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A Certain Class of Minimum Time Optimal Control Problems in 2-Banach Spaces

Utpalendu Adak[1] and Hora Krishna Samanta[2]

[1]Jadavpur University, Department of Mathematics, Kolkata-700 032, India.
E-mail: utpalendu_adak@yahoo.in.

[2]Netaji Mahavidyalaya, Department of Mathematics, Arambagh-712601, India.

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ABSTRACT

In this paper, a minimum time optimal control problem has been developed in 2-Banach spaces. Existence of the optimal control has been proved in 2-Banach space. An example is exhibited to show the technique of application of the control theory in generalized 2-normed spaces

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1. Introduction

A minimum cost control problem was formulated and solved by Minamide and Nakamura [9] in 1971 in Banach Space setting. Choudhury and Mukherjee [1,2,3] in 1981-83 developed a uniform theory of time optimal control problem for system which can be represented in terms of linear, bounded and onto transformation from a Banach space of control function to another Banach space. Important results of Functional Analysis are developed by Mehmet Acikgoz [8]; Z.Lewandowska, M.S.Moslehian, A.Saadatpour [6,7]; Freese, R., Cho, Y. [4], in 2-Banach space. They have developed a uniform theory in 2-Banach space. Optimization in 2-Banach space setting is an important area of application of functional analysis. So, it may be worthwhile to make an attempt to develop an optimization theory in 2-Banach space. In this paper, we want to formulate a certain class of minimum time optimal control problems in 2-Banach Space.

2. Some Preliminaries: Definition of 2-Normed space 2.1[12]

Let X_t be a vector space of dimension greater than one over F, where F is the real or complex number field. Suppose $N_1(.,.)$ be a non negative real valued function on $X_t \times X_t$ which satisfies the conditions: (i) $N_1(x_i,x_j)=0$ if and only if x_i and x_j are linearly dependent vectors, (ii) $N_1(x_i,x_j)= N_1(x_j, x_i)$ for all $x_i,x_j \in X_t$, (iii)

$N_1(\lambda x_i,x_j)=|\lambda|\, N_1(x_i,x_j)$ for all $\lambda\in F$ and for all $x_i,x_j\in X_t$, (iv) $N_1(x_i+x_j,z)\le N_1(x_i,z)+N_1(x_j,z)$ for all $x_i,x_j,z\in X_t$. Then $N_1(.,.)$ is called a 2-norm function defined on X_t and $(X_t, N_1(.,.))$ is called a linear 2-normed space. Also if X_t and Y are 2-Banach spaces over the field of real numbers, then $X_t\times Y$ is also 2-Banach space with respect to the 2-norm $N_3(.,.)$ where $N_3\{(x_i,y_i),(x_j,y_j)\}=\min\{N_1(x_i,x_j),N_2(y_i,y_j)\}$, i.e. $N_3(.,.)=\min\{N_1(.,.),N_2(.,.)\}$; $N_1(.,.)$ and $N_2(.,.)$ are 2-norm functions defined on the spaces X_t and Y respectively and $N_3\{(x_i,y_i),(x_j,y_j)\}=0$ iff either x_i,x_j are linearly dependent (L.D) in X_t or y_i,y_j are linearly dependent in Y, where N_1' be the 2-norms of the conjugate space of X_t. Let N_1',N_2',N_3' are the 2-norm functions defined on the spaces $X'_t,Y',(X_t\times Y)'$ respectively, where X'_t denotes the conjugate space of X_t.

We now cite the following some known 2-normed spaces.

Example 1: Consider the spaces Z where $Z=l_\infty$, c and c_0 of real sequences. Let us define: $N_1(x,y)=\sup_{i\in N}\sup_{j\in N}\left|x_iy_j-x_jy_i\right|$, where $x=(x_1,x_2,\ldots\ldots)$ and $y=(y_1,y_2,\ldots.)\in Z$. Then $N_1(.,.)$ is a 2-norm function defined on Z.

