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A New Method for European Option Pricing With Two Stocks

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ABSTRACT

Assume that the stock price obey the stochastic differential equation driven by Brownian motion, European option pricing with two stocks is considered by using stochastic dynamic theory at first time. The density function of stock process is obtained by using Fokker-Planck-Kolmogrov equation. Then, the price explicit expression of the European option is given. It provides a new method for European option pricing.

Keywords: European option pricing; stochastic dynamic theory; Fokker-Planck-Kolmogrov equation

1. Introduction

The break-through in option valuation theory started with the publication of two seminal papers by Black and Scholes (1973). In the papers the authors introduced a continuous time model of a complete friction-free market where the price of a stock follows a geometric Brownian motion. They presented a self-financing, dynamic trading strategy consisting of a riskless security and a risky stock, which replicates the payoff of an option. Then they argued that the absence of arbitrage dictates that the option price be equal to the cost of setting up the replicating portfolio. For details about this method to study option pricing, see Hull and White (1988), Becker (1991), Baksh et al. (1997), Gerber and Landry (1998), Sarwar and Krehbiel (2000), and Gencay and Salih (2003).

Bladt and Rydberg (1998) introduced a new method of option pricing. Using physical probability measure of price process and the principle of fair premium, they deal with the problems of option pricing under the unbalance, arbitrage existing and incomplete circumstance, and transform option pricing into a problem of equivalent and fair insurance premium. For details about this method to study option pricing, see Ann(2007), Paul(2005).

Since the paper of Margrabe (1978), many important extensions have been carrying on to study derivatives written on two stocks. Margrabe studied the pricing of European options for the case of two non-dividend-paying stocks driven by geometric Brownian motions.

In contrast to these references, this text aims to present the new method of option pricing by using Fokker-Planck equation. The paper is organized as follows. In section 2, we give brief review of Fokker-Planck equation. In section3, under the assumption that the stock price obeys Black-Schole model, we present the density function of stock price at every time by using Fokker-Planck equation. In section4, the option pricing base on this stock is discussed. Section 5 contains conclusions.

2. Some mathematical tools

Let's consider a simple example. Later, we'll use the result of this example to provides a new method for European claim pricing. Consider a dynamic system driven by fractional noise

$$dx_t = \alpha(t,x)dt + \beta(t,x)dW_t \tag{1}$$

where x_0 is constant, $\alpha(t,x)$ and $\beta(t,x)$ are functions dependent on t and x. $\{W_t, t \geq 0\}$ is the standard Brownian motion under the complete probability space (Ω, F, P). We endow the probability space with the filtration $\{F_t, t \geq 0\}$ generated by $\{W_t, t \geq 0\}$. So that, we get the following important.

Lemma 2.1 (FPK equation) The solution for the density function $p(t,x)$ of Eq(1) can then be written

$$\frac{\partial}{\partial t} p = -\frac{\partial}{\partial x}(\alpha(t,x)p) + \frac{\partial^2}{\partial x^2}(\beta(t,x)^2 p) . \tag{2}$$

where the density function $p(t,x)$ satisfied initialization conditions of dynamic system (1)

$P(0,x)$ is deterministic.

And, it also satisfied infinite boundary

$$\lim_{x\to\infty} p(t,x) = 0 .$$

Its demonstration see Risken(1989).

3. The Model

Consider a fractional Black-Scholes market model with a money market account and two stocks. We assume that the two stocks price S1(t), S2(t) satisfy the stochastic differential equations

$$dS_i(t) = S_i(t)[(\mu_i - \alpha_i \ln S_i(t))dt + \sigma_i dW_i(t)], \quad (i = 1,2) . \tag{3}$$

Where, return rate μ_1, μ_2 dividend rate q_1, q_2 volatility rate σ_1, σ_2 and mean value recovery rate α_1, α_2 are constant. $\{W_1(t), t \geq 0\}$ and $\{W_1(t), t \geq 0\}$ are the standard Brownian motion under the same complete probability space (Ω, F, P), respectively. Denote the natural σ -field flow by $\{F_t, t \geq 0\}$ which is generated by $\{W_1(t), t \geq 0\}$ and $\{W_2(t), t \geq 0\}$.

