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On Jordan Triple Derivations of Semiprime Γ-Rings

***A.K.Joardar*[1] *and A.C.Paul*[2]**

[1] Department of Mathematics, Islamic University, Kushtia-7003, Bangladesh
e-mail : atishjoardar@yahoo.com
[2] Department of Mathematics, University of rajshahi, Rajshahi-6205, Bangladesh
e-mail : acpaulrubd_math@yahoo.com

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ABSTRACT

In this article, we define triple derivation and Jordan triple derivation of Γ-rings as well as different types of Γ-rings, and we develop some important results relating to the concepts of triple derivation and Jordan triple derivation of gamma rings. Through every triple derivation of a gamma ring M is obviously a Jordan triple derivation of M, but the converse statement is in general not true. Here we prove that every Jordan triple derivation of a 2-torsion free semiprime gamma ring is a derivation.

Keywords: Derivation and triple derivation, Jordan triple derivation, gamma rings and semiprime rings.

1. Introduction

Let M and Γ be additive abelian groups. M is said to be a Γ-ring if there exists a mapping M× Γ×M→M (sending (x,α,y) into xαy) such that

(a) (x+y)αz = xαz+yαz
x(α+β)y=xαy+xβy
xα(y+z)=xαy+xαz

(b) (xαy)βz=xα(yβz)

For all x,y,z ∈ M and α, β ∈ Γ . This definition is due to Barnes [1]. Let M be a gamma rings. M is prime if aΓMΓb=0 with a,b∈M, implies either a=0 or b=0. M is called semiprime if aΓMΓa=0 with a∈M implies a=0. And M is called 2-torsion free if 2m=0, for m∈M implies m=0.

Let R be an associative ring. An additive mapping d: R→R is called a Triple derivation if $d(abc) = d(a)bc + ad(b)c + abd(c)$.

and Jordan Triple derivation if $d(aba) = d(a)ba + ad(b)a + abd(a)$.

It is clear that every triple derivation is a jordan triple derivation but the converse is not in general true. If R is a two torsion free semiprime ring, then every Jordan triple derivation is a derivation by M. Bresar [1] in the sence of classical rings.

If R is a two torsion free prime ring, then every Jordan triple derivation is a derivation by

Bell and Koppe [2].

N. Nobusawa [6] was first introduced the notion of gamma ring. The gamma ring due to N. Nobusawa is now denoted by Γ_N-ring. Next Barnes [3] generalized it and gave the above definition. Now a days we mean the gamma ring which is given by Barnes. It is clear that every ring is a gamma ring.

In this paper, we define triple derivation and Jordan triple derivation of a gamma ring. We give an example of triple derivation and an example of a Jordan triple derivation for gamma rings. We also prove that every Jordan triple derivation is a derivation if it is a two torsion free semiprime Γ-ring.

2. Jordan triple derivation

Let M be a Γ ring, An additive mapping d: M→M is called a triple derivation if $d(a\alpha b\beta c) = d(a)\alpha b\beta c + a\alpha d(b)\beta c + a\alpha b\beta d(c)$ for every a,b,c $\in$ M and α, β $\in$ Γ.

An additive mapping d: M→M is called a Jordan triple derivation if $d(a\alpha b\beta a) = d(a)\alpha b\beta a + a\alpha d(b)\beta a + a\alpha b\beta d(a)$ for every a,b,c $\in$ M and α, β $\in$ Γ.

It is clear that every triple derivation is a jordan triple derivation. But every Jordan triple derivation is not in general a triple derivation.

Now we give the following examples:

2.1. Example

Let R be an associative ring with unity element 1. Let M = $M_{1,2}$(R) and Γ={$\binom{n.1}{0}$, $n \in Z$)}. Then M is a Γ-ring. Let d: R→R be a derivation. Now define D((x,y))=(d(x),d(y)). Then we show that D is a triple derivation associated to jordan derivation d. For this, let a=(x_1 ,y_1), b=(x_2 , y_2), c=(x_3 , y_3), α= $\binom{n_1.1}{0}$, β= $\binom{n_2.1}{0}$. We have to prove that $D(a\alpha b\beta c) = D(a)\alpha b\beta c + a\alpha D(b)\beta c + a\alpha b\beta D(c)$.

Now we have aαbβc=($x_1 n_1 x_2 n_2 x_3$, $x_1 n_1 x_2 n_2 y_3$).