Example 2: For $X=R^3$, define:
$N_1(x,y)=\max\{|x_1y_2-x_2y_1|+|x_1y_3-x_3y_1|,|x_1y_2-x_2y_1|+|x_2y_3-x_3y_2|\}$, where $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)\in R^3$. Then $N_1(.,.)$ is a 2-norm on R^3. See more details Cho,Y [4], M. Acikgoz [8].

Reachable Region(Set) 2.2[10]: The set of all point $\xi\in D$ such that $T_tu=\xi$, $u\in U_t$ will be called the Reachable Region (set) with respect to the linear transformation T_t and will be denoted by C(t).

Unit Ball 2.3: Let $U_{Xt}=\{x_t:N_1(\alpha,x)\le 1,x\in X_t\},\alpha\in X_t$, $\alpha\ne\theta;U_Y=\{y:N_2(\beta,y)\le 1,y\in Y\}$, $\beta\in Y$, $\beta\ne\theta$ be the unit balls in X_t,Y respectively.

Definition of generalized 2-Normed space 2.4[12]: Let X and Y be real linear spaces. Denote by D a non-empty subset of $X\times Y$ such that for every $x\in X$, $y\in Y$ the sets $D_x=\{y\in Y:(x,y)\in D\}$ and $D^y=\{x\in X:(x,y)\in D\}$ are linear subspaces of the spaces Y and X, respectively. A function $N_3(.,.):D\to(0,\infty)$ will be called a generalized 2-norm on D if it satisfies the condition: (i) $N_3(x,\alpha y)=|\alpha|\,N_3(x,y)=N_3(\alpha x,y)$ for any real number α and all $(x,y)\in D$, (ii)$N_3(x,y+z)\le N_3(x,y)+N_3(x,z)$ for $x\in X$, $y,z\in Y$ with (x,y), $(x,z)\in D$, (iii) $N_3(x+y,z)\le N_3(x,z)+N_3(y,z)$ for $x,y\in X$, $z\in Y$ with (x,z), $(y,z)\in D$. Then D is called a 2-normed set. In particular, if $D=X\times Y$, the function $N_3(.,.)$ is said to be a generalized 2-norm function defined on $X\times Y$ and the pair $(X\times Y, N_3(.,.))$ is called a generalized 2-normed space. Unfortunately, there is no connection between normed spaces and 2-normed spaces, but in 1999 in order to introduce some connections between normed spaces and 2-normed spaces, Zofia Lewandowska [6] introduced generalized 2-

normed spaces, as a subspace of 2-normed spaces. If X = Y, then the generalized 2-normed space (X × X, N_1(.,.)) is denoted by (X, N_1(.,.)). In the case that X = Y, D = D^{-1}, where D^{-1} = {(y, x) : (x, y) ∈ D}, and N_3(x, y) = N_3(y, x) for all (x,y)∈D, we call N_3(.,.) a generalized symmetric 2-norm function and D a symmetric 2-norm set. Also, let (X,N(.)) be a normed space. Then N_1(x, y) = N(x). N(y), for all x, y ∈ X, is a 2-norm function defined on X × X. So, (X, N_1(.,.)) is a generalized 2-normed space. If we take as N(x)=N(y), our generalized 2-normed space will be a generalized symmetric 2-normed space with the symmetric 2-norm defined by N_1(x, y) = N(x). N(y) for all x, y ∈ X. Let us remark that a symmetric 2-normed space need not be a 2-normed space in the sense of Gahler [5]. For instance given above, x≠θ, y=kx, k≠0, we obtain $N_1(x, y)=N_1(x, kx)=|k|N_1(x,x)>0$, but inspite of this x and y are linearly dependent. So from this, we say that the 2-normed space is not a 2-normed space in the sense of definition 2.1. Each 2-normed space is a generalized 2-normed space. But, in case of X = Y, D = D^{-1}; the generalized 2-normed space is a 2-normed space.

Example 3[14]: Suppose that s be the linear space of all sequences of real numbers. Put $N_1(x,y) = \sum_{n=1}^{\infty}|x_n||y_n|$,where x={$x_n$},y={$y_n$}∈s.Then by def.2.4,D={(x,y)∈ s×s: $N_1(x,y)<\infty$} is a symmetric 2-normed set and the function $N_1(.,.):D\to(0,\infty)$ is a generalized symmetric 2-norned on D.

