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Tripled Coincidence Point Results In Partially Ordered Probabilistic Metric Spaces

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ABSTRACT

In this paper we establish a tripled coincidence point theorem in probabilistic metric spaces. Tripled fixed points are extensions of coupled fixed points, a concept which has been in focus in recent times. The result is supported with an example.

Keywords: Partially ordered set, Mixed monotone property, Tripled fixed point, Tripled coincidence point.

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1. Introduction

Coupled fixed point problems have a large share in the recent development of metric fixed point theory [1, 3, 11, 13, 15, 19]. These problems have often, but not always, been considered in metric spaces with a partial order. In fact a large number of fixed point results of different types have been treated in partially order metric spaces in recent times. The importance of these problems lie in the fact that combinations of analytic and order theoretic approaches are applied in the proofs of the theorems. There is a parallel development of fixed point theory in probabilistic metric spaces. These spaces are probabilistic generalizations of metric spaces where the metric values are distribution functions. This development was initiated in the work of Sehgal and Bharucha-Reid in [21] where they established a probabilistic version of the Banach's contraction mapping principle. Today, this line of research is a developed branch of analysis in its own right. Hadzic and Pap have given a good account of this study in their book [9]. Some more recent results are in [8, 12, 14, 16, 17, 18]. Particularly, couple fixed point in probabilistic metric spaces have been established in [5] and [10].

In a recent work [2] Berinde et al. has successfully extended the idea of the coupled fixed point to tripled fixed point and has established a tripled fixed point theorem in metric spaces with a partial ordering. The purpose of this paper is to establish a tripled fixed point result in a partially ordered probabilistic metric space. Our problem is different from that addressed by Berinde et al. in [2], not a probabilistic extension of it. The method of the proof is also different. The main result is supported with an example. In this context we note that fixed point problems in partially ordered probabilistic metric spaces have begun to be addressed in recent time in [5, 6] of which [5] is a coupled fixed point result.

2. Mathematical Preliminaries

In this section we discuss certain definitions and results which are necessary for establishing the results in the next section.

Throughout this paper $(X, \preceq)$ stands for a partially ordered set with partial order $\preceq$. By $x \succeq y$, we mean that $y \preceq x$ and by $x \prec y$, we mean that $x \preceq y$ and $x \neq y$.

Definition 2.1 [9, 20] A mapping $F : R \rightarrow R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t \in R} F(t) = 0$ and $\sup_{t \in R} F(t) = 1$, where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.

Definition 2.2 [9, 20] A binary operation $\Delta : [1,0]^2 \rightarrow [0,1]$ is called a *t*-norm if the following properties are satisfied:

(i) Δ is associative and commutative,
(ii) $\Delta(a,1) = a$ for all $a \in [0,1]$,
(iii) $\Delta(a,b) \leq \Delta(c,d)$ whenever $a \leq c$ and $b \leq d$, for all $a,b,c,d \in [0,1]$.

Typical examples of *t*-norm are $\Delta_M(a,b) = \min\{a,b\}$, $\Delta_P(a,b) = ab$.

Definition 2.3 [9, 20] A Menger space is a triplet (X, F, Δ), where X is a non empty set, F is a function defined on $X \times X$ to the set of distribution functions and Δ is a *t*-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ for all $x, y \in X$,
(ii) $F_{x,y}(s) = 1$ for all $s > 0$ and $x, y \in X$ if and only if $x = y$,
(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in X, s > 0$,
(iv) $F_{x,y}(u+v) \geq \Delta(F_{x,z}(u), F_{z,y}(v))$ for all $u, v \geq 0$ and $x, y, z \in X$.

Definition 2.4 [9, 20] Let (X, F, Δ) be a Menger space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if

$\underset{n\to\infty}{Lt}\ F_{x_n,x}(t)=1$ for all $t>0$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0<\varepsilon<1$ and $t>0$, there exists $n_0\in N$ such that $F_{x_n,x_m}(t)\geq 1-\varepsilon$ for each $n,m\geq n_0$.
(iii) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Lemma 2.5 [4] If (*X, F,* Δ) is a Menger space where Δ is continuous t-norm, then for every fixed $t>0$, if $x_n\to x, y_n\to y$, then

$$\lim_{n\to\infty} F_{x_n,y_n}(t)=F_{x,y}(t).$$

In the following lemma we note a property of a continuous function which we use in
the proof of our main result. The proof is a consequence of the definition of continuity.

Lemma 2.6 If $f:R^n\to R$ is continuous and $\{a_{i,j}\}_{i=1}^{\infty}, j=1,2,.....,n$ are such that $\liminf\limits_{i\to\infty} a_{ik}=a_k$ for all $k\neq l$ for some l and $\{a_{ii}\}_{i=1}^{\infty}$ is bounded. Then $\liminf\limits_{i\to\infty}\ f(a_{i1},a_{i2},.....,a_{in})=f(a_1,a_2,.....\liminf\limits_{i\to\infty}\ a_{il},...,a_n)$.

