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Anti-Hausdorff Spaces

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ABSTRACT

A special class of topological spaces termed anti-Hausdorff spaces has been defined and some properties of such spaces have been studied. A few characterizations of an anti-Hausdorff space have been established also. A necessary and sufficient condition for the spectrum of a commutative ring with 1 to be anti-Hausdorff has been proved.

Keywords: Anti-Hausdorff space, irreducible space, Zariski topology, spectrum

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1. Introduction

Anti-Hausdorff spaces have been defined as spaces which are completely opposite of Hausedorff spaces. These are spaces where no two distinct points can be separated by disjoint open sets. It has been proved that these spaces are precisely the irreducible spaces applied in algebraic geometry and scheme theory. A few other characterizations of such spaces have been stated and proved. The important topology used in algebraic geometry , viz., the Zariski topology on the set of prime ideals of a commutative ring with 1, is anti-Hausdorff in certain cases. This claim has also been established here. Anti-Hausdorff spaces have also been studied by the authors in [3] and [4].

2. Anti-Hausdorff spaces

Definition 2.1. A topological space X with $|X| \geq 2$ is said to be anti-Hausdorff if, for every pair of distinct points x, y in X and every pair of distinct open sets U and V such that x $\in$ U, y $\in$ V, $U \cap V \neq \Phi$, i.e. , if no two distinct points can be separated by disjoint open sets.

Clearly, a set with at least two elements and with the indiscrete topology is an anti Hausdorff space, and every infinite set with the cofinite topology is an anti Hausdorff space. Also if X={a, b, c, d} and $\Im$ = {X, Φ , {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}}, then (X, $\Im$) is an anti-Hausdorff space.

Infact, there are numerous topological spaces which are anti –Hausdorff as described in the following examples:

Example 2.2. A nonempty set with a chain topology on X is anti-Hausdorff. (Here, a chain topology on X is a topology which is totally ordered under inclusion.) This is true, since no two non-empty open sets in a chain topology are disjoint. A set with 2 elements has 3 chain topologies and a set with 3 elements has 13 chain topologies.

Example 2.3. Let (X, $\Im$) be a topological space. If A$\subseteq$X and $\Im = \widehat{S}_A(X)$ is the superset topology on X with respect to, then (X, $\Im$) is anti-Hausdorff. Here $\widehat{S}_A(X)$ is the collection of supersets of A in X together with Φ. A set with 2 elements has 3 superset topologies $\widehat{S}_A$, with A$\neq\Phi$; and a set with 3 elements has 7 such superset topologies. The reason for (X, $\Im$)'s being anti-Hausdorff is the same as that in the previous example. However, for the empty subset Φ of X, $\widehat{S}_\Phi(X)$ is a discrete space and so, X is both disconnected and Hausdorff.

We now prove the following results:

Theorem 2.4. A subspace of an anti-Hausdorff space need not be anti-Hausdorff.

Proof: Let us consider the space (X, $\Im$) where X={a, b, c, d} and $\Im$ = {X, Φ, {a}, {a, b}, {a, c}, {a, b, c}}. Then (X, $\Im$) is an anti-Hausdorff space, since there is no pair of disjoint non-empty open sets in X. Now let Y={b, c}. Then as a subspace of X ,Y has the topology $\Im^{/}$={Y, Φ, {b}, {c}}. Obviously, Y is not anti-Hausdorff.□

Theorem 2.5. If A and B are two anti-Hausdorff subspaces of a topological space X, then the subspace A$\cap$B need not be anti-Hausdorff.

Proof: Let X= {a, b, c, d, e}, $\Im$ = {X, Φ, {a}, {b},{a, b}, {a, b, c}, {a, b, d}}. Clearly $\Im$ is a topology on X. Let A = {a, c, d} and B = {b, c, d}. Then the subspace topologies $\Im_A$ and $\Im_B$ on A and B respectively are $\Im_A$= {A, Φ,{a}, {a, c}, {a, d}} and $\Im_B$={B, Φ,{b},{b, c},{b, d}}. Clearly both A and B are anti-Hausdorff. Now $A \cap B$ = {c, d} and the subspace topology on $A \cap B$ is given by $\Im_{A\cap B}$ = { $A \cap B$, Φ,{c }, {d}}. Then $A \cap B$ is Hausdorff, and so, not anti-Hausdorf. [We note that X is neither Hausdorff nor anti-Hausdorf.] □

There is a situation in which $A \cap B$ is anti-Hausdorff even if only one of A and B is so. This is shown in corollary 2.7.

