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Lattices Whose Finitely Generated n-ideals Form a Generalized Stone Lattice.

[1]***M. Ayub Ali*** **and** [2]***A.S.A Noor***

[1]Department of Mathematics
Jagannath University, Dhaka, Bangladesh.
Email: ayub_ju@yahoo.com

[2]Department of ECE
East West University,Dhaka,Bangladesh
Email: noor@ewubd.edu

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ABSTRACT

In this paper the authors have studied a lattice L whose set of finitely generated n-ideals $F_n(L)$ form a generalized Stone lattice. They have shown that $F_n(L)$ is generalized Stone if and only if $<x>_n^* \vee <x>_n^{**} = L$, which is also equivalent to $(<x>_n \cap <y>_n)^* = <x>_n^* \vee <y>_n^*$ for all $x, y \in L$. $<x>_n$ denotes the principal n-ideal generated by x and $<x>_n^*$ is the pseudo complement of $<x>_n$ in the lattice of n-ideals of L. They have also shown that $F_n(L)$ is generalized Stone if and only if $P \vee Q = L$ for any two minimal prime n-ideals P and Q of L.

Keywords: Pseudo complementation, n-ideals, Stone lattice, Generalized Stone lattice.
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1 Introduction:

Finitely generated n-ideals of a lattice were studied extensively in [3], [5] and [7]. In this paper we will study those lattices whose finitely generated n-ideals form a generalized Stone lattice and we will give generalizations of several results of generalized Stone lattices in terms of n-ideals.

For a fixed element n of a lattice L, a convex sub lattice containing n is called an n-ideal. The idea of n-ideals is a kind of generalization of both ideals and filters of a lattice. The set of all n-ideals of a lattice L is denoted by $I_n(L)$, which is an algebraic lattice under set inclusion. Moreover, $\{n\}$ and L are respectively the smallest and the largest elements of $I_n(L)$.

For any two n-ideals I and J of L, it is easy to check that -

$$I \wedge J = I \cap J = \{x \in L / x = m(i,n,j) \text{ for some } i \in I,\ j \in J\}$$

where

$$m(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$$

and

$$I \vee J = \{x \in L / i_1 \wedge j_1 \le x \le i_2 \vee j_2 \text{ for some } i_1, i_2 \in I \text{ and } j_1, j_2 \in J\}$$

The n-ideal generated by a finite number of elements is called a finitely generated n-ideal. The set of all finitely generated n-ideals is denoted by $F_n(L)$. n-ideal generated by $a_1, a_2, \ldots\ldots, a_m$ is denoted by $< a_1, a_2, \ldots\ldots\ldots\ldots a_m >_n$, which is the interval $[a_1 \wedge a_2 \wedge \ldots\ldots \wedge a_m \wedge n,\ a_1 \vee a_2 \vee \ldots\ldots\ldots \vee a_m \vee n]$.

Thus, the members of $F_n(L)$ are simply the intervals [a, b] such that $a \le n \le b$. A neat description of finitely generated n-ideals can be found in [7]. By [3] and [7], we know that $F_n(L)$ is a lattice and for $[a,b]$, $[a_1,b_1]$ $\in$ $F_n(L)$, $[a,b] \cap [a_1,b_1] = [a \vee a_1, b \wedge b_1]$ and $[a,b] \vee [a_1,b_1] = [a \wedge a_1, b \vee b_1]$.

The n-ideal generated by a single element a is called principal n-ideal, denoted by $< a >_n$. Clearly, $< a >_n = [a \wedge n, a \vee n]$.

Let L be a lattice with 0 and 1. An element $a^* \in L$ is called a pseudo complement of $a \in L$, if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \le a^*$. L is called pseudo complemented if its every element has a pseudo complement.

A lattice L with 0 is called a sectionally pseudo complemented lattice if the interval [0, x] is pseudo complemented for each $x \in L$.

A distributive lattice L with 0 and 1 is called a Stone lattice if it is pseudo complemented and for each $a \in L,\ a^* \vee a^{**} = 1$.