Definition 2.7 [7] The family of functions Φ is such that, for each $\phi\in\Phi,\phi:R^+\to R^+$ and satisfies the following conditions:
(i) ϕ is strict increasing,
(ii) ϕ is upper semi-continuous from the right
(iii) $\sum_{n=0}^{\infty}\phi^n(t)<+\infty$ for all $t>0$,
where $\phi^n(t)$ is the n-th iteration of $\phi(t)$.
It is immediate that if $\phi\in\Phi$, then $\phi(t)<t$ for all $t>0$.

Lemma 2.8 [10] Let $\{x_n\}$ be a sequence in a Menger space (*X, F,* Δ), where Δ is a minimum t-norm. If there exists a function $\phi\in\Phi$ such that
$F_{x_n,x_{n+1}}(\phi(t))\geq\min\{F_{x_{n-1},x_n}(t),F_{x_n,x_{n+1}}(t)\}$
for all $t>0, n\geq 1$. Then $\{x_n\}$ is a Cauchy sequence in *X*.

Lemma 2.9 [7] Let (*X, F,* Δ) is a Menger space. If there exists $\phi\in\Phi$ such that $F_{x,y}(\phi(t)+0)\geq F_{x,y}(t)$ for all $t>0$ and $x,y\in X$, then $x=y$.

Let $(X,\preceq)$ be a partially ordered set and $G: X \to X$ be a mapping. The mapping G is said to be non-decreasing if, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $G(x_1) \preceq G(x_2)$ and non-increasing if, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $G(x_1) \succeq G(x_2)$[1].

Definition 2.10 [2] Let $(X,\preceq)$ be a partially ordered set and $G: X \times X \times X \to X$ be a mapping. The mapping G is said to have the mixed monotone property if G is non-decreasing in its first and third arguments and is non-increasing in its second argument, that is, if, for all $x_1, x_2, y_1, y_2, z_1, z_2 \in X$,

(i) $x_1 \preceq x_2$ implies $G(x_1, y, z) \preceq G(x_2, y, z)$ for fixed $y, z \in X$,

(ii) $y_1 \preceq y_2$ implies $G(x, y_1, z) \succeq G(x, y_2, z)$ for fixed $x, z \in X$

and

(iii) $z_1 \preceq z_2$ implies $G(x, y, z_1) \preceq G(x, y, z_2)$ for fixed $x, y \in X$.

Definition 2.11 [2] Let $(X,\preceq)$ be a partially ordered set. $G: X \times X \times X \to X$ and $g: X \to X$ be two mappings. The mapping G is said to have the mixed g-monotone property if G is non-decreasing in its first and third arguments and is non-increasing in its second argument, that is, if, for all $x_1, x_2, y_1, y_2, z_1, z_2 \in X$,

(i) $gx_1 \preceq gx_2$ implies $G(x_1, y, z) \preceq G(x_2, y, z)$ for fixed $y, z \in X$,

(ii) $gy_1 \preceq gy_2$ implies $G(x, y_1, z) \succeq G(x, y_2, z)$ for fixed $x, z \in X$

and

(iii) $gz_1 \preceq gz_2$ implies $G(x, y, z_1) \preceq G(x, y, z_2)$ for fixed $x, y \in X$.

Remark. If $g = I$, the identity mapping, then Definition 2.11 reduces to Definition 2.10.

Definition 2.12 [2] Let X be a nonempty set. An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of the mapping $G: X \times X \times X \to X$ if

$$G(x, y, z) = x,\ G(y, x, y) = y \text{ and } G(z, y, x) = z.$$

Definition 2.13 [2] Let X be a nonempty set. An element $(x, y, z) \in X \times X \times X$ is called a tripled coincidence point of the mappings $G: X \times X \times X \to X$ and $g: X \to X$ if

$$G(x, y, z) = gx,\ G(y, x, y) = gy \text{ and } G(z, y, x) = gz.$$

Remark. If $g = I$, the identity mapping, then Definition 2.13 reduces to Definition 2.12.

