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Reliability Assessment for Cold Standby System Under Progressively Type-II Censoring with Binomial Removals

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ABSTRACT

The cold standby system is widely used to improve system reliability. Supposing that the lifetime of each unit follows gamma distribution, this article deals with the reliability assessment for cold standby system under Type-II progressive censoring with binomial removals. Maximum likelihood and Bayesian estimations for the scale parameter, probability of removal and reliability indices of system are derived. Besides, we demonstrate the effects of removal probability on the estimators and some numerical results by Monte Carlo simulation for illustrative purposes.

Keywords: Cold standby system; Type-II progressive censoring; Binomial removals; Gamma distribution; Maximum-likelihood estimator; Bayesian estimator

1. Introduction

The most common censoring schemes are Type-I and Type-II censoring, which don't allow for functioning items to be removed from test before the completion of the experiment. But some units need to be removed from the test in advance for time limit or other purposes. As a result, progressively Type-II censoring scheme is introduced. In recent years, many authors have developed the estimation procedures for some lifetime distributions under progressively censored scheme with pre-fixed number of removals, such as [1-6]. However, the number of removals at each stage is random in most real situations, so Yuen and Tse [7], Tse [8] et al. proposed that the number of removals at each failure time followed a discrete uniform and binomial distribution for progressively censored Weibull distribution respectively. From then on, progressively Type-II censoring scheme with random removals has been considered extensively such as [9-11]. But none of their papers involved how the removal probability p affected parameter estimations.

Non-repairable cold standby system with components having exponential time-to-failure distributions has been well studied. Ref. [12] and [13] have considered Bayesian estimation of reliability performances for cold standby series system under progressively and general progressive Type-II censored data respectively. In fact, gamma distribution can be used as a good approximation for several component lifetime distributions. But studies on cold standby system with gamma components are rare.

In this paper, we therefore consider gamma distribution for component lifetime and conduct reliability assessment for the cold standby system under progressive censoring scheme with binomial removals. Section 2 presents a brief overview of model and the system description. The Maximum likelihood estimations (MLEs) and Bayesian estimations for parameters, reliability function, mean time to failure (MTTF) and hazard rate are demonstrated in section 3 and section 4. Finally, section 5 and section 6 offer simulation results and conclusions respectively.

2. Some assumptions and model description

2.1. Some assumptions

Assume that the cold standby system under discussion consists of n components and one completely reliable switch and satisfies the following assumptions.

(1) Initially one unit is operating and the remaining n-1 units are in cold standby. The standby component will be switched to the operating state once the functioning unit fails. The system stops working only if all the components fail.

(2) Units neither fail nor age when they are in the standby state.

(3) The lifetime of each unit is identical and independent random variable with gamma distribution, which has the probability density function (PDF) as follows

$$f(t) = \lambda^{\alpha} t^{\alpha-1} e^{-\lambda t} / \Gamma(\alpha), t \geq 0, \alpha, \lambda > 0 \tag{1}$$

Here $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$, α is the shape parameter, λ is the scale parameter.

When α is a known integer, the cumulative density function (CDF) of gamma distribution has the following closed-form expression

$$F(t;\alpha,\lambda) = 1 - \sum_{i=0}^{\alpha-1} e^{-\lambda t} (\lambda t)^i / i! = \sum_{i=\alpha}^{\infty} e^{-\lambda t} (\lambda t)^i / i! \tag{2}$$

We derive the lifetime T of cold standby system with n gamma units with PDF

$$f(t) = \lambda^{n\alpha} t^{n\alpha-1} e^{-\lambda t} / \Gamma(n\alpha), t \geq 0, \alpha, \lambda > 0 \tag{3}$$

Then the reliability function, MTTF and hazard rate for cold standby system are given as follow, respectively.

$$R(t) = e^{-\lambda t} \sum_{i=0}^{n\alpha-1} (\lambda t)^i / i! \tag{4}$$

$$r(t) = \lambda^{n\alpha} t^{n\alpha-1} [(n\alpha - 1)! \sum_{i=0}^{n\alpha-1} (\lambda t)^i / i!]^{-1} \tag{5}$$

$$MTTF = n\alpha / \lambda \tag{6}$$

2.2. Model description

Assume that N units from gamma distribution with PDF (1) and CDF (2) are placed on test simultaneously. When the first failure is observed, r_1 surviving units are randomly removed from the test. Similarly, r_2 of the remaining $N-2-r_1$ units are randomly removed when the second failure is observed. This test is terminated until the m th failure is observed and $r_m = N - m - r_1 - r_2 - \cdots r_{m-1}$ surviving units are all removed. Progressively censored sample is denoted by $X = (x_1, x_2, ..., x_m)$. As well discussed as before, some surviving items are removed from the test via the binomial probability law with certain probability p .Then the probability of a removal at i th stage is