2 Sectional pseudo complementation in the lattice of finitely generated n-ideals.

For any n-ideal J of L, we denote $J^* = \{x \in L : m(x,n,j) = n \text{ for all } j \in J\}$. Observe that J^* is an n-ideal and $J \cap J^* = \{n\}$. In fact, this is the largest n-ideal which annihilates J. We call this as the pseudo complement of J in $I_n(L)$. Moreover, for a distributive lattice L, $I_n(L)$ is a distributive algebraic lattice and so it is pseudo complemented. Observe that $F_n(L)$ has always the smallest element viz. {n}. But it does not necessarily contain the largest element. So in a general distributive lattice L with $n \in L,$ we can not talk on pseudo complementation in the lattice $F_n(L)$. But we can discuss on sectional pseudo complementation in $F_n(L)$. $F_n(L)$ is called sectionally pseudo complemented if for each $[a,b] \in F_n(L)$, the interval $[\{n\},[a,b]]$ in $F_n(L)$ is pseudo complemented. That is, each finitely generated n-ideal contained in $[a, b]$ has a relative pseudo complement in $[\{n\},[a,b]]$ which is also a member of $F_n(L)$.

We shall denote the relative pseudo complement of $[c,d]$ by $[c, d]^0$, while $[c, d]^*$ denotes the pseudo complement of $[c, d]$ in $I_n(L)$.

By [7], we know that $F_n(L) \cong (n]^d \times [n)$ where $(n]^d$ denotes the dual of the lattice $(n]$. So we have the following result.

Theorem 2.1: Let L be a distributive lattice and $n \in L,$ $F_n(L)$ is sectionally pseudo complemented if and only if $(n]$ is sectionally dual pseudo complemented and [n) is sectionally pseudo complemented. □

A distributive lattice L with 0 is called a generalized Stone lattice if for each $x \in L,\ (x]^* \vee (x]^{**} = L$. By Katrinak[2], we know that a distributive lattice L with 0 is a generalized Stone lattice if and only if each interval [0, x], $x \in L$ is a Stone lattice.

The main results of this section are given in theorem 2.8 which gives several characterizations of those $F_n(L)$ which are generalized Stone and this also generalizes some of the work of [1]. To prove this theorem we need the following results. Lemma 2.2 is trivial by the fact $F_n(L) \cong (n]^d \times [n)$, while lemma 2.3 and 2.4 are due to [5].

Lemma 2.2 : Suppose $F_n(L)$ is a sectionally pseudo complemented distributive lattice. Then $F_n(L)$ is generalized stone if and only if (n] is generalized dual Stone and [n) is generalized Stone. □

Lemma 2.3: Let L be a distributive lattice and $n \in L$**.** Then for any $[a, b] \in F_n(L)$ and for any n-ideal I, $I \cap [a,b]^* \cap [a, b] = I^* \cap [a, b]$**.** □

Lemma 2.4: Suppose $F_n(L)$ is a sectionally pseudo complemented distributive lattice and $[c\ d] \subseteq [a, b]$ in $F_n(L)$ then

(i) $[c, d]^0 = [c, d]^* \cap [a, b]$ and

(ii) $[c, d]^{00} = [c, d]^{**} \cap [a, b]$. □

Suppose $0,1 \in L$. If $[n) = [n,1]$ is pseudo complemented, then for $b \in [n)$, b^+ denotes the relative pseudo complement of b in $[n)$. Also if $(n] = [0, n]$ is dual pseudo complemented, then for $a \in (n], a^{+d}$ denotes the dual pseudo complement of a in [0, n].Following result is due to [6]

Lemma 2.5: Let F_n(L) be a distributive pseudo complemented lattice (Then of course F_n(L) has a largest element, and so 0, 1 ∈ L). Then for [a, b] $\in F_n$ (L), $[a,b]^* = [a^{+d}, b^+]$. □