Definition 2.14 [2] Let X be a nonempty set and the mappings $G: X \times X \times X \to X$ and $g: X \to X$ are commuting if for all $x, y, z \in X$

$$gG(x, y, z) = G(gx, gy, gz).$$

Definition 2.15 Let (X, F, Δ) be a Menger space. The mappings g and G where $g: X \to X$ and $G: X \times X \times X \to X$, are said to be compatible if for all $t > 0$

$$\lim_{n\to\infty} F_{gG(x_n, y_n, z_n), G(gx_n, gy_n, gz_n)}(t) = 1,$$
$$\lim_{n\to\infty} F_{gG(y_n, x_n, y_n), G(gy_n, gx_n, gy_n)}(t) = 1$$

and

$$\lim_{n\to\infty} F_{gG(z_n, y_n, x_n), G(gz_n, gy_n, gx_n)}(t) = 1,$$

whenever $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are sequences in X such that $\lim_{n\to\infty} G(x_n, y_n, z_n) = \lim_{n\to\infty} gx_n = x, \lim_{n\to\infty} G(y_n, x_n, y_n) = \lim_{n\to\infty} gy_n = y$ and $\lim_{n\to\infty} G(z_n, y_n, x_n) = \lim_{n\to\infty} gz_n = z$.

3. Main Results

Theorem 3.1 Let (X, F, Δ) be a complete Menger space where Δ is a minimum t-norm on which a partial ordering $\preceq$ is defined. Let $G: X \times X \times X \to X$ and $g: X \to X$ be two mappings such that G has the mixed g-monotone property. Let there exist $\phi \in \Phi$ and $q \geq 0$ such that

$$F_{G(x,y,z),G(u,v,w)}(\phi(t)) + q(1 - \max\{F_{gx,G(u,v,w)}(\phi(t)), F_{gu,G(x,y,z)}(\phi(t))\}$$
$$\geq \min\{F_{gx,gu}(t), F_{gx,G(x,y,z)}(t), F_{gu,G(u,v,w)}(t)\}, \tag{3.1}$$

for all $t > 0$, $x, y, z, u, v, w \in X$ with $gx \succeq gu$, $gy \preceq gv$ and $gz \succeq gw$. Also g is continuous, monotonic increasing, compatible with G and such that $G(X \times X \times X) \subseteq g(X)$. Also suppose either

(a) G is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$, (3.2)

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$. (3.3)

If there are $x_0, y_0, z_0 \in X$ such that $gx_0 \preceq G(x_0, y_0, z_0), gy_0 \succeq G(y_0, x_0, y_0)$ and $gz_0 \preceq G(z_0, y_0, x_0)$, then g and G have a tripled coincidence point in X, that is, there exist $x, y, z \in X$ such that $gx = G(x, y, z), gy = G(y, x, y)$ and $gz = G(z, y, x)$.

Proof. By the condition of the theorem, there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \preceq G(x_0, y_0, z_0), gy_0 \succeq G(y_0, x_0, y_0)$ and $gz_0 \preceq G(z_0, y_0, x_0)$. Since $G(X \times X \times X) \subseteq g(X)$, it is possible to define the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X as follows:

$$gx_1 = G(x_0, y_0, z_0), gy_1 = G(y_0, x_0, y_0) \text{ and } gz_1 = G(z_0, y_0, x_0)$$
$$gx_2 = G(x_1, y_1, z_1), gy_2 = G(y_1, x_1, y_1) \text{ and } gz_2 = G(z_1, y_1, x_1)$$

and, in general, for all $n \geq 0$,

$$gx_{n+1} = G(x_n, y_n, z_n)\ ,\ gy_{n+1} = G(y_n, x_n, y_n) \text{ and } gz_{n+1} = G(z_n, y_n, x_n). \tag{3.4}$$

Next, for all $n \geq 0$, we prove that

$$gx_n \preceq gx_{n+1} \tag{3.5}$$
$$gy_n \succeq gy_{n+1} \tag{3.6}$$
$$gz_n \preceq gz_{n+1}. \tag{3.7}$$

Since $gx_0 \preceq G(x_0, y_0, z_0), gy_0 \succeq G(y_0, x_0, y_0)$ and $gz_0 \preceq G(z_0, y_0, x_0)$, in view of the facts that $gx_1 = G(x_0, y_0, z_0), gy_1 = G(y_0, x_0, y_0)$ and $gz_1 = G(z_0, y_0, x_0)$ we have $gx_0 \preceq gx_1, gy_0 \succeq gy_1$ and $gz_0 \preceq gz_1$. Therefore (3.5), (3.6) and (3.7) hold for $n = 0$. Let (3.5), (3.6) and (3.7) hold for some $n = m$, that is, $gx_m \preceq gx_{m+1}, gy_m \succeq gy_{m+1}$ and $gz_m \preceq gz_{m+1}$. As *G* has the mixed *g*-monotone property, from (3.4), we get

$$gx_{m+1} = G(x_m, y_m, z_m) \preceq G(x_{m+1}, y_m, z_m) \preceq G(x_{m+1}, y_{m+1}, z_{m+1}) = gx_{m+2}\ ,$$
$$gy_{m+1} = G(y_m, x_m, y_m) \succeq G(y_{m+1}, x_m, y_{m+1}) \succeq G(y_{m+1}, x_{m+1}, y_{m+1}) = gy_{m+2}\ ,$$
$$gz_{m+1} = G(z_m, y_m, x_m) \preceq G(z_{m+1}, y_m, x_m) \preceq G(z_{m+1}, y_{m+1}, x_{m+1}) = gz_{m+2}\ .$$

Thus (3.5), (3.6) and (3.7) hold for $n = m+1$. So, by induction, we conclude that (3.5), (3.6) and (3.7) hold for $n \geq 1$.