Example 4[13]: Let X be real inner product space. Then X is a symmetric generalized 2-normed space under the 2-norm function $N_1(x,y)=\sum_{i=1}^{n}|\langle x_i, y_i\rangle| = \sum_{i=1}^{n} x_i y_i$, ∀ $x_i,y_i \in X$, by def.2.4.

Example 5[12]: Let x, y∈C[a,b] and X denotes the set of all real-valued continuous functions x(t) defined on the closed interval [a,b]. If x ≡x(t) and y≡y(t) are in X, $N(x) = \sup_{a\le t\le b}|x(t)|$, $N(y) = \sup_{a\le t\le b}|y(t)|$, where X is a normed space, then by def.2.4, $N_1(x,y)= \sup_{a\le t\le b}|x(t)| \cdot \sup_{a\le t\le b}|y(t)|$ is a generalized 2-normed space.

For another examples of generalized 2-normed spaces, see U.Adak ([10]--[15]).

Example 6: Let X=R^3 and consider the following 2-norm function defined on X:

$$N_1(x, y) = \left|\det\begin{pmatrix} i & j & k \\ l & m & n \\ p & q & r \end{pmatrix}\right|,$$

where N(x,y)=|x×y|, x=(l,m,n) & y=(p,q,r). Then (X,N_1(.,.)) is a 2-normed Space.

Example 7: Let $X=Q^3$, the field be the rational number and consider the following 2-norm on X :

$$N_1(x,y) = \left| \det \begin{pmatrix} i & j & k \\ l & m & n \\ p & q & r \end{pmatrix} \right|$$, where $N(x,y)=|x\times y|$, x=(l,m,n) & y=(p,q,r). Then $(X,N_1(.,.))$ is a 2-normed Space.

Example 8: Let P_n denotes the set of all real polynomials of degree≤n, on the interval [0,1]. By considering usual addition and scalar multiplication, P_n is a linear vector space over the reals. Let $\{x_0,x_1,x_2,\ldots x_{2n}\}$ be distinct fixed point in [0,1] and define the following 2-norm function defined on P_n: $N_1(f,g)= \sum_{i=1}^{2n} |f(x_i), g(y_i)|$, whenever f and g are linearly independent; and $N_1(f,g)=0$ if f and g are linearly dependent. Then $(P_n,N_1(.,.))$ is a 2-normed space.

The time optimal control problem can be defined as follows:

Problem: Let B_t be a 2-Banach space depending upon the continuous parameter t. Let D be another 2-Banach space. Let T_t be a linear, bounded transformation depending upon the parameter t, mapping B_t onto D. Let $U_t \subset Bt$ be the unit ball in B_t and $\xi \in D$. The problem is to determine $u \in U_t$ such that $T_t u = \xi$ and t is minimum. Here B_t is an increasing function of t in the sense that $B_{t1} \subset B_{t2}$, whenever $t_1 \leq t_2$. Also T_{t1} can be regarded as the restriction of T_{t2} defined on B_{t2} on B_{t1}. It can be verified that $U_{t1} \subset U_{t2}$.

Theorem 2.1: The reachable region C(t) is bounded and a convex body, symmetrical with respect to the origin of D.

Proof: Since U_t is convex and bounded and T is linear and bounded, and since linear operators preserve convexity, the image $C(t) = T_t U_t$ is convex and bounded. Also, if λ is any real number with $|\lambda| \leq 1$ then for $\xi \in C(t)$ we also have $\lambda\xi \in C(t)$ because $\xi = T_t u$ for some $u \in U_t$ implies $\lambda\xi = \lambda T_t u = T_t(\lambda u)$ and $\lambda u \in U_t$ due to $N_1\{(\lambda u, \lambda u_1): \lambda u, \lambda u_1 \in U_t\} = |\lambda| N_1\{(u, u_1): u, u_1 \in U_t\} \leq 1$. Thus,C(t) is circled and symmetrical. Again, since T_t is onto, it follows from open mapping theorem that $T_t U_t = C(t)$ will contain multiple of the unit ball in D. Thus, C(t) is a convex set with nonempty interior and thus is a convex body.

