(1+r)^{-n} \qquad \text{— (5)}$$

Where,
P_0 = Current price of the bond after the increase in the rates of interest.
r_t = The new expected risk-free rate of interest per rupee prevailing at time t and applicable to bonds having maturity of n-t.
r = The new expected risk-free rate of return per rupee required by the investor over the life of the bond.

If there is an upward shift in the rates of interest, the average rate of interest required by the investor over the life of the bond will also increase. As a result, the price of the existing coupon bond will fall (i.e., $P_0 < B_0$). Since, in this case, there will be an increase in the reinvestment rates, an investor gains from reinvestment of intermediate cash inflows. So, the investor suffers from fall in price but gains from increase in the reinvestment rates at which intermediate cash inflows will be reinvested. The outcomes will be opposite if there is a downward shift in the interest rates. If the coupon bond is held till its maturity, there is no price risk but the investor has to bear the reinvestment risk. In case of zero-coupon bond, there is no reinvestment rate risk but price risk exists before maturity although at maturity, there is no price risk.

Since price risk and reinvestment rate risk have opposite effects, investors can immunize themselves from price risk and reinvestment risk arising out of fluctuations in the interest rates.

Strategy to counter the interest rate risks

In case of investment in a default risk-free zero-coupon bond, price risk arises only if the bond is sold before its maturity. Moreover, if the zero-coupon bond is sold before the maturity and the proceeds are reinvested, the investor faces the reinvestment rate risk. In order to avoid these risks, an investor should choose such default risk-free zero-coupon bonds the maturity of which matches with his investment horizon. In that case, the yield-to-maturity from investment in such bond will match his initial required rate of return.

Price risk and reinvestment rate risk are associated with investment in default risk-free coupon-bonds. But we have already seen that when there is fall in price, there is gain from reinvestment and vice versa. Thus, in that case, there exist a point of time (which is called the duration) at which these two effects will off-set each other. So, duration is a point of time at which the actual wealth of the bondholder matches with the desired wealth. It can also be defined as a point of time at which gain (loss) from price increase matches with loss (gain) from reinvestment of intermediate cash inflows. Hence, if the investment horizon of the investor matches with the duration of the default risk-free coupon-bond, there will be no price risk and reinvestment rate risk (Fischer and Jordan, 2007). Let us now derive the duration of a default risk-free coupon-bond.

Determination of actual wealth, desired wealth and a measure of duration

Let **m** be the duration of the coupon-bond. At **m** point of time the actual wealth will be equal to the desired wealth (the desired wealth generates a rate of return of i).
Now, equation (5) can be rewritten as follows –

$$P_0 = [\, I(1+r_1)^{n-1} + I(1+r_2)^{n-2} + \dots\dots\dots + I(1+r_{m-1})^{n-m+1} + I(1+r_m)^{n-m} + I(1+r_{m+1})^{n-m-1} + \dots\dots\dots + I(1+r_{n-1}) + I + F\,]\,(1+r)^{-n}$$

$$= [\, I(1+r_1)^{n-1} + I(1+r_2)^{n-2} + \dots\dots + I(1+r_{m-1})^{n-m+1} + I(1+r_m)^{n-m}\,]\,(1+r)^{-n} + [\, I(1+r_{m+1})^{n-m-1} + \dots\dots + I(1+r_{n-1}) + I + F\,]\,(1+r)^{-n} \qquad \text{— (6)}$$

Let us assume a flat term structure of interest rates and consequently assume that $r_1 = r_2 = \dots = r_m = \dots = r_n = r$. Then, we get –

$P_0 = [\ I\ (1 + r)^{n-1} + I\ (1 + r)^{n-2} + \ldots\ldots\ldots + I\ (1 + r)^{n-m+1} + I\ (1 + r)^{n-m}\ ]\ (1+r)^{-n} + [\ I\ (1 + r)^{n-m-1} + \ldots + I\ (1 + r) + I + F\ ]\ (1 + r)^{-n}$ — (7)

Now, let us assume that the intermediate cash inflows will be reinvested till the end of **m** and at that point of time the bond will be sold in the market.