If [a, b] ∈ [{n}, [c, d]]. Then {n} ⊆ [a, b] ⊆ [c, d]. The relative pseudo complement if [a, b] in above interval is denoted by $[a,b]^0$. Here $c \le a \le n \le b \le d$. a^{0d} denotes the relative dual pseudo complement of a in [c, n] and b^o denotes the relative pseudo complement of b in [n, d], if [c,n] is relatively dual pseudo complemented and [n,d] is relatively pseudo complemented. Thus by the same proof of Lemma 2.5, we have the following corollary:

Corollary 2.5.1: Let F_n **(L)** be a sectionally pseudo complemented distributive lattice. Then for $\{n\} \subseteq [a,b] \subseteq [c,d], [a,b]^0 = [a^{0d}, b^0]$. □

Theorem 2.6 : Suppose $F_n(L)$ is a sectionally pseudo complemented distributive lattice. Let x, y $\in L$ with $<x>_n \cap <y>_n = \{n\}$. Then the following conditions are equivalent:

(i) $<x>_n^* \vee <y>_n^* = L$;

(ii) For any t $\in L$; $<m(x, n, t)>_n^0 \vee <m(y,n,t)>_n^0 = <t>_n$, where $<m(x, n, t)>_n^0$

denotes the relative pseudo complement of $< m(x, n, t)>_n$ in $[\{n\}, <t>_n]$.

Proof :(i)$\Rightarrow$(ii). Suppose (i) holds. Then for any $t \in L$, using Lemma 2.4

$$< m(x, n, t)>_n^0 \vee < m(y,n,t) >_n^o$$
$$=(<x>_n \cap < t >_n)^0 \vee (< y >_n \cap < t >_n)^0$$
$$=((< x >_n \cap < t >_n)^* \cap < t >_n) \vee ((< y >_n \cap < t >_n)^* \cap < t >_n)$$
$$=((< x >_n^* \cap < t >_n) \vee (< y >_n^* \cap < t >_n)) \text{ (by Lemma 2.3)}$$
$$=((< x >_n^* \vee < y >_n^*) \cap < t >_n = L \cap < t >_n = < t >_n$$

$(ii) \Rightarrow (i)$; Suppose (ii) holds and $t \in L$. By (ii), $< m(x,n,t) >_n^0 \vee < m(y,n,t) >_n^0 = < t >_n$. Then by calculation of (i) $\Rightarrow$ (ii) , we have $(< x >_n^* \vee < y >_n^*) \cap < t >_n = < t >_n$. This implies $< t >_n \subseteq < x >_n^* \vee < y >_n^*$ and so $t \in < x >_n^* \vee < y >_n^*$. Therefore, $< x >_n^* \vee < y >_n^* = L$. □

Theorem 2.7: Let $F_n(L)$ be a sectionally pseudo complemented distributive lattice. Then the following conditions are equivalent:

- **(i)** $F_n(L)$ is generalized Stone**;**
- **(ii)** For any $x \in L, < x >_n^* \vee < y >_n^{**} = L$;
- **(iii)** For all $x, y \in L, (< x >_n \cap < y >_n)^* = < x >_n^* \vee < y >_n^*$;
- **(iv)** For all $x, y \in L, < x >_n \cap < y >_n = \{n\}$ implies that $< x >_n^* \vee < y >_n^* = L$.

Proof: (i) $\Rightarrow$ (ii) . Suppose (i) holds and $t \in L$. Then for any $x \in L$, m (x, n, t) $\in < t >_n$ and so $<m(t, n, x)>_n \in [\{n\}, < t >_n]$.

Since $F_n(L)$ is generalized Stone, so $<m(t, n, x)>_n^0 \vee < m(t,n,x) >_n^{00} = < t >_n$. Then by Lemma 2.4,

$$< t >_n = (< m(t, n, x) >_n^* \cap < t >_n) \vee (< m(t, n, x) >_n^{**}) \cap < t >_n =$$
$$((< x >_n \cap < t >_n)^* \cap < t >_n) \vee ((< x >_n \cap < t >_n)^{**} \cap < t >_n).$$

Thus by Lemma 2.3, $<t>_n = \left(<x>_n^* \cap <t>_n\right) \vee \left(<x>_n^{**} \cap <t>_n\right) =$ $= \left(<x>_n^* \vee <x>_n^{**}\right) \cap <t>_n$. This implies $<t>_n \subseteq <x>_n^* \vee <x>_n^{**}$ and so $t \in <x>_n^* \vee <x>_n^{**}$. Therefore, $<x>_n^* \vee <x>_n^{**} = L$.