Corollary 2.1: The reachable region C(t) is closed when B_t is either a reflexive space or it can be considered as a conjugate of some other 2-Banach space.

Proof: It is known that a space is reflexive if and only if its unit ball is weakly compact. So, if it is assumed that B_t is reflexive, then the unit ball U_t is weakly compact. Again, since T_t is linear and bounded, it is continuous. Also, the continuous image of weakly compact set is weakly compact. Consequently C(t) is weakly compact and hence it is weakly closed and therefore it is strongly closed. Then C(t) is closed. However, if B_t is not a reflexive space but it can be considered as a conjugate of some other 2-Banach space, then it follows that its unit ball U_t is weakly compact in some topology. Therefore by the previous analogy we can conclude that C(t) is closed in this case also. To solve the minimum time control problem, we shall first consider the following auxiliary problem.

Auxiliary problem: Let $\xi \in \delta C(t)$ where $\delta C(t)$ denotes the boundary of C(t) for some given time t. Then determine $u \in U_t$ such that $T_t u = \xi$ and $N_1\{(u, u_1): u, u_1 \in U_t\}$ is minimum. We shall call this the minimum 2-norm problem. The corresponding control will be called the optimal control. In the following theorems we shall find the form of the optimal control and also the shape of the reachable set w.r.t. the minimum time t.

Theorem 2.2: An admissible control which will be optimal must satisfy $N_1\{(u,u_1):u,u_1 \in U_t = 1$.

Proof: We have already shown that C(t) is a closed convex body. Thus if $\xi \in \delta C(t)$, then there exists a $\Phi \in D^*$, where D^* is the conjugate space to D, such that $\langle \xi, \Phi \rangle \geq \langle \eta, \Phi \rangle$ for all $\eta \in C(t)$. Let $u \in U_t$ be such that $T_t u = \xi$. Since C(t) is circled, it follows, that $\langle \xi, \Phi \rangle \geq |\langle T_t u, \Phi \rangle| = |\langle u, T_t^* \Phi \rangle|$ for all $u \in U_t$, T_t^* being the transformation adjoint to T_t. $\therefore$ $\langle \xi, \Phi \rangle \geq \sup_{N_1\{(u,u_1):u,u_1 \in U_t\} \leq 1} \left| \left\langle T_t^* u, \Phi \right\rangle \right| =$ $N_1'\{(T_t^*\phi, f):T_t^*\phi, f \in B_t^*\}$. Now let $u \in U_t$ such that $T_t u = \xi$. Then $[N_1\{(u, u_1): u, u_1 \in U_t\}\ N_1'\{(T_t^*\Phi, f):T_t^*\Phi, f \in B_t^*\}] \geq \langle u, T_t^*\Phi \rangle = \langle T_t u, \Phi \rangle = \langle \xi, \Phi \rangle \geq N_1'\{(T_t^*\Phi, f):T_t^*\Phi, f \in B_t^*\}$. $\therefore$ $N_1\{(u, u_1): u, u_1 \in U_t\} \geq 1$. Thus $N_1\{(u, u_1): u, u_1 \in U_t\} = 1$. This proves the theorem.

Theorem 2.3: Let $\xi \in \delta C(t)$ and $\Phi \in D^*$ denotes a supporting hyperplane to C(t) at ξ. Then $\langle \xi, \Phi \rangle = N_1'\{(T_t^*\Phi, f):T_t^*\Phi, f \in B_t^*\}$, where D^* is the conjugate space to D and T^* is the transformation adjoint to T.