Theorem 2.6. Every open subspace of an anti-Hausdorff space is anti-Hausdorff.

Proof: Let X be an anti-Hausdorff space and Y an open subspace of X. If Y is a singleton set, the theorem is vacuously true. So, let y_1 and y_2 be two distinct points of Y, and let H_1 and H_2 be two open sets in Y such that $y_1 \in H_1$ and $y_2 \in H_2$. Now $H_1 = G_1 \cap Y$ and $H_2 = G_2 \cap Y$ for some open sets G_1 and G_2 in X. Since $y_1 \in H_1$ and $y_2 \in H_2$ and X is anti-Hausdorff, $H_1 \cap H_2 \neq \Phi$. This proves that Y is anti-Hausdorff. □

We then have the obvious deduction:

Corollary 2.7. Let X be a topological space and let A and B be two subspaces of X such that

(i) A is anti-Hausdorff,

(ii) $A \cap B$ is an open subspace of A.

Then $A \cap B$ is anti-Hausdorff.

Remark 2.8. If A_1 and A_2 are two subspaces of a topological space X, then the subspace $A_1 \cap A_2$ may be anti-Hausdorff even if neither A_1 nor A_2 is so.

Proof: Let X = {a, b, c, d}, $\Im$ = {X, Φ, {a}, {c}, {d}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}}. Let X_1 ={a, b, d}and X_2 ={a, b, c}. Then the subspace topologies on X_1 and X_2 are $\Im_1$={X_1, Φ,{a},{b},{d},{a, d},{a, b},{b,d}} and $\Im_2$ = {X_2, Φ, {a}, {c}, {a, c}, {b, c}} respectively. However, $X_1 \cap X_2$ = {a, b} is anti-Hausdorff with the subspace topology $\Im'$ ={$X_1 \cap X_2$, Φ, {a}}.□

We next prove the following theorem.

Theorem 2.9. Let A_1 and A_2 be two anti-Hausdorff spaces with topologies $\Im_1$ *and* $\Im_2$ respectively. Then $A_1 \cup A_2, \langle \Im_1 \cup \Im_2 \rangle)$ need not be anti-Hausdorff. Here $\langle \Im_1 \cup \Im_2 \rangle$ is the topology generated by $\Im_1 \cup \Im_2$ in $A_1 \cup A_2$.

Proof: Let us consider the space A_1 ={a, b, c} with the topology $\Im_1 = \{A_1, \Phi, \{a,b\}\}$ and A_2={a, d, e}, with the topology $\Im_2 = \{A_2, \Phi, \{e\}, \{d,e\}\}$.
Let $A = A_1 \cup A_2$ = {a, b, c, d, e} and let the topology $\Im$ on A be generated by $\Im_1 \cup \Im_2$, i.e.,

$$\Im = \{A, \Phi, A_1, A_2, \{a\}, \{e\}, \{a,b\}, \{d,e\}, \{a,b,e\}, \{a,b,d,e\}, \{a,b,c,e\}, \{a,b,d,e\}\}$$

So, in (X, $\Im$), $a \in \{a\}$, $e \in \{e\}$ with $\{a\}, \{e\} \in \Im$ and $\{a\} \cap \{e\} = \Phi$. Hence (X, $\Im$) is not anti- Hausdorff. □

3. Continuous Image

Theorem 3.1. Every continuous image of an anti-Hausdorff space is anti-Hausdorff.

Proof: Let X, Y be two topological space where X is anti-Hausdorff. Let f be a continuous map of X onto Y. Let y_1 and y_2 be two distinct points of Y, and let H_1 and H_2 be two open sets in Y such that $y_1 \in H_1$, $y_2 \in H_2$. Since f is onto there exist x_1, x_2 in X such that $f(x_1) = y_1$, $f(x_2) = y_2$. Let $G_1 = f^{-1}(H_1)$, $G_2 = f^{-1}(H_2)$. Since f is continuous, both G_1 and G_2 are open. Since X is anti-Hausdorff, $G_1 \cap G_2 \neq \Phi$. Let $x \in G_1 \cap G_2$, then $f(x) \in H_1 \cap H_2$. Thus $H_1 \cap H_2 \neq \Phi$. So, Y is anti-Hausdorff. □

Corollary 3.2. If X is an anti-Hausdorff space and R is an equivalence relation on X, then quotient space X/R is anti-Hausdorff.