$$P(R_i = r_i \mid R_{i-1} = r_{i-1},...,R_1 = r_1) = \binom{N-m-\sum_{k=1}^{i-1} r_k}{r_i} p^{r_i}(1-p)^{N-m-\sum_{k=1}^{i} r_k}$$

where $r_0 = 0, 0 \le r_i \le N-m-\sum_{j=1}^{i-1} r_j, i = 1,2,...,m-1$.

Therefore the joint probability distribution of $(R_1, R_2,..., R_m)$ is given by

$$P(R,p) = P(R_{m-1} = r_{m-1} \mid R_{m-2} = r_{m-2},...,R_1 = r_1)P(R_{m-2} = r_{m-2} \mid R_{m-3} = r_{m-3},...,R_1 = r_1)...P(R_1 = r_1)$$
$$= A_1 \cdot p^{m_1}(1-p)^{m_2} \tag{7}$$

where $A_1 = (N-m)![(N-m-\sum_{j=1}^{m-1} r_j)!\prod_{i=1}^{m} r_i!]^{-1}$, $m_1 = \sum_{i=1}^{m-1} r_i$, $m_2 = (m-1)(N-m) - \sum_{i=1}^{m-1}(m-i)r_i$.

The likelihood function under progressive Type-II censoring with pre-fixed number of removals is

$$L(X \mid \lambda) = A_2 \prod_{i=1}^{m} f(x_i \mid \lambda)[1-F(x_i \mid \lambda)]^{r_i}, A_2 = N(N-1-r_1)...(N-\sum_{i=1}^{m-1}(r_i+1)) \tag{8}$$

Further, we assume that $X = (x_1, x_2,..., x_m)$ is independent with $R = (r_1,..., r_m)$, then the full likelihood function has the following expression

$$L(X;\lambda,p) = L(X \mid \lambda)P(R,p) \tag{9}$$

3. Maximum likelihood estimation

Substituting (1)-(2) and (7)-(8) into (9), we have

$$L(X;\lambda,p) = A_1A_2\frac{\lambda^{m\alpha}}{(\Gamma(\alpha))^m}e^{-\lambda T_0}\prod_{i=1}^{m}[x_i^{\alpha-1}(\sum_{k=0}^{\alpha-1}\frac{(\lambda x_i)^k}{k!})^{r_i}]p^{m_1}(1-p)^{m_2} \propto \lambda^{m\alpha}e^{-\lambda T_0}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}p^{m_1}(1-p)^{m_2} \tag{10}$$

where $T_0 = \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} r_i x_i$, $T_{i,\alpha-1} = \sum_{k=0}^{\alpha-1}(\lambda x_i)^k / k!$.

Then we derive the corresponding log-likelihood function of (10) as the following form

$$l(\lambda,p) = C + m\alpha\ln\lambda - \lambda T_0 + \sum_{i=1}^{m} r_i \ln T_{i,\alpha-1} + \sum_{i=1}^{m-1} r_i \ln p + [(N-m)(m-1) - \sum_{i=1}^{m-1}(m-i)r_i]\ln(1-p) \tag{11}$$

So the MLE of p and λ can be obtained by solving the following two equations.

$$\frac{\partial l}{\partial p} = p^{-1}\sum_{i=1}^{m-1} r_i - (1-p)^{-1}[(N-m)(m-1) - \sum_{i=1}^{m-1}(m-i)r_i] = 0 \tag{12}$$

$$\frac{\partial l}{\partial \lambda} = m\alpha/\lambda - T_0 + \sum_{i=1}^{m} r_i T_{i,\alpha-2} x_i / T_{i,\alpha-1} = 0 \tag{13}$$

where $T_{i,\alpha-2} = \sum_{k=0}^{\alpha-2}(\lambda x_i)^k / k!$. From Eq. (12), we can get MLE of p directly as

$$\hat{p} = [(N-m)(m-1) - \sum_{i=1}^{m-1}(m-i-1)r_i]^{-1}\sum_{i=1}^{m-1} r_i \tag{14}$$