(ii) $\Rightarrow$ (iii). For any $x, y \in L, (<x>_n \cap <y>_n) \cap (<x>_n^* \vee <y>_n^*) =$ $(<x>_n \cap <y>_n \cap <x>_n^*) \vee (<x>_n \cap <y>_n \cap <y>_n^*) =$ $\{n\} \vee \{n\} = \{n\}$.

Now, let $<x>_n \cap <y>_n \cap I = \{n\}$ for some n-ideal I. Then $<y>_n \cap I \subseteq <x>_n^*$. Meeting $<x>_n^{**}$ with both sides, we have $<y>_n \cap I \cap <x>_n^{**} = \{n\}$. Then $<y>_n \cap I \subseteq <x>_n^*$ and $I \cap <x>_n^{**} \subseteq <y>_n^*$.

Hence $I = I \cap L = I \cap (<x>_n^* \vee <x>_n^{**}) =$ $(I \cap (<x>_n^*) \vee (I \cap <x>_n^{**}) \subseteq <x>_n^* \vee <y>_n^*$.

Therefore, $<x>_n^* \vee <y>_n^* = \left(<x>_n \cap <y>_n\right)^*$.

(iii) $\Rightarrow$ (iv) Let $<x>_n \cap <y>_n = \{n\}$ for some x, $y \in L.$ Then by (iii), L$= \{n\}^* = \left(<x>_n \cap <y>_n\right)^* = <x>_n^* \vee <y>_n^*$. Thus (iv) holds.

(iv) $\Rightarrow$ (ii) Let $t \in L.$ By Lemma 2.3 and by Lemma 2.4, for any $x \in L., \left(<x>_n^* \vee <x>_n^{**}\right) \cap <t>_n$

$= \left(<x>_n^* \cap <t>_n\right) \vee \left(<x>_n^{**} \cap <t>_n\right)$

$= \left(\left(<x>_n \cap <t>_n\right)^* \cap <t>_n\right) \vee \left(\left(<x>_n \cap <t>_n\right)^{**} \cap <t>_n\right)$

$= \left(<m(t, n, x)>_n^* \cap <t>_n\right) \vee \left(<m(t, n, x)>_n^{**} \cap <t>_n\right)$

$= <m(x, n, t)>_n^0 \vee <m(x, n, t)>_n^{00}$. Here $<m(x, n, t)>_n^0$ is finitely generated n-ideal contained in $<t>_n$ as $F_n(L)$ is sectionally pseudo complemented. Then by [3], $<m(x, n, t)>_n^0$ is a principal n-ideal, say $<r>_n$. Now

$< m(x, n, t) >_n \cap < r >_n = \{n\}$. So by (iv) and Lemma 2.4 $< m(x, n, t) >_n^0 \vee < r >_n^0 = < t >_n$ .Therefore, $\left(< x >_n^* \vee < x >_n^{**}\right) \cap < t >_n = < t >_n$ and so $t \in < x >_n^* \vee < x >_n^{**}$. This implies $< x >_n^* \vee < x >_n^{**} = L.$ Thus (ii) holds.

To complete the proof we shall show that (iv) $\Rightarrow$ (i). Since $F_n(L)$ is sectionally pseudo complemented, so by Theorem 2.1, (n] is sectionally dual pseudo complemented and [n) is sectionally pseudo complemented.