Proof: Since the map $f : X \rightarrow X/R$ given by f(x) = cls x is continuous and onto, the proof is obvious. □

4. **Irreducible spaces**

Now we shall discuss another kind of topological spaces, called **irreducible spaces**. Such spaces have been used in Macdonald [2] and Atiyah and Macdonald [1] .

Definition 4.1. A topological space X is said to be irreducible if for every pair of non-empty open sets V, W in X, $V \cap W \neq \Phi$.

The following theorem yields a number of characterisations for an anti-Hausdorff space through identification of such spaces with irreducible spaces, some of these characterisation of irreducible spaces were mentioned in [1] without proof.

Theorem 4.2. Let X be a topological space. The following statements about X are equivalent:

(i) X is anti-Hausdorff,
(ii) X is irreducible,
(iii) Every non-empty open set in X is connected,
(iv) Every non-empty open set in X is dense in X.

Proof: To prove (i) $\Rightarrow$ (ii), let X be an anti-Hausdorff space. If possible suppose X is not irreducible. Then there exist non-empty open sets V and W in X such that $V \cap W = \Phi$. Since V and W are non-empty, there exist $x \in V$ and $y \in W$. Since $V \cap W = \Phi$, $x \neq y$. X being anti-Hausdorff, this is a contradiction. Therefore, X is irreducible.

We now prove (ii) $\Rightarrow$(i). Let X be irreducible. If possible, let X be not anti-Hausdorff. Then there exist x, y$\in$X with x $\neq$ y and open sets V and W in X with V$\cap$W = Φ and x $\in$V, y$\in$W. Since V and W are non-empty, this is a contradiction to the fact that X is irreducible. Hence X is anti-Hausdorff.

Now we shall prove that (ii) $\Rightarrow$(iii). Let X be irreducible and V be a non-empty open set in X. If V is disconnected, then there exist open sets W_1 and W_2 in X s.t. (1) $V\cap W_1 \neq \Phi$, $V\cap W_2 \neq \Phi$, (2) V= $(V\cap W_1)\cup(V\cap W_2)$ and (3) $(V\cap W_1)\cap(V\cap W_2) = \Phi$. In particular, $W_1 \neq \Phi$, $W_2 \neq \Phi$. Since X is irreducible, and V, W_1 and W_2 are open in X, the condition (2) contradicts irreducibility of X. Hence V is connected.

To prove (iii) $\Rightarrow$(ii), let every open set in X be connected. If X is not irreducible, then there exist non-empty sets V_1 and V_2 in X, such that $V_1\cap V_2 = \Phi$. This implies that the open set $V_1\cup V_2$ is a disconnected open set in X. This is a contradiction to our hypothesis. Hence X is irreducible.

We now prove (ii) $\Leftrightarrow$ (iv). Let X be an irreducible space. Let V be a non-empty open set in X and x$\in$X. Let W be an open set in X such that x$\in$W. Then W $\neq\Phi$. Since X is irreducible, W$\cap$V$\neq\Phi$. So, x$\in \overline{V}$. Thus X =$\overline{V}$. Thus (ii) $\Rightarrow$(iv).

Conversely, suppose every non-empty open set in X is dense in X. Let V and W be two non-empty open sets in X. x$\in$V. Since $\overline{W}$ =X and V is a neighbourhood of x, V$\cap$W$\neq$ Φ. So, X is irreducible. Therefore, (iv) $\Rightarrow$(ii).

The proof of the theorem is thus complete. □

5. Zariski Topology

An interesting and very useful example of anti-Hausdorff topology is Zariski topology in a special kind of rings. This topology is important in the context of algebraic geometry. For description of Zariski topology we need some elaborate background.

The following result has been stated without proof in ([1], p.11).We prove it here.

Theorem 5.1.

Let R be a commutative ring with 1 and X the set of all prime ideals of R. For any subset E of R, let F(E) be the set of all prime ideals of R which contain E. Then,

(i) F(0)=X, F(R) = Φ,

(ii) if $\{E_\alpha\}_{\alpha\in A}$ is any family of subsets of R, then $\bigcap_{\alpha\in A} F(E_\alpha)=F(\bigcup_{\alpha\in A} E_\alpha)$,

(iii) if E_1 and E_2 are subsets of R, then $F(E_1)\bigcup F(E_2)=F(E_1\bigcap E_2)$.

Proof: (i) This is obviously true.