Proof: Since $\xi \in \delta C(t)$, there is a $u_\Phi \in U_t$ such that $\xi = T_t u_\Phi$. Hence by theorem 2.2,
$\langle \xi, \Phi \rangle \geq N_1'\{(T_t^*\Phi, f):T_t^*\Phi, f \in B_t^*\}$ …………………(1).
Also $\langle \xi, \Phi \rangle = \langle T_t u_\phi, \Phi \rangle = \langle u_\Phi, T_t^*\Phi \rangle \leq$
$[N_1\{(u_\Phi,v_\Phi):u_\Phi,v_\Phi \in U_t\} N_1'\{(T_t^*\Phi,f):T_t^*\Phi,f \in B_t^*\}] \leq N_1'\{(T_t^*\Phi, f):T_t^*\Phi, f \in B_t^*\}$..(2).
From (1) and (2) $\langle \xi, \Phi \rangle = N_1'\{(T_t^*\Phi, f):T_t^*\Phi, f \in B_t^*\}$.
Again, as $0 \in$ int C(t) (C(t)is a convex body), it follows that $\langle \xi, \Phi \rangle > 0$.

Theorem 2.4: Let $\xi \in \delta C(t)$ where t is the given terminal time, and $\Phi \in D^*$ denotes a supporting hyperplane at ξ. Let u_Φ be the optimal control to reach at ξ in the above sense. Then u_Φ maximizes $\langle u, T_t^*\Phi\rangle$ where T_t^* and D^* denote the adjoint transformati-on and adjoint space to T_t and D respectively and $\langle u_\phi, T_t^*\Phi\rangle = \max_{N_1\{(u,u_1):u,u_1\in U_t\}=1}\langle u, T_t^*\Phi\rangle = N_1'\{(T_t^*\Phi,f):T_t^*\Phi,f\in B_t^*\}$ and

$N_1\{(u_\Phi,v_\Phi): u_\Phi,v_\Phi\in U_t\}=1$.

Proof: Since C(t) is closed convex body (by theorem 2.1) and $\xi\in\delta C(t)$, there exists a $\Phi\in D^*$, such that $\langle\eta,\Phi\rangle\le\langle\xi,\Phi\rangle$ for all $\eta\in C(t)$. Let $u\in U_t\subset B_t$ be such that $\eta = T_t u$. Since C(t) is circled (by theorem 2.1), it follows that $\langle\xi,\Phi\rangle \ge |\langle T_t u,\Phi\rangle| = |\langle u, T_t^*\Phi\rangle|$, for all $u\in U_t\subset B_t$.

Hence $\langle\xi,\Phi\rangle \ge \sup_{N_1\{(u,u_1):u,u_1\in U_t\}\le 1, u\in B_t}\left|\langle u, T_t^*\Phi\rangle\right| = N_1'\{(T_t^*\Phi,f):T_t^*\Phi, f\in B_t^*\}$ --(1).

Now, since $\xi\in\delta C(t)$, there is a $u_\Phi\in U_t$, such that $\xi=T_t u_\Phi$. Thus $\langle\xi,\Phi\rangle = \langle T_t u_\phi,\Phi\rangle = \langle u_\Phi, T_t^*\Phi\rangle \le N_1\{(u_\Phi,v_\Phi): u_\Phi,v_\Phi\in U_t\}\ N_1'\{(T_t^*\Phi,f):T_t^*\Phi,f\in B_t^*\}\le N_1'\{(T_t^*\Phi,f):T_t^*\Phi,f\in B_t^*\}$------(2), since $N_1\{(u_\Phi,v_\Phi): u_\Phi,v_\Phi\in U_t\}=1$, (by theorem 2.2). From (1) and (2) $\langle u_\Phi, T_t^*\Phi\rangle = N_1'\{(T_t^*\Phi, f):T_t^*\Phi, f\in B_t^*\}$ --------------(3). Again, $\langle\eta,\Phi\rangle\le\langle\xi,\Phi\rangle$ for all $\eta\in C(t)$, therefore $\langle u,T_t^*\Phi\rangle\le\langle u_\phi,T_t^*\Phi\rangle$ for all $u\in U_t\subset B_t$. $\therefore \sup_{N_1\{(u,u_1):u,u_1\in U_t\}\le 1, u\in B_t}\left|\langle u, T_t^*\Phi\rangle\right|\le$