But (13) is a nonlinear equation, we can only adopt iterative method to derive the MLE of λ . We formulate $\lambda = 1/\theta$ and the iterative formula is given by

$$\theta_{k+1}=(m\alpha)^{-1}\{T_0-\sum_{i=1}^{m} r_i x_i[1-((\alpha-1)!\sum_{k=0}^{\alpha-1}\theta_k^{\alpha-1-k}x_i^{k+1-\alpha}/k!)^{-1}]\} \tag{15}$$

According to the invariance of the MLE, substituting $\hat{\lambda}$ into (4)-(6), we can obtain the MLE of the reliability function, hazard rate and MTTF of the cold standby system as follows, respectively.

$$\hat{R}(t)=e^{-\hat{\lambda}t}\sum_{i=0}^{n\alpha-1}\left(\hat{\lambda}t\right)^i/i! \tag{16}$$

$$\hat{r}(t)=\hat{\lambda}^{n\alpha}t^{n\alpha-1}[(n\alpha-1)!\sum_{i=0}^{n\alpha-1}\left(\hat{\lambda}t\right)^i/i!]^{-1} \tag{17}$$

$$\widehat{MTTF}=n\alpha/\hat{\lambda} \tag{18}$$

4. Bayesian estimation

In this section, we discuss Bayesian estimations of parameters and reliability indices under square loss function and Linex loss function, respectively.

We take a beta distribution and a gamma distribution as the prior distribution of p and λ respectively. Then we have

$$\pi(p)=[\Gamma(c+d)/\Gamma(c)\Gamma(d)]p^{c-1}(1-p)^{d-1}, \qquad 0<p<1, c>0, d>0 \tag{19}$$

$$\pi(\lambda)=[b^a/\Gamma(a)]\lambda^{a-1}e^{-\lambda b}, \qquad a,b,\lambda>0 \tag{20}$$

We assume that p is independent of λ , then the joint prior of p and λ is

$$\pi(\lambda,p)=\pi(\lambda)\pi(p)=[b^a/\Gamma(a)]\lambda^{a-1}e^{-\lambda b}[\Gamma(c+d)/\Gamma(c)\Gamma(d)]p^{c-1}(1-p)^{d-1} \tag{21}$$

Thus, we obtain the posterior distribution for p and λ of the following form respectively

$$\pi(p\mid X,\lambda)=[\Gamma(u+v)/\Gamma(u)\Gamma(v)]p^{u-1}(1-p)^{v-1} \tag{22}$$

$$\pi(\lambda\mid p,X)=\lambda^{m\alpha+a-1}e^{-\lambda(T_0+b)}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}/\int_0^\infty\lambda^{m\alpha+a-1}e^{-\lambda(T_0+b)}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda \tag{23}$$

where $u=\sum_{i=1}^{m-1}r_i+c, v=(m-1)(N-m)-\sum_{i=1}^{m-1}(m-i)r_i+d$.

4.1. Bayesian estimates under square loss function

Reference [14] indicates that the Bayesian estimation of parameter under square loss function $L(\theta,d)=(\theta-d)^2$ has the form $\hat{\theta}_{BS}=\int_\Theta\theta\pi(\theta\mid X)d\theta$, where $\pi(\theta\mid X)$ and Θ are the posterior density function and domain of the parameter, respectively.

Thus, the Bayesian estimates of the parameters are given by

$$\hat{p}_{BS}=u/(u+v) \tag{24}$$

$$\hat{\lambda}_{BS}=\int_0^\infty\lambda^{m\alpha+a}e^{-(b+T_0)\lambda}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda/\int_0^\infty\lambda^{m\alpha+a-1}e^{-(b+T_0)\lambda}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda \tag{25}$$

Similarly, we can derive Bayesian estimations of the reliability, hazard rate and MTTF of the cold standby system by

$$\hat{R}_{BS}(t)=\int_0^\infty R(t)\lambda^{m\alpha+a-1}e^{-(b+T_0)\lambda}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda/\int_0^\infty\lambda^{m\alpha+a-1}e^{-(b+T_0)\lambda}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda \tag{26}$$

$$\hat{r}_{BS}(t)=\int_0^\infty r(t)\lambda^{m\alpha+a-1}e^{-(b+T_0)\lambda}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda \Big/ \int_0^\infty \lambda^{m\alpha+a-1}e^{-(b+T_0)\lambda}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda \tag{27}$$

$$\widehat{MTTF}_{BS}=\int_0^\infty n\alpha\lambda^{m\alpha+a-2}e^{-(b+T_0)\lambda}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda \Big/ \int_0^\infty \lambda^{m\alpha+a-1}e^{-(b+T_0)\lambda}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda \tag{28}$$

It's clear that (25)-(28) can't be evaluated analytically, so we will use numerical technique and the results are presented in section 5.

