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The EB Estimation of Scale-parameter for Two-parameter Exponential Distribution Under the Type-I Censoring Life Test

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ABSTRACT

This paper is devoted to the estimation of scale-parameter for two-parameter exponential distribution using empirical Bayes procedure. Under the type-I censoring life test, we first choose the prior distribution of the scale-parameter and find its Bayes estimation, and then we use maximum likelihood method to obtain the estimation of the super-parameter which is included in the prior distribution. finally, the empirical estimation of Scale-parameter is derived from current samples. Moreover, an illustrative example is examined numerically by means of the Monte-Carlo simulation, and this shows that such EB estimation is simple and efficient.

1. Introduction

The idea of Bayes and empirical Bayes (EB) approach is due to Robbines [1]. In recent years, A number of paper investigated the Bayes or empirical Bayes estimation of unknown parameter for specific distribution families, for example, Rayleigh [2], Gamma [3], Weibull [4], normal [5], Uniform [6], ect. Parameter estimation problems associated with the exponential distribution are of obvious interest in applied work. A.M.Sarhan [7] studied EB estimates in one-parameter exponential reliability model, Zhou [8-9] considered Bayes estimation and prediction for one-parameter and two exponential distribution, Suppose that the prior distribution of unknown parameter is unknown, Ye and Yang [10] considered the EB estimation of location parameter of two- parameter exponential lifetime distribution under type-II censoring model, and they obtained the convergence rate of EB estimation.

The purpose of this paper is to investigate the EB estimation of scale-parameter for two- parameter exponential distribution under the type-I censoring life test. A type-I censoring life test (n, t_0, n_0) means that there are n units placed on the test which is terminated at a fixed time t_0. The failure units are not replaced during the test.

The EB method has three steps. Firstly, we choose the prior distribution of the scale-parameter and get its Bayes estimation; then, use maximum likelihood method to obtain the estimation of the super-parameter which is included in the prior distribution; finally, the empirical estimation of Scale-parameter is derived from current samples.

2. Bayes estimation of scale-parameter

Suppose that the life distribution of unit X follows the two-parameter exponenttial distribution with the probability density function (pdf) $p(t \mid \mu,\sigma)$ given by

$$p(t \mid \mu,\sigma) = \sigma \exp\{-\sigma(t-\mu)\} \quad t > \mu > 0, \sigma > 0 . \tag{1}$$

where μ is location-parameter, and σ is scale-parameter. Assume that the minimum life (i.e.ensured life) μ is a known constant.

In the special case ($\mu = 0$), the distribution reduces to be ordinary one-parameter exponential distribution, whose density function is

$$p_0(t \mid \sigma) = \sigma \exp(-\sigma t) , \qquad t > 0, \sigma > 0 .$$

We now place n units and perform the type-I censoring life test. Suppose that the failure units number is r in time interval (0, t_0). Where t_0 is a fixed time. Failure times of r units are denoted by order statistics $t_{(1)} \le t_{(2)} \le \cdots \le t_{(r)} \le t_0$, $1 \le r \le n$. From [11] we can prove that the associated density function of $(t_{(1)}, t_{(2)}, \cdots, t_{(r)})$ is

$$f(t_1, t_2, \cdots, t_r \mid \sigma) = \frac{n!}{(n-r)!} \prod_{i=1}^{n} p(t_i \mid \sigma)[1 - F(t_0 \mid \sigma)]^{n-r}, \quad t_1 \le t_2 \le \cdots \le t_r \le t_0, 1 \le r \le n . \tag{2}$$

where $p(t_i \mid \sigma)$ and $F(t_0 \mid \sigma)$ respectively are regarded as density function and distribution function of the unit X_i , $i = 1, 2, \cdots, r$.

$$p(t_i \mid \mu,\sigma) = \sigma \exp\{-\sigma(t_i - \mu)\} \quad t_i > \mu > 0, \sigma > 0 .$$

$$F(t_0 \mid \sigma) = 1 - \exp\{-\sigma(t_0 - \mu)\} \quad t_0 > \mu > 0 .$$

Let t=($t_1, t_2, \cdots, t_r$) , then we can get $f(t \mid \sigma)$.

$$\begin{aligned} f(t \mid \sigma) &= \frac{n!}{(n-r)!} \prod_{i=1}^{n} p(t_i \mid \sigma)[1 - F(t_0 \mid \sigma)]^{n-r} \\ &= \frac{n!}{(n-r)!} \prod_{i=1}^{r} \sigma \exp\{-\sigma(t_i - \mu)\}[\exp\{-\sigma(t_0 - \mu)\}]^{n-r} \\ &= \frac{n!}{(n-r)!} \sigma^r \exp\{-\sigma \sum_{i=1}^{r} (t_i - \mu)\}[\exp\{-\sigma(t_0 - \mu)\}]^{n-r} \\ &= \frac{n!}{(n-r)!} \sigma^r \exp\{-\sigma[\sum_{i=1}^{r} (t_i - \mu) + (n-r)(t_0 - \mu)]\} \ . \end{aligned}$$