We also assume that a money market account B_t, with dynamics

$$dB_t = rB_t dt, \quad B_0 = 1, \quad 0 \leq t \leq T. \tag{4}$$

so that $B_t = \exp\{rt\}$, where the interest rate r is constant. We introduce an new probability measure Q such that

$$\left.\frac{dQ}{dP}\right|_T = \exp\{-\theta W(T) - \frac{1}{2}\theta^2 T\},$$

Where

$$\theta = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \mu_1 - \alpha_1 \ln S_1(t) - r \\ \mu_2 - \alpha_2 \ln S_2(t) - r \end{pmatrix}, W(t) = (W_1(t), W_2(t)), \ 0 \leq t \leq T.$$

Let

$$\widetilde{W}(t) = \theta t + W(t), 0 \leq t \leq T.$$

Girsanov's theorem shows that $\widetilde{W}(t) = (\widetilde{W}_1(t), \widetilde{W}_2(t))$ is a two-dimensional standard Brownian motion under probability measure Q ,and

$$dS_i(t) = S_i(t)[rdt + \sigma_i d\widetilde{W}_i(t)], (i = 1,2). \tag{5}$$

Theorem 3.1 By the FPK equation, the density function $p_i(t,x)$ of the $S_i(t)$ satisfy

$$-\frac{\partial}{\partial t} p_i + \frac{\sigma_i^2}{2} x^2 \frac{\partial^2}{\partial x^2} p_i + (2\sigma_i^2 - r) x \frac{\partial}{\partial x} p_i + (\sigma_i^2 - r) p_i = 0, (i = 1,2), \tag{6}$$

$$\lim_{t \to 0} \int_{S_0 - \varepsilon}^{S_0 + \varepsilon} p_i(t,x) dx = 1, \text{ for all } \varepsilon > 0. \tag{7}$$

$$p_i(t, x | F_0) = 0, \text{for all } x \leq 0, \tag{8}$$

$$\lim_{x \to +\infty} p_i(t, x | F_0) = 0, \text{ for all } x > 0. \tag{9}$$

Proof. By the lemma2.1, Eq(5) have the following FPK equation

$$\frac{\partial}{\partial t} p_i = -\frac{\partial}{\partial x}(rxp_i) + \frac{1}{2} \frac{\partial^2}{\partial x^2}(\sigma_i^2 x^2 p_i), \tag{10}$$

now

$$\frac{\partial}{\partial x}(rxp_i) = rp_i + rx \frac{\partial}{\partial x} p_i, \ \frac{\partial^2}{\partial x^2}(\sigma_i^2 x^2 p_i) = 2\sigma_i^2 p_i + 4\sigma_i^2 x \frac{\partial}{\partial x} p_i + \sigma_i^2 x^2 \frac{\partial^2}{\partial x^2} p_i,$$

Since

$$-\frac{\partial}{\partial t} p_i + \frac{\sigma_i^2}{2} x^2 \frac{\partial^2}{\partial x^2} p_i + (2\sigma_i^2 - r) x \frac{\partial}{\partial x} p_i + (\sigma_i^2 - r) p_i = 0.$$

To see that $S_i(0)$ is deterministic, such that

$$\lim_{t \to 0} \int_{S_0 - \varepsilon}^{S_0 + \varepsilon} p_i(t,x) dx = 1, \text{ for all } \varepsilon > 0.$$