Multiplying both sides with $(1+ r)^m$ we get –

$P_0\ (1 + r)^m = [\ I\ (1 + r)^{n-1} + I\ (1 + r)^{n-2} + \ldots + I\ (1 + r)^{n-m+1} + I\ (1 + r)^{n-m}]\ (1 + r)^{-n}\ (1 + r)^m + [\ I\ (1 + r)^{n-m-1} + \ldots + I\ (1 + r) + I + F\ ]\ (1 + r)^{-n}\ (1 + r)^m$

$= [\ I\ (1+ r)^{n-1-n+m} + I\ (1+ r)^{n-2-n+m} + \ldots + I\ (1+ r)^{n-m+1-n+m} + I\ (1 + r)^{n-m-n+m}\ ] + [\ I\ (1 + r)^{n-m-1-n+m} + \ldots + I\ (1 + r)^{1-n+m} + I\ (1 + r)^{-n+m} + F\ (1+ r)^{-n+m}\ ]$

$= [\ I\ (1 + r)^{m-1} + I\ (1 + r)^{m-2} + \ldots + I\ (1 + r) + I\ ] + [\ I\ (1 + r)^{-1} + \ldots + I\ (1 + r)^{-n+m+1} + I\ (1 + r)^{-n+m} + F\ (1 + r)^{-n+m}\ ]$

Let P_m be the actual price of the bond at **m** point of time. Where $P_m = [\ I\ (1 + r)^{-1} + \ldots + I\ (1 + r)^{-n+m+1} + I\ (1 + r)^{-n+m} + F\ (1 + r)^{-n+m}\ ]$

Then, we get –

$P_0\ (1 + r)^m = [\ I\ (1 + r)^{m-1} + I\ (1 + r)^{m-2} + \ldots + I\ (1 + r) + I\ ] + P_m$ — (8)

Thus, **P_0 (1 + r)^m indicates the actual wealth of the investor at time m**. But we know that the reinvestment rates may be different. So, we can rewrite the above equation as follows –

$P_0\ (1 + r)^m = [\ I\ (1 + r_1)^{m-1} + I\ (1 + r_2)^{m-2} + \ldots + I\ (1 + r_{m-1}) + I\ ] + P_m$ — (9)

Now, let us find the desired wealth of the investor at **m**. Let us again assume that $i_1 = i_2 = \ldots = i_m = \ldots = i_n = i$. Now, equation (2) can be rewritten as follows –

$B_0 = [I\ (1 + i)^{n-1} + I\ (1 + i)^{n-2} + \ldots + I\ (1 + i)^{n-m+1} + I\ (1 + i)^{n-m} + I\ (1 + i)^{n-m-1} + \ldots + I\ (1 + i) + I + F\ ]\ (1 + i)^{-n}$

Multiplying both sides with $(1 + i)^m$ we get –

$B_0\ (1 + i)^m = [\ I\ (1 + i)^{m-1} + I\ (1 + i)^{m-2} + \ldots + I\ (1 + i) + I\ ] + [\ I\ (1 + i)^{-1} + \ldots + I\ (1 + i)^{m+1-n} + I\ (1 + i)^{m-n} + F\ (1 + i)^{m-n}\ ]$ —(10)

Let $B_m = [\ I\ (1 + i)^{-1} + \ldots + I\ (1 + i)^{m+1-n} + I\ (1 + i)^{m-n} + F\ (1 + i)^{m-n}\ ]$.

Then we get –

$B_0\ (1+i)^m = [\ I\ (1 + i)^{m-1} + I\ (1 + i)^{m-2} + \ldots + I\ (1 + i) + I\ ] + B_m$ —(11)

The desired wealth is represented by B_0 (1+i)^m.

According to our definition of duration (i.e., **m**), we can write –

Actual wealth at m = Desired wealth at m

Or, $P_0\ (1+ r)^m = B_0\ (1+i)^m$ — (12)

Dividing both sides with $B_0 (1 + r)^m$, we get –

Or, $P_0 / B_0 = [(1 + i) / (1 + r)]^m$

Or, (Actual Price at t=0) / (Desired Price at t=0) = {(1+ Desired return per rupee)/ (1+Actual return per rupee)} ^{m}

Taking natural logarithm (denoted by ln) of both sides we get –

Or, $\ln (P_0 / B_0) = m \ln \{(1 + i) / (1 + r)\}$

Or, $m = [\ln (P_0 / B_0)] / [\ln \{(1 + i) / (1 + r)\}]$ — (12a)

Using equation (12a) we can find duration very easily. If the investment horizon and the duration of the bond match, there will be no interest rate risk. But the problem with equation (12a) is that when there is no change in the rate of interest, **m** cannot be determined. In that case, we can take the average of the durations at the rates of interest which precede and follow the desired rate of interest.