We choose the prior distribution of σ is

$$\pi(\sigma) = \beta \exp\{-\beta\sigma\} , \quad \sigma > 0 \tag{3}$$

Where β is a super-parameter. So, the posterior distribution of σ is

$$h(\sigma \mid t)=\frac{f(t \mid \sigma)}{\int_0^{+\infty} f(t \mid \sigma)\pi(\sigma)d\sigma}=\frac{\sigma^r \exp\{-\sigma[\sum_{i=1}^{r}(t_i-\mu)+(n-r)(t_0-\mu)+\beta]\}}{\int_0^{+\infty}\sigma^r \exp\{-\sigma[\sum_{i=1}^{r}(t_i-\mu)+(n-r)(t_0-\mu)+\beta]\}d\sigma}$$

$$=\sigma^r[\sum_{i=1}^{r}(t_i-\mu)+(n-r)(t_0-\mu)+\beta]^{r+1}\exp\{-\sigma[\sum_{i=1}^{r}(t_i-\mu)+(n-r)(t_0-\mu)+\beta]\}[\Gamma(r+1)]^{-1}$$

Under the square error loss, The Bayes estimation of scale parameter σ is

$$\hat{\sigma}_B=E(\sigma \mid t)=\int_0^{+\infty}\sigma h(\sigma \mid t)d\sigma$$

$$=\int_0^{+\infty}\sigma^{r+1}[\sum_{i=1}^{r}(t_i-\mu)+(n-r)(t_0-\mu)+\beta]^{r+1}\exp\{-\sigma[\sum_{i=1}^{r}(t_i-\mu)+(n-r)(t_0-\mu)+\beta]\}[\Gamma(r+1)]^{-1}d\sigma$$

$$=(r+1)[\sum_{i=1}^{r}(t_i-\mu)+(n-r)(t_0-\mu)+\beta]^{-1} \qquad (4)$$

3. Empirical Bayes estimation of scale-parameter

As β is an unknown constant, $\hat{\sigma}$ can not be used. In order to estimate β, we need to use the the maximum likelihood method. Since the life distribu- tion of every unit X follows two-parameter exponential distribution, and its probability density function is given by (1). So, the margin density function of X is as follows

$$f_X(t)=\int_0^{\infty}p(t \mid \mu,\sigma)\pi(\sigma)d\sigma=\int_0^{\infty}\beta\exp(-\sigma\beta)\sigma\exp[-\sigma(t-\mu)]d\sigma$$

$$=\int_0^{\infty}\beta\sigma\exp[-\sigma(t-\mu+\beta)]d\sigma=\beta(t-\mu+\beta)^{-2}.$$

$$1-F_X(t_0)=\int_{t_0}^{\infty}f_X(t)dt=\int_{t_0}^{\infty}\beta(t-\mu+\beta)^{-2}dt=\beta(t_0-\mu+\beta)^{-1}.$$

Hence, the associate density function of $(t_{(1)},t_{(2)},\cdots,t_{(r)})$ is

$$L=\frac{n!}{(n-r)!}[\prod_{i=1}^{r}f_X(t_i)][1-F_X(t_0)]^{n-r}=\frac{n!}{(n-r)!}\beta^r[\prod_{i=1}^{r}(t_i-\mu+\beta)^{-2}]\beta^{n-r}[\beta+t_0-\mu]^{-(n-r)}.$$

$$\lg L=\lg\frac{n!}{(n-r)!}+r(\lg\beta)-2\sum_{i=1}^{r}\lg(t_i-\mu+\beta)+(n-r)[\lg\beta-\lg(t_0-\mu+\beta)].$$

$$\frac{d\lg L}{d\beta}=\frac{r}{\beta}-2\sum_{i=1}^{r}\frac{1}{(t_i-\mu+\beta)}+(n-r)(\frac{1}{\beta}-\frac{1}{t_0-\mu+\beta})=g_1(\beta)-g_2(\beta).$$

Where $g_1(\beta)=\frac{r}{\beta}+(n-r)(\frac{1}{\beta}-\frac{1}{t_0-\mu+\beta})$ □ $g_2(\beta)=2\sum_{i=1}^{r}\frac{1}{t_i-\mu+\beta}$, $t_0 \ge t_i > \mu$.

In order to obtain the maximum likelihood estimation of β, we just draw a conclusion that the equation $g_1(\beta)=g_2(\beta)$ has unique solution. The reason is as follows.