Note that

$$S_i(t) > 0, \text{ for all } t > 0,$$

since

$$p_i(t,x|F_0)=0\text{ , for all } x\le 0 .$$

Because $p(t,x)$ is a density function of $S_i(t)$, so that

$$\lim_{t\to 0}\int_{S_0-\varepsilon}^{S_0+\varepsilon} p_i(t,x)dx=1\text{ , for all } \varepsilon>0\text{ , }\lim_{x\to+\infty} p_i(t,x|F_0)=0 .\square$$

Theorem 3.2 The density function of the Eq(5) can be written as follow

$$p_i(t,x)=\frac{1}{\sqrt{2\pi}\sigma_i x}\exp\{-\frac{1}{2\sigma_i^2}(\ln x-\ln S_0-rt+\frac{1}{2}\sigma_i^2 t)^2\} .$$

Proof. Assume that $r=q_1$, we will find that $S_1(t)$ can be changed as follow

$$dS_1(t)=\sigma_1 S_1(t)dW_t .$$

Then, $S_1(t)$ obey the normal state distribution of logarithm. Let

$$\xi \square N(a_t,\sigma_t)\text{ , } X=e^{\xi} .$$

Consequently , we can take steps to assume that $S_1(t)=e^{\xi}$. Then

$$p_1(t,x)=\frac{1}{\sqrt{2\pi}}\exp\{-\ln x-\frac{1}{2}\ln\sigma_t-\frac{1}{2\sigma_t}(\ln x-a_t)^2\} ,$$

by the Eq(6), such that

$$\frac{\partial}{\partial t}p_1=p_1(\frac{(\ln x-a_t)a_t'}{\sigma_t}+\frac{(\ln x-a_t)^2 a_t'}{\sigma_t^2}-\frac{\sigma_t'}{2\sigma_t}\text{ , }\frac{\partial}{\partial x}p_1=-\frac{1}{x}p_1(1+\frac{\ln x-a_t}{\sigma_t}) ,$$

$$\frac{\partial^2}{\partial x^2}p_1=\frac{1}{x^2}p_1(1+\frac{\ln x-a_t}{\sigma_t})^2+\frac{1}{x^2}p_1(1+\frac{\ln x-a_t}{\sigma_t})-\frac{1}{x^2}\frac{p_1}{\sigma_t} .$$

So by Theorem 3.1, we have

$$[-\frac{a_t'}{\sigma_t}+\frac{\sigma_1^2}{\sigma_t}+\frac{\sigma_1^2}{2\sigma_t^2}-\frac{2\sigma_1^2-r}{\sigma_t}](\ln x-a_t)+[\frac{\sigma_1^2}{2\sigma_t^2}-\frac{\sigma_t'}{\sigma_t^2}](\ln x-a_t)^2$$
$$+\frac{\sigma_t'}{\sigma_t}+\sigma_1^2-\frac{\sigma_1^2}{2\sigma_t^2}-(2\sigma_1^2-r)+(\sigma_1^2-r)=0.$$

By the arbitrariness of $\ln x-a_t$, we have

$$-\frac{a_t'}{\sigma_t}+\frac{\sigma_1^2}{\sigma_t}+\frac{\sigma_1^2}{2\sigma_t^2}-\frac{2\sigma_1^2-r}{\sigma_t}=0,\frac{\sigma_1^2}{2\sigma_t^2}-\frac{\sigma_t'}{\sigma_t^2}=0 ,$$

$$\frac{\sigma_t'}{\sigma_t}+\sigma_1^2-\frac{\sigma_1^2}{2\sigma_t^2}-(2\sigma_1^2-r)+(\sigma_1^2-r)=0 .$$

and therefore, since

$$\sigma_t'=\frac{\sigma_1^2}{2}\text{ , }\sigma_t=\frac{\sigma_1^2}{2}t\text{ , }a_t'=r-\frac{\sigma_1^2}{2} .$$

Let $a_t=c+rt-\frac{\sigma_1^2}{2}t$, where c is constant, since

$$p_1(t,x)=\frac{1}{\sqrt{2\pi t}\sigma_1 x}\exp\{-\frac{1}{2\sigma_1^2 t}(\ln x-c-rt+\frac{1}{2}\sigma_1^2 t)^2\}.$$