Duration and term structure of interest rates

While deriving equation (12a) we have assumed that the rates of interest applicable to investments in bonds having different maturities are the same. Thus, we have assumed that there may be a change in the level of interest rate but the new term structure of interest rates will again be flat. When the term structures of interest rates are not flat, the calculation becomes too tedious. Replacing P_0 from equation (5) and B_0 from equation (3) in equation (12a) we get –

$m = [\text{Ln} [\{ I (1 + r_1)^{n-1} + I (1 + r_2)^{n-2} + \ldots + I + F \} (1 + r)^{-n}] - \text{Ln} [\{ I (1 + i_1)^{n-1} + I (1 + i_2)^{n-2} + \ldots + I + F \} (1 + i)^{-n}]] / \text{Ln} \{(1 + i) / (1 + r)\}$ —(13)

Equation – (13) shows the complexity associated with the computation of duration when interest rates applicable to bonds having different maturities vary.

Let us show the computation of duration assuming that the term structures of interest rates will be flat.

Example

Let us consider the numerical example mentioned earlier. The default risk-free coupon-bond has a term to maturity of 6 years. The redemption price at maturity is Rs 1,000. Annual interest of Rs 100 is payable at the end of each year. The initial required rate of return (or interest) of the investors was 10%. The current rate of interest is 9%. The term structures of interest rates are assumed to be flat.

Table – 1

Computation of Actual Wealth, Desired Wealth and Duration

Rs.

Particulars	At the end of year						
	0	1	2	3	4	5	6
Annual interest		100	100	100	100	100	100
Repayment of principal							1000
Cash inflows		100	100	100	100	100	1100
Computation of actual wealth							
Current rate of interest = 9%							
Reinvestment rates		9%	9%	9%	9%	9%	9%
Actual price	1044.86	1038.90	1032.40	1025.31	1017.59	1009.17	1000.00
Accumulated interest	0.00	100.00	209.00	327.81	457.31	598.47	752.33
Actual wealth	**1044.86**	**1138.90**	**1241.40**	**1353.12**	**1474.90**	**1607.64**	**1752.33**
Computation of desired wealth							
Desired rate of interest = 10%							
Desired reinvestment rates		10%	10%	10%	10%	10%	10%
Desired price	1000.00	1000.00	1000.00	1000.00	1000.00	1000.00	1000.00
Accumulated interest	0.00	100.00	210.00	331.00	464.10	610.51	771.56
Desired wealth	**1000.00**	**1100.00**	**1210.00**	**1331.00**	**1464.10**	**1610.51**	**1771.56**
Computation of duration							
Gain from price increase (Actual Price – Desired Price)	44.86	38.90	32.40	25.31	17.59	9.17	0.00
Loss from reinvestment (Actual Accumulated Interest – Desired Accumulated Interest)	0.00	0.00	–1.00	–3.19	6.79	–12.04	–19.23
Net gain (Actual wealth – Desired wealth)	44.86	38.90	31.40	22.12	10.80	–2.87	–19.23
No. of years needed to match actual wealth with desired wealth	0	1	1	1	1	0.790	
Duration = 4.790 years						*	

* It is assumed that during the 5th year, wealth in terms of present value is generated evenly throughout the 5th year.

Table – 1 shows actual and desired wealth. Due to fall in the rate of interest, the price of the bond increases but at the same time the reinvestment rate also falls. As a result, although initially the actual wealth is higher than the desired wealth but afterwards the actual wealth becomes lower than the desired wealth. Consequently, we get positive differences between actual wealth and desired wealth initially but afterwards the difference becomes negative. Duration is that period at end of which the difference between actual wealth and desired wealth is zero. From Table – 1, it is seen that the actual wealth equals to the desired wealth between t=4 and t=5. Assuming a linear relationship between time and the difference between actual wealth and desired wealth (within t=4 and t=5), we find duration through interpolation.

$$\text{Duration (in years)} = 4 + \{10.80 / (10.80+2.87)\} = 4.790 \qquad \text{— (12b)}$$

Again, we can find duration by using equation (12a) as follows:-

$$\text{Duration (in years)} = \ln (1044.86/1000) / \ln (1.10 / 1.09)$$
$$= 0.043882 / 0.009132 = 4.805$$

Since, equation (12b) has an inherent limitation, it gives an approximate value of duration and hence this equation is inferior to equation (12a).