Suppose $n \le b \le d$. Let b^0 be the relative pseudo complement of b in [n, d]. Now $b^0 \wedge b^{00} = n$. Thus $< b^0 >_n \cap < b^{00} >_n = [n, b^0 \wedge b^{00}] = \{n\}$. Also, $< b^0 >_n, < b^{00} >_n \subseteq < d >_n$. Then by the equivalent condition of Theorem 2.7, we have $< m\left(b^0, n, d\right) >_n^0 \vee < m(b^{00}, n, d) >_n^0 = < d >_n$. But $m\left(b^0, n, d\right) = b^0$ and $m\left(b^{00}, n, d\right) = b^{00}$ as $n \le b^0, b^{00} \le d.$ But by corollary 2.6 $< b^0 >_n^0 = < b^{00} >_n$ and $< b^{00} >_n^0 = < b^{000} >_n = < b^0 >_n$. Therefore, $< d >_n = < b^{00} >_n \vee < b^0 >_n = < b^0 \vee b^{00} >_n$ which gives $b^0 \vee b^{00} = d$. This implies [n, d] is a Stone lattice. That is [n) is generalized Stone.

A dual proof of above shows that (iv) also implies that (n] is a generalized dual Stone lattice. There fore, by Lemma 2.2, $F_n(L)$ is generalized Stone.

3 Minimal prime n-ideals: A prime n-ideal P of a lattice L is called a minimal prime n-ideal if there exists no prime n-ideal Q such $Q \ne P$ and $Q \subseteq P$. The following characterization of minimal prime n-ideals is due to [5].

Theorem 3.1: Let $F_n(L)$ be a sectionally pseudo complemented distributive lattice and P be a prime n-ideal of L. Then the following conditions are equivalent**.**

i) P is minimal**;**

ii) $x \in P$ implies $< x >_n^* \not\subseteq P$ **;**

iii) $x \in P$ implies $< x >_n^{**} \subseteq P$ **;**

iv) $P \cap D(< t >_n) = \phi$ for all $t \in L - P$

where **;**

$$D(<t>_n) = \{x \in <t>_n : <x>_n = \{n\}\}.$$

For a prime ideal P of a distributive lattice L with 0, Cornish in [1] has defined 0(P) = $\{x \in L : x \wedge y = 0 \text{ for some } y \in L - P\}$. Clearly, 0(P) is an ideal and 0(P) $\subseteq P$. Cornish in [1] has shown that 0(P) is the intersection of all the minimal prime ideals of L which are contained in P.

For a prime n-ideal P of a distributive lattice L, we write n(P)= $\{y \in L : m(y,n,x) = n \text{ for some } x \in L - P\}$. Clearly, $n(P)$ is an n-ideal and $n(P) \subseteq P$.

Lemma 3.1.1: Let P be a prime n-ideal in a distributive lattice L. Then each minimal prime n-ideal belonging to n(P) is contained in P.

Proof: Let Q be a minimal prime n-ideal belonging to n(P). If $Q \not\subseteq P$, then choose $y \in Q - P.$ By [5] we know that Q is either an ideal or a filter. Without loss of generality suppose Q is an ideal. Now let $S = \{s \in L : m(y,n,s) \in n(P)\}.$ We shall show that $S \not\subseteq Q$**.** If not, let $D = (L - Q) \vee [y).$ Then $n(P) \cap D = \phi.$ For otherwise, $y \wedge r \in n(P)$ for some $r \in L - Q$. Then by convexity, $y \wedge r \leq m(y,n,r) \leq (y \wedge r) \vee n$ implies $m(y,n,r) \in n(P).$ Hence $r \in S \subseteq Q$**,** which is a contradiction. Thus, by Stone's separation theorem for n-ideals in [4] there exists a prime n-ideal R containing n(P) disjoint to D. Then $R \subseteq Q$**.** Moreover, $R \neq Q$ as $y \notin R,$ this shows that Q is not a minimal prime n-ideal belonging to n(P), which is a contradiction. Therefore, $S \not\subseteq Q$**.** Hence there exists $z \notin Q$ such that $m(y,n,z) \in n(P)$. Thus $m(m(y,n,z),n,x) = n \text{ for some } x \in L - P.$ It is easy to see that $m(m(y,n,z),n,x) = m(m(y,n,z),n,z).$ Hence $m(m(y,n,x),n,z) = n.$ Since P is prime and y, $x \notin P$, so $m(y,n,x) \notin P$. Therefore, $z \in n(P) \subseteq Q$, which is a contradiction. Hence $Q \subseteq P$**.**

