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On Some Exact Solutions for a Generalized form of Yang's Euclidean R-Gauge Equations and their Relation with Painleve' Property

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ABSTRACT

It is difficult to get solutions for non-linear partial differential equations. A strategy for overcoming the difficulty in getting solutions that works well in some situation may not work equally well in some other (apparently) similar situation. Such an example for a generalized version of Yang's Euclidean R-gauge equations is presented here. In one situation one gets solutions in explicit quadrature form and in another situation (different only for the presence of an innocent looking parameter) one cannot invert and therefore cannot represent the solution even in quadrature form. It is observed from a comparative study that the equations admit Painleve' properties in the previous situation when we get explicit solutions and in the later case the equations are deprived of Painleve' properties when we do not get solutions even in quadrature form.

Keywords: SU(2) gauge field; self-duality; exact solutions; Painleve' analysis

1. Introduction

The generalized form of Yang's equations discussed here originates from the Yang's equations [1] for three variables or two variables one real and one complex and are obtained when the condition of self-duality for a SU(2) R-gauge field on Euclidean four-dimensional flat space is integrated once.

The equations are given by

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - \phi_y \phi_{\bar{y}} - \phi_z \phi_{\bar{z}} + \rho_y \bar{\rho}_{\bar{y}} + \rho_z \bar{\rho}_{\bar{z}} = 0 \tag{1a}$$

$$\phi(\rho_{y\bar{y}} + \rho_{z\bar{z}}) - 2\rho_y \phi_{\bar{y}} - 2\rho_z \phi_{\bar{z}} = 0 \tag{1b}$$

where an over bar denotes the complex conjugate, ϕ and ρ are functions of $y, \bar{y}, z, \bar{z}$, ϕ is real, ρ is complex and $\sqrt{2}\, y = x^1 + i x^2$, $\sqrt{2}z = x^3 - i x^4$, x^1, x^2, x^3, x^4 are real.

Once one has found ρ and ϕ, the corresponding R-gauge potentials are given by

$\phi\vec{b}_y = (i\rho_y, \rho_y, - i\phi_y), \phi\vec{b}_{\bar{y}} = (-i \bar{\rho}_{\bar{y}}, \bar{\rho}_y, i\phi_y)$ (2a)

$\phi\vec{b}_z = (i\rho_z, \rho_z, - i\phi_z), \phi\vec{b}_{\bar{z}} = (-i \bar{\rho}_{\bar{z}}, \bar{\rho}_{\bar{z}}, i\phi_z)$ (2b)

and R-gauge field strengths $F_{\mu\upsilon}$ are given by

$F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - B_\mu B_\nu + B_\nu B_\mu$ (3a)

$B_\mu = b^i_\mu X_i$ (3b)

and $X_i = - \left({}^1/_2\right) i\sigma_i$ (3c)

where σ_i are 2 × 2 Pauli matrices.

All such solutions represent the condition of self-duality except when ϕ is zero. Because when ϕ is zero $F_{\mu\nu}$ becomes singular and the solutions can only be treated as solutions of Yang's R-gauge equations and not self-dual solutions unless a transformation like $F'_{\mu\nu} \rightarrow U^{-1} F_{\mu\nu} U$ removes the singularities.

Yang [1] and several other authors [2,3,4,5,6,7] have presented solutions to (1 a,b) or its equivalent in real variables. Chakraborty and Chanda [8] reported some graphical representation of one these exact solutions. It is observed from there that the solutions represent spreading wave with solitary profile and spreading wave packet. These profiles of solitary wave and wave packet tend to vanish as time tends to infinity.

Jimbo, Kruskal and Miwa [9] adopted the algorithm of Weiss, Tabor and Carnavalel [10] and showed that the eqs (1) and (2) pass the Painleve test for integrability. Using the same algorithm, Chakraborty and Chanda [11] have found that the real form of eqs (1a) and (1b), i.e. equations 4a and 4b (stated subsequently) with ε =1 and k' =1 pass the Painleve test for integrability and admit truncation of series leading to non-trivial exact solutions obtained previously and auto-Backlund transformation between two pairs of these solutions (see, for example, the work of Larsen [12] and Roychowdhury [13]). An important aspect of the work of Chakraborty and Chanda [11] was that they had analyzed the equation keeping the singularity manifold completely general, whereas Jimbo et al. [9] analysed the same equation with a restricted nature of singularity manifold.