Since duration based investment strategy attempts to eliminate the interest rate risk which may be represented by the volatility in the price of the bond due to change in the rate of interest, derivation of a measure of volatility will be very useful. Let us derive a measure of sensitivity of price to change in interest rate.

Volatility in price and interest rate

Volatility in the price of a bond may be defined as the percentage change in the price of the bond due to one percentage change in the actual interest rate from the rate of interest which was desired at the time of making the investment. Thus, volatility can be computed as follows:

V = (Percentage change in the price of the bond) / (Percentage change in the rate of interest)

$$= [\{(P_0 – B_0) \text{ X } 100\}/ B_0] / [\{(r \text{ x } 100 – i \text{ x } 100) \text{ X } 100\}/ i \text{ x } 100]$$
$$= [\{(P_0 – B_0) \text{ X } 100\}/ B_0] / [\{(r – i)\ 100 \text{ X } 100\}/ i \text{ x } 100]$$
$$= (B / B_0) / (i / i)$$
$$= (B / i) (i / B_0)$$
$$= (B / B_0) (i / i) \qquad \text{—(14)}$$

Where, V = Volatility in the price of the bond due to change in the rate of interest

P_0 = Current price of the bond

B_0 = Initial price of the bond
r = The new risk-free rate of interest per rupee
i = The initial risk-free rate of interest per rupee
ΔB = The change in price due to change in the risk-free rate of interest
Δi = The change in the risk-free rate of interest per rupee.

This measure of volatility depends on price, change in the rate of interest and the initial rate of interest. Price, again, depends on the cash flow stream comprising of interests and price at which the bond will be redeemed and the term to maturity of the bond. Thus, volatility depends on the change in the rate of interest, level of the rate of interest, coupon, redemption price and term to maturity (Homer and Leibowitz, 1971 quoted in Hopewell and Kaufman, 1973). This paper attempts to study the effect of the change in the rate of interest and the level of rate of interest on the price of a bond for the given amount of annual interest, redemption price and maturity. If r is greater than i, P_0 will be lower than B_0 and vice versa. So, $(\Delta B/\Delta i)$ and *V* will be negative. Since, we are interested to know the sensitivity of the price of the bond to the change in the rate of interest, we shall consider the absolute value of *V*.

Generally, people are interested to know the variability in the price of the bond due to one percentage point change in the rate of interest. Thus, **volatility can be defined as a percentage point change in the price of the bond due to one percentage point change in the rate of interest**. So, volatility can be computed as follows:

V = (Percentage point change in the price of the bond) / (Percentage point change in the rate of interest)

$$= [100\,(P_0 - B_0) / B_0] / (100\,r - 100\,i)$$
$$= [100\,(P_0 - B_0) / B_0] / [(r - i)\,100]$$
$$= (\Delta B / B_0) / \Delta i$$
$$= (\Delta B / \Delta i)\,(1 / B_0) \qquad \text{— (14a)}$$

When the change in the interest rate is very small, we can write the above equation as follows:

$$V = (\Delta B / \Delta i)\,(1 / B_0) \qquad \text{— (14b)}$$

Example

We have considered the numerical example mentioned earlier and computed durations using equations (12a) and (12b) and the volatility in price using equation (14a). Let us assume that the current interest rate may take any discreet value between 8% and 15%. The results are given in Table – 2.

From Table – 2, it is observed that as interest rate increases price of the bond falls (Columns 1 and 2) but at a decreasing rate (Column 4). Durations as per equation (12a) are also computed and given in column (5). It is seen that as interest rate increases duration falls because the gain from reinvestment of intermediate cash inflows increases. Equation (12b) is based on the assumption that within two time points the relationship between time and the difference between actual wealth and desired wealth is linear. Because of this assumption, there are differences between the durations computed as per the equations (12a) and (12b).