Now, for all $t > 0$, $n \geq 1$, we have

$$\begin{aligned}
F_{gx_n, gx_{n+1}}(\phi(t)) &= F_{G(x_{n-1}, y_{n-1}, z_{n-1}), G(x_n, y_n, z_n)}(\phi(t)) \qquad \text{(by(3.4))}\\
&\geq \min\{F_{gx_{n-1}, gx_n}(t), F_{gx_{n-1}, G(x_{n-1}, y_{n-1}, z_{n-1})}(t), F_{gx_n, G(x_n, y_n, z_n)}(t)\}\\
&\qquad -q(1-\max\{F_{gx_{n-1}, G(x_n, y_n, z_n)}(\phi(t)), F_{gx_n, G(x_{n-1}, y_{n-1}, z_{n-1})}(\phi(t))\})\\
&\qquad\qquad \text{(by(3.1))}\\
&= \min\{F_{gx_{n-1}, gx_n}(t), F_{gx_{n-1}, gx_n}(t), F_{gx_n, gx_{n+1}}(t)\}\\
&\qquad -q(1-\max\{F_{gx_{n-1}, gx_{n+1}}(\phi(t)), F_{gx_n, gx_n}(\phi(t))\})\\
&= \min\{F_{gx_{n-1}, gx_n}(t), F_{gx_n, gx_{n+1}}(t)\} - q(1-1)
\end{aligned}$$

$$= \min\{F_{gx_{n-1},gx_n}(t), F_{gx_n,gx_{n+1}}(t)\}.$$

Then, by Lemma 2.8, we conclude that $\{gx_n\}$ is a Cauchy sequence.

Again, for all $t>0$, $n \geq 1$, we have

$$F_{gy_n,gy_{n+1}}(\phi(t)) = F_{G(y_{n-1},x_{n-1},y_{n-1}),G(y_n,x_n,y_n)}(\phi(t)) \qquad \text{(by(3.4))}$$
$$\geq \min\{F_{gy_{n-1},gy_n}(t), F_{gy_{n-1},G(y_{n-1},x_{n-1},y_{n-1})}(t), F_{gy_n,G(y_n,x_n,y_n)}(t)\}$$
$$-q(1-\max\{F_{gy_{n-1},G(y_n,x_n,y_n)}(\phi(t)), F_{gy_n,G(y_{n-1},x_{n-1},y_{n-1})}(\phi(t))\}) \qquad \text{(by(3.1))}$$
$$= \min\{F_{gy_{n-1},gy_n}(t), F_{gy_{n-1},gy_n}(t), F_{gy_n,gy_{n+1}}(t)\}$$
$$-q(1-\max\{F_{gy_{n-1},gy_{n+1}}(\phi(t)), F_{gy_n,gy_n}(\phi(t))\})$$
$$= \min\{F_{gy_{n-1},gy_n}(t), F_{gy_n,gy_{n+1}}(t)\} - q(1-1)$$
$$= \min\{F_{gy_{n-1},gy_n}(t), F_{gy_n,gy_{n+1}}(t)\}.$$

Then, by Lemma 2.8, we conclude that $\{gy_n\}$ is a Cauchy sequence.

For all $t>0$, $n \geq 1$, we have

$$F_{gz_n,gz_{n+1}}(\phi(t)) = F_{G(z_{n-1},y_{n-1},x_{n-1}),G(z_n,y_n,x_n)}(\phi(t)) \qquad \text{(by(3.4))}$$
$$\geq \min\{F_{gz_{n-1},gz_n}(t), F_{gz_{n-1},G(z_{n-1},y_{n-1},x_{n-1})}(t), F_{gz_n,G(z_n,y_n,x_n)}(t)\}$$
$$-q(1-\max\{F_{gz_{n-1},G(z_n,y_n,x_n)}(\phi(t)), F_{gz_n,G(z_{n-1},y_{n-1},x_{n-1})}(\phi(t))\}) \qquad \text{(by(3.1))}$$
$$= \min\{F_{gz_{n-1},gz_n}(t), F_{gz_{n-1},gz_n}(t), F_{gz_n,gz_{n+1}}(t)\}$$
$$-q(1-\max\{F_{gz_{n-1},gz_{n+1}}(\phi(t)), F_{gz_n,gz_n}(\phi(t))\})$$
$$= \min\{F_{gz_{n-1},gz_n}(t), F_{gz_n,gz_{n+1}}(t)\} - q(1-1)$$
$$= \min\{F_{gz_{n-1},gz_n}(t), F_{gz_n,gz_{n+1}}(t)\}.$$

Then, by Lemma 2.8, we conclude that $\{gz_n\}$ is a Cauchy sequence.