By the Eq(7),we have

$$\lim_{t\to 0}\int_{S_0-\varepsilon}^{S_0+\varepsilon}\frac{1}{\sqrt{2\pi t}\sigma_1 x}\exp\{-\frac{1}{2\sigma_1^2 t}(\ln x-c-rt+\frac{1}{2}\sigma_1^2 t)^2\}dx=1,$$

Then

$$\lim_{t\to 0}\int_{A_2(t)}^{A_1(t)}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\}dx=1, \tag{11}$$

where

$$A_1(t)=\frac{\ln(S_1(0)+\varepsilon)-c+rt-\frac{1}{2}\sigma_1^2 t}{\sigma_1\sqrt{t}},\quad A_2(t)=\frac{\ln(S_1(0)-\varepsilon)-c+rt-\frac{1}{2}\sigma_1^2 t}{\sigma_1\sqrt{t}}.$$

and

$$\lim_{t\to 0}A_1(t)=+\infty,\ \lim_{t\to 0}A_2(t)=-\infty, \tag{12}$$

By the arbitrariness of $\varepsilon>0$, we have $\ln S_1(0)=c$. If not, we can assume that $\ln S_1(0)>c$, let $\varepsilon=\frac{S_1(0)-c}{2}$, we will find

$$\ln(S_1(0)+\varepsilon)-c>0,\quad \ln(S_1(0)-\varepsilon)-c=\ln(\frac{S_1(0)}{2}+\frac{e^c}{2})-c>\ln(\frac{e^c}{2}+\frac{e^c}{2})-c=0$$

It means

$$\lim_{t\to 0}A_1(t)=+\infty,\ \lim_{t\to 0}A_2(t)=+\infty,$$

This is inconsistent with Eq(8). Since

$$p_1(t,x)=\frac{1}{\sqrt{2\pi t}\sigma_1 x}\exp\{-\frac{1}{2\sigma_1^2 t}(\ln x-\ln S_1(0)-rt+\frac{1}{2}\sigma_1^2 t)^2\}.$$

Then, by using the same way, the analytic formula of $p_1(t,x)$ is hold. □

4. Option Pricing

In what follows we introduce some relevant derivatives of two stocks, and show how to obtain the formulae for the value of these derivatives. Let

$$a_1=[\ln S_1(0)-\ln S_2(0)+\frac{1}{2}(\sigma_1^2+\sigma_2^2)T]\Big/\sqrt{(\sigma_1^2+\sigma_2^2)T},\ a_2=a_1-\sqrt{(\sigma_1^2+\sigma_2^2)T},,$$

$$b_1=[\ln S_1(0)-\ln K+rT+\frac{1}{2}\sigma_1^2 T]\Big/\sqrt{\sigma_1^2 T},\ b_2=b_1-\sqrt{\sigma_1^2 T},$$

$$c_1=[\ln K-\ln S_2(0)-rT-\frac{1}{2}\sigma_2^2 T]\Big/\sqrt{\sigma_2^2 T},\ c_2=c_1-\sqrt{\sigma_2^2 T},$$

$$d_1=\ln S_1(0)+rT-\frac{1}{2}\sigma_1^2 T,\ d_2=\ln S_1(0)+rT-\frac{1}{2}\sigma_1^2 T.$$

Lemma 4.1 The price at time $t=0$ of a bounded F_0–measurable claim $F\in L^2(Q)$, whose derivative of the two stocks $S_1(T), S_2(T)$, is given by

$$v = e^{-rT} E[F | F_0] .$$

Consider the derivative with the pay off function

$$g(x, y) = (x - y)^+ ,$$

called the Swap Option to exchange a stock $S_2(T)$ for another stock $S_1(T)$.