Table – 2
Duration and Volatility at Different Rate of Interest

Interest rate (%)	Current Price (Rs)	Change in price from initial price (Rs)	Difference in the change in price (Rs)	Duration as per Eq. (12a) (in years)	Duration as per Eq. (12b) (in years)	Volatility as per Eq. (14a)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
8	1092.46	92.46	47.60	4.8193	4.8062	4.623
9	1044.86	44.86	44.86	4.8051	4.7904	4.486
10	**1000.00**	**0.00**	**0.00**	**4.7908***	**4.7746***	**0.00***
11	957.69	-42.31	-42.31	4.7765	4.7587	4.231
12	917.77	-82.23	-39.92	4.7621	4.7427	4.111
13	880.07	-119.93	-37.70	4.7477	4.7267	3.998
14	844.45	-155.55	-35.62	4.7333	4.7106	3.889
15	810.78	-189.22	-33.67	4.7189	4.6946	3.784

Eq. stands for equation. Initial price is Rs 1000 since the initial interest rate is 10%.

* There is no change in price and amount from reinvestment since interest rate remains the same. The average of the durations at 9% and 11% rates of interest may be taken as an approximate duration at 10% rate of interest. Such approximate duration at 10% interest rate as per Eq. (12a) is 4.7908.

It is seen from Table – 2, that volatility in price falls as interest rate increases (Column 7). Equation (14a) gives a measure of volatility which is based on the assumption that volatility varies linearly with change in the rate of interest between two given rates of interest. So, equation (14a) does not give the true volatility because volatility does not vary linearly with the change in interest rate. Moreover, volatility also depends on the level of interest rate.

Thus, duration can be accurately and easily computed using equation (12a). But one should be cautious in interpreting the volatility computed as per equation (14a).

Macaulay's Measure of Duration

Now, let us see the Macaulay's measure of duration. "Duration is the weighted average of a bond's life where the proportions of the present values of the cash inflows likely to be generated at various time points to the current bond price are used as weights" (Gultekin and Rogalski, 1983; Fischer and Jordan, 2007). Now, let us see the derivation of this measure of duration.

Assuming a flat term structure of interest rates and rearranging equation (5) we get –

$P = I\,(1 + r)^{-1} + I\,(1 + r)^{-2} + \ldots. + I\,(1 + r)^{-n} + F\,(1 + r)^{-n}$

Where, P = Price of the bond for a required rate of return r.

The sensitivity of price to the rate of interest can be found out by taking the first order derivative of the above equation with respect to interest rate.

$$\delta P / \delta r = -1\,I\,(1 + r)^{-2} - 2\,I\,(1 + r)^{-3} - 3\,I\,(1 + r)^{-4} - \ldots\ldots - n\,I\,(1+ r)^{-n-1} - nF\,(1 + r)^{-n-1}$$

$$= -(1 + r)^{-1}\,[1\,I\,(1 + r)^{-1} + 2\,I\,(1 + r)^{-2} + 3\,I\,(1 + r)^{-3} + \ldots + n\,I\,(1 + r)^{-n} + n\,F\,(1 + r)^{-n}] \quad \text{— (15)}$$

Dividing both sides with P_0 (let P_0 stands for the initial price) and multiplying both sides with δr we get –

$$\delta P / P_0 = -\,\delta r\,(1 + r)^{-1}\,[1\,I\,(1 + r)^{-1} + 2\,I\,(1 + r)^{-2} + 3\,I\,(1 + r)^{-3} + \ldots + n\,I\,(1 + r)^{-n} + n\,F\,(1 + r)^{-n}] / P_0 \quad \text{— (15a)}$$

$$\text{Let } D = [\,1\,I\,(1 + r)^{-1} + 2\,I\,(1 + r)^{-2} + 3\,I\,(1 + r)^{-3} + \ldots\ldots\ldots + n\,I\,(1 + r)^{-n} + n\,F\,(1 + r)^{-n}\,] / P_0 \quad \text{— (15b)}$$

The equation (15b) gives the duration as defined by Macaulay (Hopewell and Kaufman, 1973).