(ii) Let $P\in\bigcap_{\alpha\in A} F(E_\alpha)$. Then for each $\alpha\in A, P\in F(E_\alpha)$ and so, for each $\alpha\in A$, $P\supseteq E_\alpha$. Therefore, $P\supseteq\bigcup_{\alpha\in A} E_\alpha$, i.e., $P\in F(\bigcup_{\alpha\in A} E_\alpha)$.

So, $\bigcap_{\alpha\in A} F(E_\alpha) \subseteq F(\bigcup_{\alpha\in A} E_\alpha)$

Now let $P \in F(\bigcup_{\alpha\in A} E_\alpha)$.Then P is a prime ideal such that, $P \supseteq \bigcup_{\alpha\in A} E_\alpha$. Hence for each $\alpha \in A$, $P \supseteq E_\alpha$, i.e., $P \in F(E_\alpha)$, for each $\alpha \in A$. So, $P \in \bigcap_{\alpha\in A} F(E_\alpha)$.

Hence, $\bigcap_{\alpha\in A} F(E_\alpha) = F(\bigcup_{\alpha\in A} E_\alpha)$.

iii) Let $P \in F(E_1) \bigcup F(E_2)$. Then either $P \in F(E_1)$ or $P \in F(E_2)$, i.e., either $P \supseteq E_1$ or $P \supseteq E_2$, $P \supseteq E_1 \bigcap E_2$, i.e., $P \in F(E_1 \bigcap E_2)$.

So, $F(E_1) \bigcup F(E_2) \subseteq F(E_1 \bigcap E_2)$.

Now, let $P \in F(E_1 \bigcap E_2)$. Then $P \supseteq E_1 \bigcap E_2$. If $P \notin F(E_1) \bigcup F(E_2)$, then $P \notin F(E_1)$ and $P \notin F(E_2)$, i.e., $E_1 \not\subset P$ and $E_2 \not\subset P$. Therefore there exist x in E_1 and y in E_2 such that $x \notin P$ and $y \notin P$.

However, $xy \in \langle xy \rangle \subseteq \langle E_1 \bigcap E_2 \rangle \subseteq P$, where $\langle xy \rangle$ and $\langle E_1 \cap E_2 \rangle$ denote the ideals generated by xy and $E_1 \cap E_2$ respectively. Since P is prime, this is a contradiction. Therefore, $P \in F(E_1) \bigcup F(E_2)$.

So, $F(E_1 \bigcap E_2) \subseteq F(E_1) \bigcup F(E_2)$.

Thus $F(E_1) \bigcup F(E_2) = F(E_1 \bigcap E_2)$.□

The above theorem shows that {F(E)| E $\subseteq$ R} is the collection of all closed sets with respect to some topology on X. This topology on X is called Zariski topology after the algebraic geometer Oscar Zariski. X together with the Zariski topology is called the spectrum of R and written as Spec(R).

We now state a result of Atiyah and Macdonald [1]. We shall prove the result by ourselves.

Theorem 5.2. Spec (R) is anti-Hausdorff if and only if the nilradical of R is a prime ideal.

Proof: Let Spec (R) be irreducible and N the nilradical of R. Then N= the intersection of all prime ideals of R. Let a, b $\in$ R such that a $\notin$ N, b $\notin$ N. Then there are prime ideals P_1 and P_2 such that a $\notin$ P_1, b $\notin$ P_1. Let G and H be the sets of prime ideals of R which do not contain {a} and {b} respectively. By definition of Spec (R), G and H are open sets in Spec (R). Since $P_1 \in$ G and $P_2 \in$ H, both G and H are non-empty. Since Spec (R) is irreducible, $G \bigcap H \neq \Phi$.

Let P $\in$ $G \bigcap H$. Then a $\notin$ P, b $\notin$ P. P being prime, ab $\notin$ P. Hence ab $\notin$ N. Therefore N is a prime ideal.

Conversely, let the nilradical N of R be a prime ideal, say P_o. Let G and H be two non-empty open sets in Spec(R) then there are prime ideals P_1 and P_2 in G and H respectively.

Now there exist sets B and C such that G = the set of all prime ideals P such that $B \not\subset P$, H= the set of all prime ideals *P'* such that $B \not\subset P'$. Hence $B \not\subset P_1, C \not\subset P_2$. It follows that neither B nor C is contained in $P_1 \cap P_2$. Since $P_0 = N, P_o \subseteq P_1 \cap P_2$, and therefore, neither B nor C is contained in P_0 so that P_0 belongs to both G and H. Thus $G \cap H \neq \Phi$. Hence Spec (R) is irreducible. □

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