Since X is complete, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} gx_n = x, \lim_{n\to\infty} gy_n = y \text{ and } \lim_{n\to\infty} gz_n = z,$$

that is, $\lim_{n\to\infty} G(x_n, y_n, z_n) = \lim_{n\to\infty} gx_{n+1} = x, \lim_{n\to\infty} G(y_n, x_n, y_n) = \lim_{n\to\infty} gy_{n+1} = y$ and

$$\lim_{n\to\infty} G(z_n, y_n, x_n) = \lim_{n\to\infty} gz_{n+1} = z. \qquad (3.8)$$

By continuity of g we get

$$\lim_{n\to\infty} g(gx_n) = gx, \lim_{n\to\infty} g(gy_n) = gy \text{ and } \lim_{n\to\infty} g(gz_n) = gz.$$

Since (g, G) is a compatible pair and using continuity of g, we have

$$gx = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} g(G(x_n, y_n, z_n)) = \lim_{n\to\infty} G(gx_n, gy_n, gz_n), \qquad (3.9)$$

$$gy = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} g(G(y_n, x_n, y_n)) = \lim_{n\to\infty} G(gy_n, gx_n, gy_n) \tag{3.10}$$

and

$$gz = \lim_{n\to\infty} g(gz_{n+1}) = \lim_{n\to\infty} g(G(z_n, y_n, x_n)) = \lim_{n\to\infty} G(gz_n, gy_n, gx_n). \tag{3.11}$$

Next we show that $gx = G(x, y, z), gy = G(y, x, y)$ and $gz = G(z, y, x)$.
Let the assumption (a) holds. From (3.9), (3.10) and (3.11), by (3.8) and continuity of G, we get

$$gx = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} G(gx_n, gy_n, gz_n) = G(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n, \lim_{n\to\infty} gz_n) = G(x, y, z)$$

$$gy = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} G(gy_n, gx_n, gy_n) = G(\lim_{n\to\infty} gy_n, \lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n) = G(y, x, y)$$

and

$$gz = \lim_{n\to\infty} g(gz_{n+1}) = \lim_{n\to\infty} G(gz_n, gy_n, gx_n) = G(\lim_{n\to\infty} gz_n, \lim_{n\to\infty} gy_n, \lim_{n\to\infty} gx_n) = G(z, y, x).$$

Next we assume that (b) holds. By (3.5), (3.6), (3.7) and (3.8), we have $\{gx_n\}$ is non-decreasing sequence with $gx_n \to x$, $\{gy_n\}$ is non-increasing sequence with $gy_n \to y$ as $n \to \infty$ and $\{gz_n\}$ is non-decreasing sequence with $gz_n \to z$. Then, by (3.2) and (3.3) we have for all $n \geq 0$,

$$gx_n \preceq x,\ gy_n \succeq y \text{ and } gz_n \preceq z.$$

Since, g is monotonic increasing, so

$$g(gx_n) \preceq gx,\ g(gy_n) \succeq gy \text{ and } g(gz_n) \preceq gz. \tag{3.12}$$

Then, for all $t > 0$, $n \geq 0$, we have for $0 < k < 1$
$F_{gx,G(x,y,z)}(\phi(t)) \geq \Delta\{F_{gx,g(gx_{n+1})}(\phi(t) - \phi(kt)), F_{g(gx_{n+1}),G(x,y,z)}(\phi(kt))\}$.
Again, (3.9) implies that for all $t > 0$,

$$\lim_{n\to\infty} F_{gx,g(gx_{n+1})}(t) = 1. \tag{3.13}$$

Taking limit on both sides of above inequality, for all $t > 0$, we have

$$\begin{aligned} F_{gx,G(x,y,z)}(\phi(t)) &\geq \liminf_{n\to\infty} \Delta\{F_{gx,g(gx_{n+1})}(\phi(t) - \phi(kt)), F_{g(gx_{n+1}),G(x,y,z)}(\phi(kt))\} \\ &= \Delta\{\lim_{n\to\infty} F_{gx,g(gx_{n+1})}(\phi(t) - \phi(kt)), \liminf_{n\to\infty} F_{g(gx_{n+1}),G(x,y,z)}(\phi(kt))\} \\ &\qquad \text{(by the continuity } \Delta\text{, (3.13) and Lemma 2.6)} \\ &= \min\{1, \liminf_{n\to\infty} F_{G(gx_n, gy_n, gz_n),G(x,y,z)}(\phi(kt))\} \end{aligned}$$