Following Chua (1984) we can simplify the calculation of duration.

$$\text{Let } Y = 1\,I\,(1 + r)^{-1} + 2\,I\,(1 + r)^{-2} + 3\,I\,(1 + r)^{-3} + \ldots + n\,I\,(1 + r)^{-n} \quad \text{—(15c)}$$

Then, we get –

$$D = [Y + n\,F\,(1 + r)^{-n}] / P_0 \quad \text{—(15d)}$$

Multiplying both sides of equation (15c) with $(1+r)^{-1}$, we get –

$$Y\,(1 + r)^{-1} = 1\,I\,(1 + r)^{-2} + 2\,I\,(1 + r)^{-3} + 3\,I\,(1 + r)^{-4} + \ldots + n\,I\,(1+r)^{-n-1} \quad \text{—(15e)}$$

Deducting (15e) from (15c) we get –

$Y - Y(1+r)^{-1} = I(1+r)^{-1} + I(1+r)^{-2} + I(1+r)^{-3} + \ldots\ldots\ldots\ldots\ldots + I(1+r)^{-n} - nI(1+r)^{-n-1}$

Or, $Y(1+r-1)(1+r)^{-1} = I[(1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + \ldots\ldots + (1+r)^{-n}] - nI(1+r)^{-n-1}$

Multiplying both sides with $(1+r)$ we get –

$$Y r = I(1+r)[(1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + \ldots + (1+r)^{-n}] - nI(1+r)^{-n} \quad \text{----(15f)}$$

Let $U = (1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + \ldots + (1+r)^{-n+1} + (1+r)^{-n}$ —(15g)

Multiplying both sides with $(1+r)$ we get –

$$U(1+r) = 1 + (1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + \ldots\ldots + (1+r)^{-n+1} \quad \text{—(15h)}$$

Subtracting (15g) from (15h) we get –

$U(1+r) - U = 1 - (1+r)^{-n}$

Or, $U(1+r-1) = [1 - (1+r)^{-n}]$

Or, $U = [1 - (1+r)^{-n}]\, r^{-1}$ ----(15i)

Replacing the value of U in equation (15f) and rearranging it, we get –

$Y r = I(1+r)[\{1 - (1+r)^{-n}\} r^{-1}] - nI(1+r)^{-n}$

Or, $Y = I(1+r)[\{1 - (1+r)^{-n}\} r^{-2}] - nI r^{-1}(1+r)^{-n}$

Adding $nF(1+r)^{-n}$ to both sides, we get –

$$\begin{aligned} Y + nF(1+r)^{-n} &= I(1+r)\{1 - (1+r)^{-n}\}r^{-2} - nIr^{-1}(1+r)^{-n} + nF(1+r)^{-n} \\ &= I(1+r)^{-n}(1+r)\{(1+r)^{n} - 1\}r^{-2} - nIr^{-1}(1+r)^{-n} + nF(1+r)^{-n} \\ &= (1+r)^{-n} r^{-2}[I(1+r)\{(1+r)^{n} - 1\} - nIr + nFr^{2}] \quad \text{—(15j)} \end{aligned}$$

Following equation (1b), we get –

$$\begin{aligned} P_0 &= Ir^{-1}\{1 - (1+r)^{-n}\} + F(1+r)^{-n} \\ &= Ir^{-1}(1+r)^{-n}\{(1+r)^{n} - 1\} + F(1+r)^{-n} \\ &= (1+r)^{-n} r^{-2}[Ir\{(1+r)^{n} - 1\} + Fr^{2}] \quad \text{— (15k)} \end{aligned}$$

Replacing the values of $Y + nF(1+r)^{-n}$ and P_0 as per equations (15j) and (15k) in equation (15d), we get –

$$\begin{aligned} D &= (1+r)^{-n} r^{-2}[I(1+r)\{(1+r)^{n} - 1\} - nIr + nFr^{2}] / (1+r)^{-n} r^{-2}[Ir\{(1+r)^{n} - 1\} + Fr^{2}] \\ &= [I(1+r)\{(1+r)^{n} - 1\} - nIr + nFr^{2}] / [Ir\{(1+r)^{n} - 1\} + Fr^{2}] \\ &= [I\{(1+r)^{n+1} - (1+r) - nr\} + nFr^{2}] / [Ir\{(1+r)^{n} - 1\} + Fr^{2}] \quad \text{—(15}\ell\text{)} \end{aligned}$$

Equation (15ℓ) looks more complex than equation (12a).

Volatility and Macaulay's measure of duration

A measure of duration will be more useful if volatility can be measured on the basis of such duration. From equations (15a) and (15b) we get –

$$\delta P / P_0 = -\delta r\, D / (1+r) \quad \text{— (16)}$$

From equation (14b) we get –

$V = (\delta P / \delta r)\,(1 / P_0)$

$\quad = (\delta P / P_0)\,(1 / \delta r)$

Replacing $\delta P / P_0$ from equation (16), we get –

$V = -\,\{\delta r\, D / (1 + r)\}\,(1 / \delta r)$

$\quad = -\,D / (1 + r)$

The change in the interest rate may be positive or negative and so the sign of the right hand side is not important to us. So, we get –

$$V = D / (1 + r) \qquad \text{— (16a)}$$

Thus, volatility in the price of the bond can be easily calculated by using equation (16a) provided the duration as per equation (15b) and the required rate of return of the investor are known.