Theorem 4.1 The value of the Swap Option, which exchange a stock $S_2(T)$ for another stock $S_1(T)$ at time 0 is given by

$$v = e^{-rT} E_Q[(S_1(T) - S_2(T))^+ | F_t] = S_1(0)\Phi(a_1) - S_2(0)\Phi(a_2) .$$

Proof. By the Theorem 3.2, we have

$$v = e^{-rT} E_Q[(S_1(T) - S_2(T))^+ | F_0] = e^{-rT} \iint_{x>y} (x-y) p_1(T,x) p_2(T,y) dxdy$$

$$= e^{-rT} \iint_{x>y} (x-y) \frac{1}{2\pi\sigma_1\sigma_2 xy} \exp\{-\frac{1}{2\sigma_1^2 T}(\ln x - d_1)^2 - \frac{1}{2\sigma_2^2 T}(\ln y - d_2)^2\} dxdy$$

$$= e^{-rT} \iint_{x>y} (e^x - e^y) \frac{1}{2\pi\sigma_1\sigma_2} \exp\{-\frac{1}{2\sigma_1^2 T}(x - d_1)^2 - \frac{1}{2\sigma_2^2 T}(y - d_2)^2\} dxdy$$

$$= S_1(0)\Phi(a_1) - S_2(0)\Phi(a_2) \quad \square$$

Corollary 4.1 The value of the European Call Option of the stock $S_1(T)$ with exercise price K, at time t=0, is given by

$$v = e^{-rT} E_Q[(S_1(T) - K)^+] = S_1(0)\Phi(b_1) - Ke^{-rT}\Phi(b_2) .$$

Proof. Let $\sigma_2 = 0$, $S_2(T) = K$, In theorem 4.1, we have

$$S_2(T) = Ke^{-rT} , a_1 = b_1,\ a_2 = b_2.$$

Corollary 4.2 The value of the European Put Option of the stock $S_1(T)$ with exercise price K, at time t=0, is given by

$$v = e^{-rT} E_Q[(K - S_2(T))^+] = Ke^{-rT} Q(c_2) - S_2(0) Q(c_1) .$$

Proof. Let $\sigma_1 = 0$, $S_1(T) = K$, In theorem 4.1, we have

$$S_1(T) = Ke^{-rT} , a_1 = c_1,\ a_2 = c_2.$$

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REFERENCES

1. D.S.Ann and H.Bart,2007, ‘Distribution-free option pricing’. *Insurance: Mathematics and Economics*, 40(2), 179-199.
2. G.Baksh, C.Cao and A, Chen, 1997, ‘Empirical performance of alternative option pricing models’ *Finance*, 52 (5), 2003–2049.

3. F. Black and M. Scholes,1973. ‘The pricing of options and corporate liability’ *Economy*, 81(3), 637-654.
4. M.T. Bladt and H. Rydberg, 1998, ‘An actuarial approach to option pricing under the physical measure and without market assumptions’ *Insurance: Mathematics and Economics,* 22(1), 65-73.
5. D.Becker , 1991, ‘Statistical tests of the lognormal distribution as a basis for interest rate changes’ *Transactions of the Society of Actuaries*, 43, 7–72.
6. R.Gencay and A.Salih,2003, ‘Degree of mispricing with the Black-Scholes model and nonparametric cures’ *Annals of Economics and Finance*, 4, 73–101.
7. J.C. Hull and A.White,1988, ‘An analysis of the bias in option pricing caused by a stochastic volatility’ *Advances in Options and Futures Research* 3, 489–502.
8. W. Margrabe, 1978, ‘The value of an option to exchange one asset for another’ *Finance*, 33, 177-186.
9. M.J. Paul□K. Stephen , S. Osman and M Steven,2005, ‘Actuarial pricing of energy efficiency projects: lessons foul and fair’ *Energy Policy*, 33(10), 1319-1328.
10. Risken H, The Fokker-Planck Equation[M]. Springer, Berlin 1989.