So D(aαbβc)=(d($x_1 n_1 x_2 n_2 x_3$), d($x_1 n_1 x_2 n_2 y_3$)).

Similarly, we get

$D(a)\alpha b\beta c + a\alpha D(b)\beta c + a\alpha b\beta D(c) = (d(x_1 n_1 x_2 n_2 x_3), d(x_1 n_1 x_2 n_2 y_3))$.

2.2. Example

Let M be a Γ-ring defined as an example 2.1. Let N ={(x, x) : xϵM}. Then N is a Γ-ring contained in M. Let d be a derivation given in example 2.1. Define D: N→N by D((x,x))=(d(x),d(x)). Then we show that D is a Jordan triple derivation. Note that it is not a triple derivation.

To show this, let a=(x, x), b=(y, y), α= $\binom{n_1.1}{0}$, β= $\binom{n_2.1}{0}$. We have to prove $D(a\alpha b\beta a) = D(a)\alpha b\beta a + a\alpha D(b)\beta a + a\alpha b\beta D(a)$.

Now we have $a\alpha b\beta a=(xn_1yn_2x,\ xn_1yn_2y)$
So $D(a\alpha b\beta a)=(d(xn_1yn_2x),\ d(xn_1yn_2y))$. Similarly,
we get $D(a)\alpha b\beta a + a\alpha D(b)\beta a + a\alpha b\beta D(a) = (d(xn_1yn_2x), d(xn_1yn_2y))$.
Now we prove some lemma which are essential to prove our main theorem.

Lemma 2.1. Let M be a Γ-ring and d be a Jordan triple derivation of a Γ-ring M. Then $d(a\alpha b\beta c + c\alpha b\beta a) = d(a)\alpha b\beta c + d(c)\alpha b\beta a + a\alpha d(b)\beta c + c\alpha d(b)\beta a + a\alpha b\beta d(c) + c\alpha b\beta d(a)$ for all a,b,c ϵ M.

Proof. Computing $d((a+c)\alpha b\beta(a+c))$ and canceling the like terms from both sides, then we get prove of the lemma.

Definition 2.1. Let M be a Γ-ring. Then for all a, b, c ϵM and α, β ϵ Γ we define
$[a, b, c]_{\alpha,\beta} = a\alpha b\beta c - c\alpha b\beta a$.

Lemma 2.2. If M is a Γ-ring, then for all a, b, c ϵM and α, β ϵ Γ

(1) $[a, b, c]_{\alpha,\beta} + [c, b, a]_{\alpha,\beta} = 0$
(2) $[a+c, b, d]_{\alpha,\beta} = [a, b, d]_{\alpha,\beta} + [c, b, d]_{\alpha,\beta}$
(3) $[a, b, c+d]_{\alpha,\beta} = [a, b, c]_{\alpha,\beta} + [a, b, d]_{\alpha,\beta}$
(4) $[a, b+d, c]_{\alpha,\beta} = [a, b, c]_{\alpha,\beta} + [a, d, c]_{\alpha,\beta}$
(5) $[a, b, c]_{\alpha+\beta,\gamma} = [a, b, c]_{\alpha,\gamma} + [a, b, c]_{\beta,\gamma}$
(6) $[a, b, c]_{\alpha,\beta+\gamma} = [a, b, d]_{\alpha,\beta} + [a, b, c]_{\alpha,\gamma}$

Proof. Obvious

Definition 2.2. Let d be a Jordan triple derivation of a Γ-ring M. Then for all a, b, c ϵM and α, β ϵ Γ we define $G_{\alpha,\beta}(a\alpha b\beta c) = d(a\alpha b\beta c) - d(a)\alpha b\beta c - a\alpha d(b)\beta c - a\alpha b\beta d(c)$.

Lemma 2.3. Let d be a Jordan triple derivation of a Γ-ring M. Then for all a, b, c ϵM and α, β ϵ Γ, we have