Example

Let us take the numerical example mentioned earlier. Using equations (15b) and (16a), let us compute the Macaulay's measures of duration and volatility of the bond.

Table – 3

Macaulay's Measures of Duration and Volatility

Interest rate (%)	Current Price (Rs)	Change in price (Rs)	Duration as per Eq. (15b) (in years)	Change in Duration (in years)	Volatility as per Eq. (16a) (%)	Change in Volatility
(1)	(2)	(3)	(4)	(5)	(6)	(7)
8	1092.46		4.8474		4.4884	
9	1044.86	-47.60	4.8192	-0.0282	4.4213	-0.0671
10	1000.00	-44.86	4.7908	-0.0284	4.3553	-0.0660
11	957.69	-42.31	4.7621	-0.0287	4.2902	-0.0651
12	917.77	-39.92	4.7331	-0.0290	4.2260	-0.0642
13	880.07	-37.70	4.7040	-0.0291	4.1628	-0.0632
14	844.45	-35.62	4.6746	-0.0294	4.1005	-0.0623
15	810.78	-33.67	4.6450	-0.0296	4.0392	-0.0613

Table – 3 shows that as the interest rate increases the price of the bond falls but at a decreasing rate (See columns 1, 2 & 3). The relationship between interest rate, price and change in price evidences that the curve showing the relationship between interest rate and price is downward sloping and is convex to the origin (Also see Figure – 1). It is also observed (See columns 1, 4 & 5) that as interest rate increases duration falls in an increasing rate. Increase in interest rate implies increase in reinvestment rate for the intermediate cash inflows. As a result, there is an increase in gains from reinvestment of cash inflows which occur earlier. Since the importance (or weights) of the cash inflows which occur earlier increases, the duration of the bond falls. The relationship between interest rate and duration shows that duration depends on the level of interest rate. As interest rate increases volatility falls at a decreasing rate (See columns 1, 6 and 7) since price falls at a decreasing rate with the interest rate.

Volatility and duration measured by m

Since establishing a direct relationship between **m** and volatility is not easy, we have derived the following measure of volatility. From equation (15d) and (15j), we get

$$D = (1 + r)^{-n} r^{-2} [I (1 + r) \{(1 + r)^{n} - 1\} - n I r + n F r^{2}] / P_0$$

Putting the value of D in equation (16a), we get –

$$V = (1 + r)^{-n} r^{-2} \quad [I (1 + r) \{(1 + r)^{n} - 1\} - n I r + n F r^{2}] / \{P_0 (1 + r)\}$$
$$= [I (1 + r)^{n+1} - I (1 + r) - n I r + n F r^{2}] / \{P_0 (1 + r)^{n+1} r^{2}\} \qquad \text{—(16b)}$$

This measure of volatility given by equation (16b) appears to be more complex than the measure of volatility given by equation (16a). Moreover, volatility will be different for different interest rates all other things remaining the same. Therefore, a true measure of volatility is given by equation (14b).

Problems of the Macaulay's measure of duration

Since the relationship between price of the bond and interest rate is not linear but is represented by a curve which is convex to the origin, the sensitivity of price to the rate of interest will be different for different levels of interest rates in case of a fixed-income bond. Duration is influenced by both the change in the interest rate and the level of interest rate.