4.2. Bayesian estimates under Linex loss function

As is proved in reference [15], the Bayesian estimate of the parameter under Linex loss function $L(\theta-\hat{\theta})\propto e^{l(\theta-\hat{\theta})}-l(\theta-\hat{\theta})-1$, $l\neq 0$ has the form $\hat{\theta}_{BL}(X)=-l^{-1}\ln E(e^{-l\theta}\mid X)$

Therefore, the Bayesian estimation of p and λ has the form

$$\hat{p}_{BL}=-l^{-1}[\ln\Gamma(u+v)-\ln\Gamma(u)-\ln\Gamma(v)+\ln[\int_0^1 e^{-lp}p^{u-1}(1-p)^{v-1}dp] \tag{29}$$

$$\hat{\lambda}_{BL}=-l^{-1}\left\{\ln[\int_0^\infty \lambda^{m\alpha+a-1}e^{-\lambda(T_0+b+l)}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda]-\ln[\int_0^\infty \lambda^{m\alpha+a-1}e^{-\lambda(T_0+b)}\prod_{i=1}^{m}T_{i,\alpha-1}^{r_i}d\lambda]\right\} \tag{30}$$

Also, we adopt numerical method to derive solutions and the results can be found in section 5.

5. Simulation results

By using the algorithm proposed by Balakrishnan [16], we generate a progressive Type-II censoring sample from the gamma distribution for fixed values of $N=50$, $m=20$ $n=5$, $\alpha=3,\lambda=8$, $a=3,b=0.2;c=1,d=1$, t=1 , $l=0.5$ .Substituting the above datum in (4)-(6), we can derive the real values of reliability indices, namely $R(t)=0.9827, r(t)=0.1378, MTTF$=1.875.

Setting the probability of removal $p=0.5$, we first generate a sample and obtain the MLE and Bayesian estimator of λ and p. Then we repeat the above steps 1000 times. In the end, we calculate the relative error (RE) and mean square error (MSE) of parameters and reliability indices of the cold standby system. The results are provided in Table 1 and Table 2.

Evaluation	λ			p		
	MLE	BS	BL	MLE	BS	BL
RE	0.0262	0.0095	0.0131	0.0199	0.0187	0.0167
MSE	0. 0657	0.0456	0.0476	0.0042	0.0039	0.0039

Table 1: The RE and MSE of the estimator for λ and p

Evaluation	$R(t)$		$r(t)$		$MTTF$	
	MLE	BS	MLE	BS	MLE	BS
RE	0.0175	0.0087	0.7999	0.3928	0.0362	0.0098
MSE	0.0016	0.0002	0.0635	0.0104	0.0535	0.0294

Table 2: The RE and MSE of the estimator for reliability indices

In order to illustrate the influence of *p* on the evaluations, we plot the variation trends of RE and MSE of the estimator of λ with *p* ranging from 0.1 to 0.9. (See the Fig. 1and Fig.2).