(1) $G_{\alpha,\beta}(a\alpha b\beta c) + G_{\alpha,\beta}(c\alpha b\beta a) = 0$
(2) $G_{\alpha,\beta}((a+c)\alpha b\beta d) = G_{\alpha,\beta}(a\alpha b\beta d) + G_{\alpha,\beta}(c\alpha b\beta d)$
(3) $G_{\alpha,\beta}(a\alpha b\beta(c+d)) = G_{\alpha,\beta}(a\alpha b\beta c) + G_{\alpha,\beta}(a\alpha b\beta d)$
(4) $G_{\alpha,\beta}(a\alpha(b+c)\beta d) = G_{\alpha,\beta}(a\alpha b\beta d) + G_{\alpha,\beta}(a\alpha c\beta d)$
(5) $G_{\alpha+\beta,\gamma}(a\alpha b\beta c) = G_{\alpha,\gamma}(a\alpha b\beta c) + G_{\beta,\gamma}(a\alpha b\beta c)$
(6) $G_{\alpha,\beta+\gamma}(a\alpha b\beta c) = G_{\alpha,\beta}(a\alpha b\beta c) + G_{\alpha,\gamma}(a\alpha b\beta c)$

Proof. Obvious

Lemma 2.4. If M is a Γ-ring, then $G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta[a, b, c]_{\alpha,\beta} + [a, b, c]_{\alpha,\beta}\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c) = 0$ for all x ϵM and γ, δ ϵ Γ.

Proof. First we compute $d(a\alpha(b\beta c\gamma x\delta c\alpha b)\beta a + c\alpha(b\beta a\gamma x\delta a\alpha b)\beta c)$ by using the definition of Jordan triple derivation we get $d(a)\alpha b\beta c\gamma x\delta c\alpha b\beta a+a\alpha d(b)\beta c\gamma x\delta c\alpha b\beta a+a\alpha b\beta d(c)\gamma x\delta c\alpha b\beta a+a\alpha b\beta c\gamma d(x)\delta c\alpha b\beta a+a\alpha b\beta c\gamma x\delta d(c)\alpha b\beta a+a\alpha b\beta c\gamma x\delta c\alpha d(b)\beta a+a\alpha(b\beta c\gamma x\delta c\alpha b\beta d(a)+d(c)\alpha b\beta a\gamma x\delta a\alpha b\beta c+c\alpha d(b)\beta a\gamma x\delta a\alpha b\beta c+c\alpha b\beta d(a)\gamma x\delta a\alpha b\beta c+c\alpha b\beta a\gamma d(x)\delta a\alpha b\beta c+c\alpha b\beta a\gamma x\delta d(a)\alpha b\beta c+c\alpha b\beta a\gamma x\delta a\alpha d(b)\beta c+c\alpha b\beta a\gamma x\delta a\alpha b\beta d(c)$. On the other hand, we have $d((a\alpha b\beta c)\gamma x\delta(c\alpha b\beta a)+(c\alpha b\beta a)\gamma x\delta(a\alpha b\beta c))$ and using lemma 2.1, we get$d(a\alpha b\beta c)\gamma x\delta c\alpha b\beta a+ d(c\alpha b\beta a)\gamma x\delta a\alpha b\beta c + a\alpha b\beta c\gamma d(x)\delta c\alpha b\beta a + c\alpha b\beta a\gamma d(x)\delta a\alpha b\beta c + a\alpha b\beta c\gamma x\delta d(c\alpha b\beta a) + c\alpha b\beta a\gamma x\delta d(a\alpha b\beta c)$. Since these two are equal, cancelling the like terms from both sides of this equality and rearranging them,we get $G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta[a, b, c]_{\alpha,\beta}+[a, b, c]_{\alpha,\beta}\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c)=0$

Lemma 2.5. let M be a 2-torsion free semiprime Γ-ring and suppose that a, b ϵM. If $a\Gamma m\Gamma b + b\Gamma m\Gamma a = 0$ for all m ϵ M, then $a\Gamma m\Gamma b = b\Gamma m\Gamma a = 0$.

Proof. Let m and m′ be two arbitrary elements of M. Then by hypothesis, we have $(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b)= -(b\Gamma m\Gamma a)\Gamma m'\Gamma(a\Gamma m\Gamma b)= -(b\Gamma(m\Gamma a)\Gamma m')\Gamma a)\Gamma m\Gamma b= (a\Gamma(m\Gamma a\Gamma m')\Gamma b)\Gamma m\Gamma b = a\Gamma m\Gamma(a\Gamma m'\Gamma b)\Gamma m\Gamma b = -a\Gamma m\Gamma(b\Gamma m'\Gamma a)\Gamma m\Gamma b = -(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b)$. This implies, $2(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b)=0$. Since M is a 2-torsion free , $(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b)=0$. By semiprimeness of M, $a\Gamma m\Gamma b=0$ for all mϵM. Hence we get $a\Gamma m\Gamma b = b\Gamma m\Gamma a =0$ for all mϵM.