The present values of cash inflows to current price are used as weights under the Macaulay's measure of duration. These weights are multiplied with discrete time points at which those cash inflows are likely to occur. The sum of the weighted time points gives the Macaulay's duration. Thus, Macaulay's duration is a weighted average of the discrete time points. Equations (15c) and (15d) can be written as follows.

$$D = [1I (1 + r)^{-1} + 2I (1 + r)^{-2} + 3I (1 + r)^{-3} + \ldots + nI (1 + r)^{-n} + n F (1 + r)^{-n}] / P_0$$

Or, $D = 1\ W_1 + 2\ W_2 + \ldots\ldots + n\ W_n + n\ W_F$

Where,

$$W_t = I\,(1 + r)^{-t} / P_0$$
$$W_F = F\,(1 + r)^{-t} / P_0$$

Macaulay's duration is a linear combination of discrete time points. Thus, it implicitly assumes that within two weighted discrete time points, wealth varies linearly with time. This assumption is not true in case of continuous compounding. When the computed duration does not match with any weighted discrete time point, the implicit linearity assumption makes the computed duration incorrect. Thus, the Macaulay's measure [i.e., duration as per equation (15b)] gives an approximate value of duration. Moreover, it is difficult to derive duration by using equation (15b) when the term structure of interest rates is not flat.

Comparison between durations computed as per equations (15b) and (12a)

Based on the example mentioned earlier, we can compute the durations of the bond using equations (15b) and (12a). The initial rate of interest is taken as 10%.

Table – 4

Comparison of the Durations Computed As Per Equations (15b) and (12a)

Interest rate (%)	Current Price (Rs)	Duration (in years)	
		As per Eq. (15b) (Macaulay's Measure)	As per Eq. (12a)
(1)	(2)	(3)	(4)
8	1092.46	4.8474	4.8193
9	1044.86	4.8192	4.8051
10	**1000.00**	**4.7908**	**4.7908***
11	957.69	4.7621	4.7765
12	917.77	4.7331	4.7621
13	880.07	4.7040	4.7477
14	844.45	4.6746	4.7333
15	810.78	4.6450	4.7189

* There is no interest rate risk. See Table – 2.

Table – 4 gives the results of our calculations. Table – 4 shows that when the actual interest rate is lower than the initial rate of interest, the duration computed as per equation (15b) is higher than the true duration computed as per equation (12a). But, when the actual interest rate is higher than initial interest rate, the relationship is reversed. Interestingly, as the actual interest rate increases above the initial interest rate, the difference between duration computed as per equation (15b) and duration computed as per equation (12a) also increases.

An approximate solution to the problem of the Macaulay's measure of duration

So, a difference between the duration computed as per equation (15b) and the duration computed as per equation (12a) exists. We can take the average of the durations computed as per equation (15b) at the initial rate of interest and the current rate of interest and compare that average with the duration computed as per equation (12a).

Average duration = [Duration at the original rate of interest + Duration at the current rate of interest] / 2 — (15q)

Table – 5

Comparison of the Average Duration computed as per Equation (15q) and Duration computed as per Equation (12a)

Interest rate (%)	Current Price (Rs)	Duration (in years)		
		As per Equation (15b)	As per Equation (15q) (Average of the Macaulay's Duration)	As per Equation (12a)
(1)	(2)	(3)	(4)	(5)
6	1196.69	4.9030	4.8469	4.8476
7	1143.00	4.8754	4.8331	4.8335
8	1092.46	4.8474	4.8191	4.8193
9	1044.86	4.8192	4.8050	4.8051
10	**1000.00**	**4.7908**	**4.7908**	**4.7908***
11	957.69	4.7621	4.77645	4.7765
12	917.77	4.7331	4.76195	4.7621
13	880.07	4.7040	4.7474	4.7477
14	844.45	4.6746	4.7327	4.7333
15	810.78	4.6450	4.7179	4.7189

* There is no interest rate risk. See Table – 2

Table – 5 shows the average durations computed as per equation (15q) and durations computed as per equation (12a) for various rates of interest. It is observed that although the difference between the average duration computed as per equation (15q) and the duration computed as per equation (12a) is very low but as the current interest rate moves away from the initial interest rate, the difference between the average duration and the duration as per equation (12a) increases. Thus, the average duration as per equation (15q) still gives approximate results although the approximate results almost match with the true durations computed as per equation (12a).

Conclusion

Thus, duration based strategy can be useful to the investors in mitigating the interest rate risk associated with investment in default risk-free bonds. Equation (12a) can be used for computing duration accurately when the term structures of interest rates are flat. But the computation of duration becomes tedious when the term structures of interest rates are not flat. Computers can be useful in calculating duration under such a situation where term structures of interest rates are not flat.

The Macaulay's measure of duration computed as per equation (15b) does not match with the true duration although the average duration computed as per equation (15q) gives close approximations to the true duration. Thus, equation (12a) seems to be a superior measure of duration with respect to ease of computation and accuracy.

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