With this background and success, Saha and Chanda [14] generalized the eqns (1a,b) in real form to the equations (4a), (4b) and (4c) and called the generalized set of equations as the *Generalized Yang's equations*.

$\phi_{11} + \phi_{22} + \phi_{33} + \varepsilon\phi_{44} =$

$k'[(1/\phi) (\phi_1^2 + \phi_2^2 + \phi_3^2 +\varepsilon\ \phi_4^2) - (1/\phi) (\psi_1^2 + \psi_2^2 + \psi_3^2 + \varepsilon\psi_4^2)$

$- (1/\phi) (\chi_1^2 + \chi_2^2 + \chi_3^2 +\varepsilon\ \chi_4^2) - (2/\phi) (\psi_1\chi_2 - \psi_2\chi_1 - \psi_4\chi_3 - \psi_3\chi_4)]$ (4a)

$\psi_{11} + \psi_{22} + \psi_{33} +\varepsilon\ \psi_{44} =$

$k'[(2/\phi) (\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \varepsilon\phi_4\psi_4) + (2/\phi) (\phi\chi_2 - \phi_2\chi_1 + \phi_4\chi_3 - \phi_3\chi_4)]$ (4b)

$\chi_{11} + \chi_{22} + \chi_{33} + \varepsilon\chi_{44} =$

$k'[(2/\phi) (\phi_1\chi_1+\phi_2\chi_2+\phi_3\chi_3+\varepsilon\phi_4\chi_4) + (2/\phi) (\phi_2\psi_1-\phi_1\psi_2+\phi_3\psi_4-\phi_4\psi_3)]$ (4c)

where $\varepsilon = \pm 1$, k' are arbitrary constants.
Saha and Chanda [14] presented exact solutions along with their graphical representation for the generalized Yang equations with (i) $\varepsilon = 1, k' = 1$, (ii) $\varepsilon = 1, k' = 1/2$, (iii) $\varepsilon = -1, k' = 1$, (iv) $\varepsilon = -1, k' = 1/2$. They followed the procedure similar to that of Ray[16]. The solutions represent interesting physical characteristics like waves with solitary profile, spreading wave packets, waves with pulsating solitary profile (between zero and maxima), waves with solitary profile and chaos. Saha and Chanda commented in another study [15] on some interesting results in relation to the Painleve' properties for the equations (4).

Here we try to arrive at exact solution to (4 a,b,c) using the procedure adopted by Chakraborty, Chanda and Ray [6]. Instead of getting an explicit solution we arrive at a well defined problem that we feel to be important for reporting. We have tried to understand the problem by associating it to the Painlev e' property of the equations. In this paper we restrict our discussion for $\varepsilon = 1$ only.

2. Observations in relation to exact solutions

Ansatz used by Saha and Chanda [15] can be written as

$$\phi = \phi(u), \psi = \psi(u), \chi = \chi(u) \tag{5a}$$

where u is an unspecified function of x^1, x^2, x^3, x^4.
Ansatz used by Chakraborty, Chanda and Ray[6] can be written as

$$\phi = \phi(\tau, \sigma) \tag{6a}$$

$$\psi = \psi(\tau, \sigma) \tag{6b}$$

$$\chi = \chi(\tau, \sigma) \tag{6c}$$

$$\tau = \tau(x^1, x^2) \tag{6d}$$

$$\sigma = \sigma(x^3, x^4) \tag{6e}$$

Here we have proceeded with ansatz used by Chakraborty and Chanda, i.e (6a,b,c,d,e).
We have used the same procedure that has been used by Chakraborty, Chanda and Ray [6].

$$\Phi_{XX} = -[k' K_{19}^2 (K_{15}^2 + 1) + 2k' K_{19} K_{15} + 1] \exp(2(2k' - 1)\Phi)$$
$$- [k' K_{16}^2 (K_{15}^2 + 1) + (1 - k')\Phi_X^2] \exp(-2\Phi) \tag{7a}$$

$$\psi = K_{15}\chi + u(\Phi) \tag{7b}$$

$$\chi = K_{19} \int \exp(2k'\Phi) dX + K_{16} Y + K_{18} \tag{7c}$$

where X and Y are mutually conjugate Laplace solutions in x^1 and x^2 and $\phi = \exp(\Phi)$

One can note that one must find Φ first so as to enable one to find ψ and χ. But it appears that (7a) cannot be represented even in the quadrature form.