Proposition 3.1.2: If P is a prime n-ideal in a distributive lattice L, then n(P) is the intersection of all minimal prime n-ideals contained in P**.**

Proof: Clearly n(P) is contained in any prime n-ideal which is contained in P. Hence n(P) is contained in the intersection of all minimal prime n-ideals contained in P. Since L is distributive, so by [4], n(P) is the intersection of all minimal prime n-deals belonging to it. By [6] as each prime n-ideal contains a minimal prime n-ideal, above remarks and Lemma 2.2 establish the proposition. □

Theorem 3.4 gives another characterization of those F_n (L) which are generalized Stone in terms of minimal prime n-ideals. To prove this we need the help of the following result which is due to [6].

Theorem 3.2: Let F_n (L) be sectionally pseudo complemented distributive Lattice. Then the following conditions are equivalent**:**

- **(i)** For any $x \in L, <x>_n^* \vee <x>_n^{**} = L$**;**
- **(ii)** For all $x,\ y \in L, <x>_n \cap <y>_n = \{n\}$ implies that $<x>_n^* \vee <y>_n^* = L$**.** □

Theorem 3.3: Let F_n (L) be a sectionally pseudo complemented distributive Lattice. Then the following conditions are equivalent.

- **(i)** For any $x \in L, <x>_n^* \vee <x>_n^{**} = L$**,** equivalently**,** F_n (L) is generalized Stone**;**
- **(ii)** For any two minimal prime n-ideals P and Q**,** $P \vee Q = L$**;**
- (iii) Every prime n-ideal contains a unique minimal prime n-ideal;
- **(iv)** For each prime n-ideal P, n(P) is a prime n-ideal.

Proof: $(i) \Rightarrow (ii)$. Let $x \in P - Q$. then $<x>_n \subseteq P - Q$. Now, $<x>_n \cap <x>_n^* = \{n\} \subseteq Q$. So $<x>_n^* \subseteq Q$ as Q is prime. Again $x \in P$ implies $<x>_n^{**} \subseteq P$ by theorem 3.1. Hence by (i), $L = <x>_n^* \vee <x>_n^{**} \subseteq P$. Therefore, $P \vee Q = L$**.**

$(ii) \Leftrightarrow (iii)$ is trivial.

$(iii) \Rightarrow (iv)$ is direct consequence of Proposition 3.3

$(iv) \Rightarrow (i)$. Suppose (iv) holds. First we shall show that for all x, $y \in L$ with $<x>_n \cap <y>_n = \{n\}$ implies $<x>_n^* \vee <y>_n^* = L$. If it does not hold, then there exists $x, y \in L$ with $<x>_n \cap <y>_n = \{n\}$ such that $<x>_n^* \vee <y>_n^* \neq L$. As L is distributive, so by Stone's separation theorem, there is a prime n-ideal P such that $<x>_n^* \vee <y>_n^* \subseteq P$. Then $<x>_n^* \subseteq P$ and $<y>_n^* \subseteq P$ imply $x \notin n(P)$ and

$y \notin n(P)$. But n(P) is prime and so m(x,n,y) =n $\in n(P)$ is contradictory.

Thus for all x, y $\in L$ with $<x>_n \cap <y>_n = \{n\}$ implies that $<x>_n^* \vee <y>_n^* = L$. Hence by equivalent conditions of theorem 3.4, (i) holds. □

We conclude the paper with the following result is an immediate consequence of above theorem. This has also been proved separately in [6].

Theorem 3.4: Let F_n (L) be an pseudo complemented distributive Lattice**.** Then the following conditions are equivalent**:**

(i) F_n (L) is Stone**;**

(ii) For any two minimal prime n-ideals P and Q**,** $P \vee Q = L,$ that is, they are comaximal;

(iii) Every prime n-ideal contains a unique minimal prime n-ideal;

(iv) For each prime n-ideal P, n(P**)** is a prime n-ideal. □

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