$$\begin{aligned}
&= \liminf_{n\to\infty} F_{G(gx_n,gy_n,gz_n),G(x,y,z)}(\phi(kt)) \\
&\geq \liminf_{n\to\infty}[\min\{F_{g(gx_n),gx}(kt), F_{g(gx_n),G(gx_n,gy_n,gz_n)}(kt), F_{gx,G(x,y,z)}(kt)\} \\
&\qquad -q(1-\max\{F_{g(gx_n),G(x,y,z)}(\phi(kt)), F_{gx,G(gx_n,gy_n,gz_n)}(\phi(kt))\}] \\
&\qquad\qquad \text{(by (3.1) and (3.12))} \\
&= \min\{\lim_{n\to\infty} F_{g(gx_n),gx}(kt), \liminf_{n\to\infty} F_{g(gx_n),G(gx_n,gy_n,gz_n)}(kt), F_{gx,G(x,y,z)}(kt)\} \\
&\qquad -q(1-\max\{\lim_{n\to\infty} F_{g(gx_n),G(x,y,z)}(\phi(kt)), \lim_{n\to\infty} F_{gx,G(gx_n,gy_n,gz_n)}(\phi(kt))\}) \\
&\qquad\qquad \text{(by lemma 2.5 and (3.13))} \\
&= \min\{F_{gx,gx}(kt), F_{gx,gx}(kt), F_{gx,G(x,y,z)}(kt)\} \\
&\qquad -q(1-\max\{F_{gx,G(x,y,z)}(\phi(kt)), F_{gx,gx}(\phi(kt))\}) \\
&= \min\{F_{gx,gx}(kt), F_{gx,G(x,y,z)}(kt)\} \\
&\qquad -q(1-\max\{F_{gx,G(x,y,z)}(\phi(kt)), F_{gx,gx}(\phi(kt))\}) \\
&\geq \min\{1, F_{gx,G(x,y,z)}(kt)\} - q(1-1) \\
&\geq F_{gx,G(x,y,z)}(kt) .
\end{aligned}$$

The value of k being arbitrarily in $(0,1)$, taking $k \to 1$, and using the left continuity of F, we have

$$F_{gx,G(x,y,z)}(\phi(t)) \geq F_{gx,G(x,y,z)}(t) .$$

Since ϕ is increasing, $\phi(t)+0 \geq \phi(t)$. Again F is monotone increasing.
Therefore $F_{gx,G(x,y,z)}(\phi(t)+0) \geq F_{gx,G(x,y,z)}(\phi(t)) \geq F_{gx,G(x,y,z)}(t)$.
Then, by an application of lemma 2.9, we get $gx = G(x,y,z)$.
Similarly, we can show that $gy = G(y,x,y)$ and $gz = G(z,y,x)$, that is, g and G have a tripled coincidence point in X.
This completes the proof of the Theorem 3.1.

Corollary 3.2 Let (X,F,Δ) be a complete Menger space where $\Delta = \Delta_M$, the minimum t-norm, on which a partial ordering $\preceq$ is defined.
Let $G: X\times X\times X \to X$ and $g: X \to X$ be two mappings such that G has the mixed g-monotone property. Let there exists $\phi \in \Phi$ such that

$$F_{G(x,y,z),G(u,v,w)}(\phi(t)) \geq \min\{F_{gx,gu}(t), F_{gx,G(x,y,z)}(t), F_{gu,G(u,v,w)}(t)\}, \quad (3.14)$$

for all $t > 0$, $x,y,z,u,v,w \in X$ with $gx \succeq gu, gy \preceq gv$ and $gz \succeq gw$. Also g is continuous, monotonic increasing, commutating with G and such that $G(X\times X\times X) \subseteq g(X)$. Also suppose either
(a) G is continuous or
(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$.

If there are $x_0, y_0, z_0 \in X$ such that $gx_0 \preceq G(x_0, y_0, z_0), gy_0 \succeq G(y_0, x_0, y_0)$ and $gz_0 \preceq G(z_0, y_0, x_0)$, then g and G have a tripled coincidence point in X, that is, there exist $x, y, z \in X$ such that $gx = G(x, y, z), gy = G(y, x, y)$ and $gz = G(z, y, x)$.

Proof. Since commutativity implies compatibility, the proof is completed by an application of theorem 3.1 in case where $q = 0$.

We have the following corollary if we take $\phi(t) = kt$ in Theorem 3.1.

Corollary 3.3 Let (X, F, Δ) be a complete Menger space where $\Delta = \Delta_M$, the minimum t-norm, on which a partial ordering $\preceq$ is defined.