Thus in order to investigate the integrable situations we take up only the equation (7a) which can be rewritten using $\phi = \exp(\Phi)$ as

$$\phi_{XX}\phi+\left[\frac{1-k^{'}}{\phi^{2}}-1\right]\phi_{X}^{2}+\left(k^{'}A+1\right)\phi^{4k^{'}}+k^{'}B=0 \tag{8}$$

where

$A=K_{19}^{2}(K_{15}^{2}+1)+2K_{19}K_{15}+1$

$B=K_{16}^{2}(K_{15}^{2}+1)$

We term eqn (8) as the *First Reduced Equation* obtained from the *Generalized Yang's equation* (4 a,b,c)

It is clear from the third term (from left) of the equation (8) that the minimum integer (excluding zero) requirement for $k^{'}$ is that it should be of the form $\frac{n}{4}$, where n = positive or negative integer (excluding zero).

For $k^{'}$ =1, the equation (8) reduces to

$$\phi_{XX}\phi-\phi_{X}^{2}+\left(A+1\right)\phi^{4}+B=0 \tag{9}$$

We may call this equation as the *first Reduced equation* obtained from the *Yang's equation* (4 a,b,c with $k^{'}=1$ and $\varepsilon=1$). In this situation the equation (9) could be integrated and inverted and Chanda, Chakraborty and Ray [6] obtained explicit exact solutions.

3. Observations in relation to Painleve' property

Painleve' property, in simple terms, means the absence of any singularity other than poles (in case of nonlinear ordinary differential equations). The property was first successfully used by Kovalevskaya [17] for identifying integrable nonlinear ordinary differential equations. Ablowitz, Ramani and Segur (ARS) formulated [19,20] an algorithm for identifying ordinary differential equations. They extended the idea to nonlinear partial differential equations for which all the reduced nonlinear ordinary differential equations possess the Painleve' property and such set of nonlinear partial differential equations are integrable. According to this conjecture even if a particular reduced nonlinear ordinary equation be identified to be one which does not possess the Painleve' property the originating non-linear partial differential equations should not be stated to have Painleve' property and to be completely integrable. One should be careful about having full faith upon this conjecture. It has become successful in numerous situations and failed to do in others. Keeping this constraint in mind and having faith upon numerous cases of successes we have checked the trouble making reduced equation (8) for having Painleve' property with the help of ARS algorithm.

3.1 Painleve' property for the First reduced equation (Eq.9) from Yang's Euclidean R-Gauge equations (1a,b) i.e. 4a,b,c with $\varepsilon=1,k^{'}=1$

Leading Order Analysis

In the equation (9), we put

$$\phi=\sum\phi_{j}u^{j+\alpha} \tag{10}$$

where $u=x-x_{0},\phi=\phi(x),\phi_{j}=$ constant

x_0 = constant

$\alpha =$negative integers

Balancing the most singular terms we get

$$\alpha = -1 \tag{11}$$

Equating the co-efficients of the terms to zero (called leading order terms) that balance one another to zero we get,

$$\alpha(\alpha-1)\phi_0^2 - \alpha^2\phi_0^2 + (A+1)\phi_0^4 = 0 \tag{12a}$$

$$\phi_0^2 = -\frac{1}{A+1} \tag{12b}$$

Thus with the expansion (10) now becomes

$$\phi = \sum_{j=0}^{\infty} \phi_j u^{j-1} \tag{13}$$

where $u = x - x_0, \phi = \phi(x), \phi_j =$ constant, x_0 = constant

$$\phi_0^2 = -\frac{1}{A+1}$$

Resonance Analysis

Now we equate the co-efficients of u^{r-4} source of most singular terms after putting (10) in the equations with $\alpha = -1$ to zero, where we get

r = -1,2

To check whether the Expansion (13) permits two arbitrary functions

We put (13) in (12a) and equate the coefficients of u^{r-4} , j = 1, 2 to zero and get

$\phi_1 = 0$, $\phi_2 =$ arbitrary

The arbitrary constants are x_0 (corresponding to r = -1) and ϕ_2 (corresponding to r = 2)

Thus it is found that the number of arbitrary functions in the expansion (10) are equal to that required for a general solution of (9).