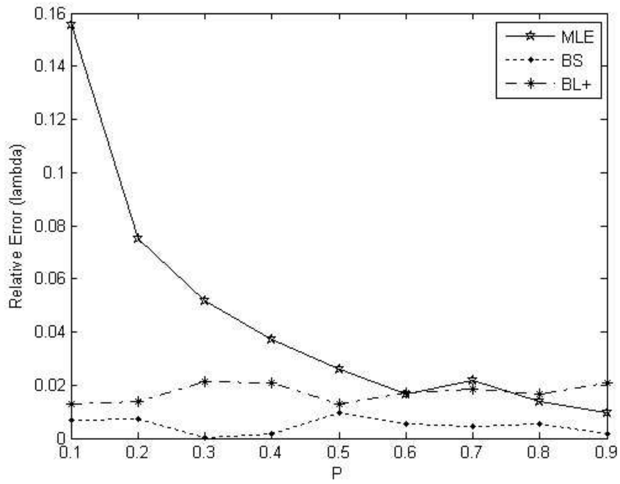

Figure 1: The variation trends of RE for $\hat{\lambda}$

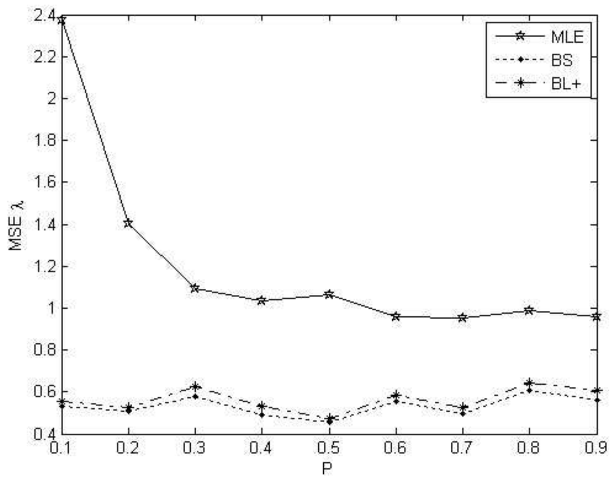

Figure 2: The variation trends of MSE for $\hat{\lambda}$

From Table 1, we can see Bayesian estimators of λ perform better than the MLE, especially under square loss function. Under Linex loss function, the Bayesian estimation of p is better than other two estimates. From Table 2, it is obvious to know that the Bayesian estimations of reliability indices are preferred to the MLE.

Fig.1 and Fig.2 tell us the Bayesian estimators vary hardly, while the MLE performs better and better with *p* ranging from 0.1 to 0.9. So the Bayesian estimations are robust.

6. Conclusion

In this paper, we discuss the MLEs and Bayesian estimates of parameters and reliability indices of the cold standby system based on progressively Type-II censored samples with binomial removals. The numerical simulations indicate the Bayesian method is more satisfactory. What's more, we conclude that the value of p don't have any effects on the Bayesian estimation, while the MLEs of parameters perform better and better with p increasing.

REFERENCES

1. X.Q.Qin, Y.M.Shi, N.Gao and J.M.Hu, Statistical analysis of the mixed accelerated life test for the type ii progressively censored sample, *Journal of Physical Sciences*, 12(2008) 23-32.
2. Y.M.Ma and Y.M.Shi, Inference for lomax distribution based on type-ii progressively hybrid censored data, *Journal of Physical Sciences*, 17(2013) 33-41.
3. H.Krishna and K.Kumar, Reliability estimation in Lindley distribution with progressively type II right censored sample, *Mathematics and Computers in Simulation*, 82(2011) 281-294.
4. B.Indrani and N.Balakrishnan, Estimation for the three-parameter gamma distribution based on progressively censored data, *Statistical Methodology*, 9 (2012) 305-319.
5. M.K.Rastogi and Y.M.Tripathi, Estimating the parameters of a Burr distribution under progressive type II censoring, *Statistical Methodology*, 9(2012)381-391.
6. J.Fu, A.Xu and Y.Tang, Objective Bayesian analysis of Pareto distribution under progressive Type-II censoring, *Statistics & Probability Letters*, 82(2012)1829-1836.
7. H.K.Yuen and S.K.Tse, Parameters estimation for Weibull distributed lifetimes under progressive censoring with random removals, *Journal of Statistical Computation and Simulation*, 55 (1996) 57-71.
8. S.K.Tse, C.Yang and H.K.Yuen, Statistical analysis of Weibull distributed lifetime data under Type II progressive censoring with binomial removals, *Journal of Applied Statistics*, 27 (2000) 1033-1043.
9. S.J.Wu, Estimation for the two parameter Pareto distribution under progressive censoring with uniform removals, *Journal of Statistical Computation and Simulation*, 73 (2003) 125-134.
10. S.J.Wu, Y.J.Chen and C.T.Chang, Statistical inference based on progressively censored samples with random removals from the Burr type XII distribution, *Journal of Statistical Computation and Simulation*, 77 (2007) 19-27.
11. D.Sanku and D.Tanujit, Statistical inference for the Rayleigh distribution under progressively Type-II censoring with binomial removal, *Applied Mathematical Modelling*, 38 (2014) 974-982.
12. D.R.Barot and M.N.Patel, Bayesian, Estimation of reliability indexes for cold standby system under general progressive Type-II censored data, *International Journal of Quality & Reliability Management*, 31 (2014) 311-343.
13. L.Li, Y.M.Shi and M.H.Li, Bayes estimators of the reliability performance for cold standby series system based on progressively Type-II censored tests, *Chinese Journal of Engineering Mathematics*, 24 (2007) 895-901.
14. S.S.Mao, J.Wang and X.Pu, *Advanced mathematical statistics*, Higher Education

Press, Beijing, China, 1998.
15. X.Z.Wu, *Modern Bayesian Statistics (in Chinese)*, China Statistics Press, Beijing, China, 2000.
16. N.Balakrishnan and R.A.Sandhu, A simple simulation algorithm for generating progressive Type II censored samples, *The American Statistician*, 49 (1995) 229-230.