Let $G: X \times X \times X \to X$ and $g: X \to X$ be two mappings such that G has the mixed g-monotone property. Let there exist $\phi \in \Phi$ and $q \geq 0$ such that

$$F_{G(x,y,z),G(u,v,w)}(kt) + q(1 - \max\{F_{gx,G(u,v,w)}(kt), F_{gu,G(x,y,z)}(kt)\}$$
$$\geq \min\{F_{gx,gu}(t), F_{gx,G(x,y,z)}(t), F_{gu,G(u,v,w)}(t)\},$$

for all $t > 0$, $x, y, z, u, v, w \in X$, $0 < k < 1$ with $gx \succeq gu, gy \preceq gv$ and $gz \succeq gw$.

Also g is continuous, monotonic increasing, compatible with G and such that $G(X \times X \times X) \subseteq g(X)$. Also suppose either

(a) G is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$.

If there are $x_0, y_0, z_0 \in X$ such that $gx_0 \preceq G(x_0, y_0, z_0), gy_0 \succeq G(y_0, x_0, y_0)$ and $gz_0 \preceq G(z_0, y_0, x_0)$, then g and G have a tripled coincidence point in X, that is, there exist $x, y, z \in X$ such that $gx = G(x, y, z), gy = G(y, x, y)$ and $gz = G(z, y, x)$.

Example 3.4 Let $(X, \preceq)$ is the partially ordered set with $X = [0,1]$ and the natural Ordering $\leq$ of the real numbers as the partially ordering $\preceq$. Let $F_{x,y}(t) = e^{-\frac{|x-y|}{t}}$ for all $x, y \in X$ and $\Delta = \Delta_M$, the minimum t-norm, then (X, F, Δ) is a complete Menger space.

Let the mapping $g: X \to X$ be defined as

$$gx = x \text{ for all } x \in X$$

and the mapping $G: X \times X \times X \to X$ be defined as

$$G(x,y,z)=\begin{cases}\dfrac{x-y+z}{6}, & if\ \ x,y,z\in[0,1], x\geq y\geq z,\\ 0, & otherwise.\end{cases}$$

Here G satisfies the mixed g-monotone property. $G(X\times X\times X)\subseteq g(X)$, g is continuous, monotonic increasing and commutating with G and ϕ is a Φ-function with $\phi(t)=\frac{2}{3}t$ for $t\in[0,\infty)$.

Let $x_0=0, z_0=0$ and $y_0=c>0$.

Then $gx_0=g0=G(0,c,0)=G(x_0,y_0,z_0)$,

$gy_0=gc=c>0=G(c,0,c)=G(y_0,x_0,y_0)$ and

$gz_0=g0=G(0,c,0)=G(z_0,y_0,x_0)$.

Thus x_0, y_0 and z_0 satisfy their requirements in corollary 3.2.

Let $x,y,z,u,v,w\in X$ are such that $gx\geq gu$, $gy\leq gv$ and $gz=gw$, that is, $x\geq u$, $y\leq v$ and $z=w$.

We show that the inequality (3.14) is satisfied for all $t>0$ and x,y,z,u,v,w chosen to satisfy the above requirements.

Let $M=\min\{e^{-\frac{|x-u|}{t}}, e^{-\frac{|x-G(x,y,z)|}{t}}, e^{-\frac{|u-G(u,v.w)|}{t}}\}$.

We consider the following possible cases.

Case I. $x\geq y\geq z$ and $u\geq v\geq w$,

$$\begin{aligned}F_{G(x,y,z),G(u,v,w)}(\phi(t)) &= e^{-\frac{|G(x,y,z)-G(u,v,w)|}{\phi(t)}}\\ &= e^{-\frac{|\frac{x-y+z}{6}-\frac{u-v+w}{6}|}{\phi(t)}}\\ &= e^{-\frac{|(x-u)-(y-v)+(z-w)|}{6\phi(t)}}\\ &= e^{-\frac{|(x-u)-(y-v)+(z-w)|}{4t}} \qquad \text{(since } \phi(t)=\tfrac{2}{3}t\text{)}\\ &\geq e^{-\frac{|(x-u)|+|(y-v)|+|(z-w)|}{4t}}\\ &\geq e^{-\frac{|(x-v)|+|(x-v)|+|(z-w)|}{4t}}\end{aligned}$$

(since $x\geq y\geq z$ and $u\geq v\geq w$)

$$= e^{-\frac{|(x-v)|}{4t}}.e^{-\frac{|(x-v)|}{4t}}.e^{-\frac{|(z-w)|}{4t}}$$

$$= e^{-\frac{|(x-v)|}{4t}}.e^{-\frac{|(x-v)|}{4t}} \qquad \text{(by taking } w = z \text{)}$$
$$= e^{-\frac{|(x-v)|}{2t}}$$
$$\geq e^{-(\frac{x}{t}-\frac{x-y+z}{6t})} \qquad \text{(since } (\frac{x}{t}-\frac{x-y+z}{6t}-\frac{x-v}{2t}) > 0 \text{)}$$
$$= M$$