This satisfies the criteria of having the Painleve' property according to ARS algorithm [19,20].

3.2. Painleve' analysis for reduced generalized Yang's equations (8)

3.2.1. *For* $k' = \frac{1}{2}$

The reduced generalized Yangs equation assumes the form

$$2\phi_{XX}\phi^3 + \phi_X^2 - 2\phi^2\phi_X^2 + (A+2)\phi^4 + B\phi^2 = 0 \tag{14}$$

Leading Order Analysis

In the equation (14) we put

$\phi = \sum \phi_j u^{j+\alpha}$, where $u = x - x_0, \phi = \phi(x), \phi_j =$ constant, x_0 = constant (15)

In the leading order analysis for $k' = \frac{1}{2}$ we find that α appears to be arbitrary. However when the co-efficients of leading order terms are equated to zero α comes out to be zero which could not be avoided with other choices.

Thus the Reduced equation for Generalized Yang equation (4a,b,c) for $k' = \frac{1}{2}$ does not have the Painleve' property. And according to ARS conjecture the generalized Yang equations (4a,b,c) for $k' = \frac{1}{2}$ do not have Painleve property.

3.2.2. *For* $k' = \frac{1}{4}$

The reduced generalized Yangs equation assumes the form

$$4\phi_{XX}\phi^3 + 3\phi_X^2 - 4\phi^2\phi_X^2 + (A+4)\phi^3 + B\phi^2 = 0 \tag{16}$$

Leading Order Analysis:

In the equation (16) we put

$$\phi = \sum \phi_j u^{j+\alpha}\text{ , where } u = x - x_0, \phi = \phi(x), \phi_j = \text{ constant, } x_0 = \text{constant} \tag{17}$$

Performing the leading order analysis we get $\alpha = -2$.

Resonance Analysis:

Now we equate the co-efficient of u^{r-10} source of most singular terms after putting (17) with $\alpha = -2$ in the equation to zero.

$$4[(r+\alpha)(r+\alpha-1)\phi_r\phi_0^3 + \alpha(\alpha-1)\phi_r\phi_0^3 + \alpha(\alpha-1)\phi_r\phi_0^3 + \alpha(\alpha-1)\phi_r\phi_0^3]$$

$$-4[\alpha^2\phi_r\phi_0^3 + \alpha^2\phi_r\phi_0^3 + (r+\alpha)\alpha\phi_r\phi_0^3 + (r+\alpha)\alpha\phi_r\phi_0^3] + 5(3A+4)\phi_r\phi_0^3 = 0$$

and get

$$4r^2 - 4r + (15A+52) = 0$$

which leads to

$$r = \frac{4 \pm i\sqrt{816+240A}}{8}$$

$$A = K_{19}^2(K_{15}^2+1) + 2K_{19}K_{15}$$

In order to avoid the imaginary part one can choose A to be negative. But the restrictions on A as imposed at the time of arriving at (8) shows that only those negative values of A are permitted for which $|A| < 1$.

Thus, $816 + 240A$ is always positive.

r is always complex, so the resonance analysis fails.

Thus the Reduced equation (16) for Generalized Yang Equation (4 a,b,c) with $k' = \frac{1}{4}$ and $\varepsilon = 1$ is not having Painleve' property. And, according to ARS conjecture the Generalized Yang Equations (4 a,b,c) for $k' = \frac{1}{4}$ do not have Painleve' property.

4. Conclusion

Thus we see that the Reduced Generalized Yang's equation satisfy Painleve' criteria for $k' = 1$ and does not satisfy the same for $k' = \frac{1}{2}, \frac{1}{4}$. Interestingly, the solutions in closed form are obtainable for $k^{'} = 1$ and not obtainable for $k^{'} = \frac{1}{2}, \frac{1}{4}$. This again establishes the strength of the relationship between integrability and existence of Painleve property. One question, however, remains. How can one get some information for the equations like (14), (16) etc ? One answer could be the application of the Krylov-Bogoliubov-Mitroploskii (KBM) [21,22] method with the subsequent developments [23]. One of our ongoing research work is involved in that direction.

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