For any $\beta>0$, $g_1(\beta)>0, g_1(\beta)\to 0(\beta\to\infty), g_1(\beta)\to\infty(\beta\to 0)$

$$g_1^{(1)}(\beta)=-\{(r\beta^{-2}+(n-r)[\beta^{-2}-(\beta+t_0-\mu)^{-2}]\}<0$$

$$g_1^{(2)}(\beta)=2r\beta^{-3}+(n-r)[2\beta^{-3}-2(\beta+t_0-\mu)^{-3}]\}>0.$$

Where $g_1^{(k)}(\beta)=\frac{d^k g_1(\beta)}{d\beta^k}, k=1,2.$ One arrives that $g_1(\beta)$ is strict monotone increasing concave function in $(0,+\infty)$. Similarly

For any $\beta>0,\quad g_2(\beta)>0,\quad g_2(\beta)\to 0, (\beta\to\infty), and\ g_2(\beta)\to 2\sum_{i=1}^{r}\frac{1}{2t_i-\mu}, (\beta\to 0)$

$$g_2^{(1)}(\beta)=-2\sum_{i=1}^{r}(t_i-\mu+\beta)^{-2}<0 \text{£ } g_2^{(2)}(\beta)=4\sum_{i=1}^{r}(t_i-\mu+\beta)^{-3}>0 .$$

Where $g_2^{(k)}(\beta)=\frac{d^k g_2(\beta)}{d\beta^k}, k=1,2.$ We get that $g_2(\beta)$ is also strict monotone increasing concave function in $(0,+\infty)$. Moreover

$$\lim_{\beta\to\infty}\frac{g_1(\beta)}{g_2(\beta)}=\lim_{\beta\to\infty}[\frac{r}{\beta}+(n-r)(\frac{1}{\beta}-\frac{1}{t_0-\mu+\beta})](2\sum_{i=1}^{r}\frac{1}{t_i-\mu+\beta})^{-1}=\frac{1}{2}<1 \cdot$$

From above conclusion, we could conclude that the equation $\frac{\mathrm{d}\lg L}{\mathrm{d}\beta}=0$ (i.e) $g_1(\beta)=g_2(\beta)$) has unique solution. From the equation $\frac{\mathrm{d}\lg L}{\mathrm{d}\beta}=0$, we can get

$$\beta=r[2\sum_{i=1}^{r}\frac{1}{t_i-\mu+\beta}-(n-r)\frac{t_0-\mu}{\beta(t_0-\mu+\beta)}]^{-1}$$

Using iterative computing method to obtain the solution, the iteration formula is as follows

$$\beta^{(k+1)}=r[2\sum_{i=1}^{r}\frac{1}{t_i-\mu+\beta^{(k)}}-(n-r)\frac{t_0-\mu}{\beta^{(k)}(t_0-\mu+\beta^{(k)})}]^{-1} . \qquad (5)$$

Where $\beta^{(k)}$ is kth iteration value ($k=1,2,\cdots$). $\beta^{(0)}$ is an initial value. If the iteration solution is denoted by $\hat{\beta}$ and replacing β in (4) by $\hat{\beta}$, then we can obtain the EB estimator of the Scale-parameter σ.

As a result, we have the following important theorem.

Theorem Let loss function be square errors loss. Under the type-I censoring life test (n, t_0,n_0)**,**the EB estimation of scale-parameter σ in two- parameter exponent -tial distribution (1)(μ is a known constant) is given as following expression (6), if the prior distribution is exponential distribution (3) and super-parameter is given by the maximum likelihood estimation $\hat{\beta}$.

$$\hat{\sigma}_{EB}=(r+1)[\sum_{i=1}^{r}(t_i-\mu)+(n-r)(t_0-\mu)+\hat{\beta}]^{-1} . \qquad (6)$$

Corollary Let loss function be square errors loss. Under the type-I censoring life test (n, t_0,n_0),the EB estimation of scale-parameter σ in two- parameter exponent-tial distribution (μ is a known constant) is given as following expression (7), if the prior distribution is exponential distribution (3) and super-parameter is given by the maximum likelihood estimation $\hat{\beta}$.

$$\hat{\sigma}_{EB} = (r+1)[\sum_{i=1}^{r} t_i + (n-r)t_0 + \hat{\beta}]^{-1} . \quad (7)$$

Where $\hat{\beta}$ is iteration solution of the following equation:

$$\beta^{(k+1)} = r[2\sum_{i=1}^{r}\frac{1}{t_i+\beta^{(k)}} - (n-r)\frac{t_0}{\beta^{(k)}(t_0+\beta^{(k)})}]^{-1}, \quad k = 0,1,2,\cdots$$

Where $\beta^{(0)}$ is an initial value.

4. Numerical example

Suppose that the prior distribution function of σ is given by (3), and probability density function of two- parameter exponential distribution is

$$p(t) = \sigma \exp(-\sigma(t-3)), \quad t > 3 .$$

Let n=13, t_0=16 and σ=0.085. By using Monte-Carlo simulation, we can get the failure times of type-I censoring life test (n, t_0,n_0**).** In the time interval (0, t_0), failure times are

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
3.1005	4.0536	5.2314	6.5667	8.1082	9.9314	12.1629	15.7296

Programming with C language to do the iteration. When the iteration times N=100, we get the iterative solution $\hat{\beta}$=8.387356.

From the expression (6) of the theorem, we obtain that $\hat{\sigma}_{EB} = 0.079$, which is close to true value of σ .

5. Conclusion

Using Bayes and maximum likelihood estimation method, we study the empirical Bayes estimation of scale- parameter for two- parameter exponent- tial distribution under the type –I censoring life test. The Monte-carlo simula- tion is used to examine the result of the empirical Bayes estimation. The simulation result shows that such EB estimation is simple and straightforward, and its precision is good.

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