Case II. $x \geq y \geq z$ and $v \geq u \geq w$ ($x \geq y \geq z$ and $v \geq w \geq u$),

or

$u \geq v \geq w$ and $x \geq z \geq y$ ($u \geq v \geq w$ and $z \geq x \geq y$).

$$F_{G(x,y,z),G(u,v,w)}(\phi(t)) = e^{-\frac{|G(x,y,z)-G(u,v,w)|}{\phi(t)}}$$
$$= e^{-\frac{|\frac{x-y+z}{6}-0|}{\phi(t)}}$$
$$= e^{-\frac{|x-y+z|}{6\phi(t)}}$$
$$= e^{-\frac{|x-y+z|}{4t}} \qquad \text{(since } \phi(t) = \frac{2}{3}t \text{)}$$
$$\geq e^{-(\frac{x}{t}-\frac{x-y+z}{6t})} \qquad \text{(since } (\frac{x}{t}-\frac{x-y+z}{6t}-\frac{x-y+z}{4t}) > 0 \text{)}$$
$$= M$$

Case III. $x < y < z$ and $u < v < w$ ($x < y < z$ and $u < w < v$)

or

$u < v < w$ and $y < z < x$ ($u < v < w$ and $y < x < z$).

In this case the inequality (3.14) is trivially satisfied.

Taking into account all the three cases mentioned above, we conclude that the inequality (3.14) is satisfied by x, y, z, u, v, w chosen according to the conditions given in corollary 3.2 and for all $t > 0$. Thus all the conditions of corollary 3.2 are satisfied. Then, by an application of the corollary 3.2, we conclude that g and G have a tripled coincidence point. Here $(0,0,0)$ is a tripled coincidence point of g and G in X.

Remark

Since by omitting the third variable, coupled coincidence point results are obtained from tripled coincidence point results, so our result is a genuine extension of the result proved by Hu and Ma[10].

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REFERENCES

1. T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., 65 (2006) 1379–1393.
2. V. Berinde and M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal., 74 (2011) 4889-4897.
3. B.S. Choudhury and A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal., 73 (2010) 2524-2531.
4. S.S. Chang, Y.J. Cho, S.M. Kang, Probablistic Metric Spaces and Nonlinear Operator Theory, Sichuan University Press, Chengdu, China, 1994.
5. L. Ćirić, R.P. Agarwal, B. Samet, Mixed monotone generalized contractions in partially ordered Probabilistic metric spaces, Fixed point theory and Applications, 2011:56.
6. L. Ćirić, D. Mihet, R. Saadati, Monotone generalized contractions in partially ordered Probabilistic metric spaces, Topology and its Applications, 156 (2009) 2838-2844.
7. J.X. Fang, Common fixed point theorems of compatible and weak compatible maps in Menger spaces, Nonlinear Anal., 71 (2009) 1833-1843.
8. J.X. Fang, Y. Gao, Common fixed point theorems under strict contractive conditions in Menger spaces, Nonlinear Anal., 70 (2009) 184-193.
9. O. Hadzic, E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic Publishers, 2001.
10. X.Q. Hu, X.Y. Ma, Coupled coincidence point theorems under contractive conditions in partially ordered probabilistic metric spaces, Nonlinear Anal., 74 (2011) 6451-6458.
11. V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal., 70 (2009) 4341-4349.
12. Y. Liu, Z. Li, Coincidence point theorem in probabilistic and fuzzy metric spaces, Fuzzy Sets and Systems, 158 (2009) 58-70.
13. N.V. Luong, N.X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal., 74 (2011) 983-992.
14. D. Mihet, Altering distances in probabilistic Menger spaces, Nonlinear Anal., 71 (2009) 2734-2738.
15. H.K. Nashine, W. Shatanawi, Coupled common fixed point theorems for a pair of commuting mappings in partially ordered complete metric spaces, Comput. Math. Appl., 62 (2011) 1984-1993.

16. D. O'Regan, R. Saadati, Nonlinear contraction theorems in probabilistic spaces, Appl. Math. Comput., 195 (2008) 86-93.
17. V. Radu, Sehagal contractions on Menger spaces, Fixed Point Theory, 7 (2006) 315-322.
18. A. Razani, K. Fouladgar, Extension of contractive maps in the Menger probabilistic metric space, Chaos, Solitons and Fractals, 34 (2007) 1724-1731.
19. B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal., 72 (2010) 4508-4517.
20. B. Schweizer, A. Sklar, Probabilistic Metric Spaces, Elsevier, North-Holland, 2009.
21. V.M. Sehgal, A.T. Bharucha-Reid, Fixed point of contraction mappings on PM space, Math. Sys. Theory, 6(2) (1972) 97-100.