Lemma 2.6. Let M is a 2-torsion free semiprime Γ-ring. Then for all a, b, c, xϵM and α, β, γ, δ ϵ Γ. If $G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta[a, b, c]_{\alpha,\beta}+[a, b, c]_{\alpha,\beta}\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c)=0$, then $G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta[a, b, c]_{\alpha,\beta}=0$, $[a, b, c]_{\alpha,\beta}\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c)=0$.

Proof. $G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta[a, b, c]_{\alpha,\beta}+[a, b, c]_{\alpha,\beta}\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c)=0$. Now by above lemma, we get $G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta[a, b, c]_{\alpha,\beta}=0$, $[a, b, c]_{\alpha,\beta}\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c)=0$

Now we have the position to prove our main theorem

Theorem 2.1. Any Jordan triple derivation of a 2-torsion free semiprime Γ-ring is a derivation.

Proof. From lemma 2.6 we get $G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta[a, b, c]_{\alpha,\beta} = 0$ (1)

And $[a, b, c]_{\alpha,\beta}\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c)=0$ (2)

Our first goal will be show that d is a derivation. Hence $2 G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c) = (G_{\alpha,\beta}(a\alpha b\beta c) + G_{\alpha,\beta}(a\alpha b\beta c))\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c) =(d([a, b, c]_{\alpha,\beta}) + [c, b, d(a)]_{\alpha,\beta} + [c, d(b), a]_{\alpha,\beta} + [d(c), b, a]_{\alpha,\beta})\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c) = d([a, b, c]_{\alpha,\beta})\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c)$ by using lemma 2.6.

$2 G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c) = d([a, b, c]_{\alpha,\beta})\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c)$ (3)

Similarly, $2G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c) = G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta d([a, b, c]_{\alpha,\beta})$ (4)

Next we have $0 = d(G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta [a, b, c]_{\alpha,\beta} + [a, b, c]_{\alpha,\beta}\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c)) = d(G_{\alpha,\beta}(a\alpha b\beta c))\gamma x\delta [a, b, c]_{\alpha,\beta} + d([a, b, c]_{\alpha,\beta})\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c) + G_{\alpha,\beta}(a\alpha b\beta c)\gamma d(x)\delta [a, b, c]_{\alpha,\beta} + [a, b, c]_{\alpha,\beta}\gamma d(x)\delta G_{\alpha,\beta}(a\alpha b\beta c) + G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta d([a, b, c]_{\alpha,\beta}) + [a, b, c]_{\alpha,\beta}\gamma x\delta d(G_{\alpha,\beta}(a\alpha b\beta c))$ and according to (1), (2) $0 = 4G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta G_{\alpha,\beta}(a\alpha b\beta c) + d(G_{\alpha,\beta}(a\alpha b\beta c))\gamma x\delta [a, b, c]_{\alpha,\beta}+$

$[a, b, c]_{\alpha,\beta}\gamma x\delta d(G_{\alpha,\beta}(a\alpha b\beta c))$ we multiply the above relation from the left by $G_{\alpha,\beta}(a\alpha b\beta c)\gamma x\delta\, G_{\alpha,\beta}(a\alpha b\beta c)\lambda y\mu$ and by eqn. (1) and (2). Since M is a two torsion free semiprime ring, the we get $G_{\alpha,\beta}(a\alpha b\beta c) = 0$

i.e. $d(a\alpha b\beta c) = d(a)\alpha b\beta c + a\alpha d(b)\beta c + a\alpha b\beta d(c)$ (5)

Now we consider w = d(aα(bγxδa)αb) by equation (5)

= d(a)αbγxδaαb + aαd(bγxδa)αb + aαbγxδaαd(b) = d(a)αbγxδaαb + aαd(b)γxδaαb + aαbγd(x)δaαb + aαbγxδd(a)αb + aαbγxδaαd(b) again, w = d((aαb)γxδ(aαb)) = d(aαb)γxδaαb + aαbγd(x)δaαb + aαbγxδd(aαb) comparing the two expression, we obtain (d(aαb) – d(a)αb – aαd(b))γxδaαb + aαbγxδ(d(aαb) – d(a)αb – aαd(b)) = 0. Again by semiprimeness of M, d(aαb) – d(a)αb – aαd(b) = 0, i.e. d is a derivation.

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