$\langle u_\phi,T_t^*\Phi\rangle = N_1'\{(T_t^*\Phi,f):T_t^*\Phi,f\in B_t^*\}$, by (3),-------------(4). Again since $U_t\subset B_t$ is weakly compact set, and $\langle u,T_t^*\Phi\rangle$ is strongly continuous function of u, therefore

$\sup_{N_1\{(u,u_1):u,u_1\in U_t\}\le 1, u\in B_t}\left|\langle u, T_t^*\Phi\rangle\right| = \sup_{N_1\{(u,u_1):u,u_1\in U_t\}=1, u\in B_t}\left|\langle u, T_t^*\Phi\rangle\right|$ will be attained at some point $u_\Phi\in U_t\subset B_t$, $N_1\{(u_\Phi,v_\Phi): u_\Phi,v_\Phi\in U_t\}=1$, which proves the theorem.

Theorem 2.5: Let K be a weakly compact, convex set in a 2-Banach space D and let Φ be any element $\in D^*$, the conjugate space to D. Then there exists a point $\eta_0\in K$, such that Φ denotes a supporting hyperplane to K at $\eta_0\in\delta K$.

Proof: If Φ is a supporting hyperplane to K at η_0, then the theorem is proved. So let us suppose that Φ is not a supporting hyperplane at any $\xi\in K$. Now, because $\Phi\in D^*$ and K is bounded, therefore $\langle\eta,\phi\rangle\le C$ for all $\eta\in K$ where $C>0$ is some constant. Thus $\sup_{\eta\in K}\langle\eta,\Phi\rangle$ will exist. Put $\sup_{\eta\in K}\langle\eta,\Phi\rangle = M$. Then there will exists a sequence $\{\eta_n;\ \eta_n\in K\}$ such that $\langle\eta_n,\Phi\rangle > M-(1/n)$ for $n\ge N$. Again, since K is weakly compact, there will exists a subsequence $\left\{\eta_{n_k}\right\}$ such that $\left\{\eta_{n_k}\right\}$ converges weakly to some $\eta_0 \in K$. Therefore $\left\langle \eta_{n_k}, \Phi \right\rangle > M - 1/n_k$. Therefore $\lim \left\langle \eta_{n_k}, \Phi \right\rangle = \langle \eta_0, \Phi \rangle$ $\geq$ M. On the other hand, $\left\langle \eta_{n_k}, \Phi \right\rangle \leq$ M (because M is the least upper bound). $\therefore \lim \left\langle \eta_{n_k}, \Phi \right\rangle = \langle \eta_0, \Phi \rangle \leq M$. Hence $\langle \eta_0, \Phi \rangle = M$. Thus $\langle \eta, \Phi \rangle \leq \langle \eta_0, \Phi \rangle$ for all $\eta \in K$. Also the above relation shows that the functional Φ assumes its maximum value on K at the vector η_0, which together with the fact that any linear functional maps open sets into open sets, shows that η_0 cannot belong to interior of K. Consequently $\eta_0 \in \delta K$.

Theorem 2.6: Let $\xi \in C(t_1) \cap \delta C(t_1)$ where $C(t_1)$ is the reachable region. Then $\max\limits_{\psi} \dfrac{\langle \xi, \psi \rangle}{N_1^{'}\{(T_{t2}^{*}\psi, f) : T_{t2}^{*}\psi, f \in B_t^{*}\}}$ is ≤ 1 or ≥ 1 according as $t_2 \geq t_1$ or $t_2 \leq t_1$. (Here B_t is to be considered as in Corollary of Theorem 2.1). To prove this we require the following corollary.

Corollary 2.2: Let $\xi \in \delta C(t_1)$, $t_2 \geq t_1$. Then the ray $k\xi$, $k > 0$ intersects $\delta C(t_2)$ at some point $\eta = \ell\xi$, $\ell \geq 1$.

Proof: Since $C(t_2)$ is bounded (by Theorem 2.1), there will exist a $k > 0$, say $k = k_0$, such that $k_0\xi \notin C(t_2)$. Considered the position of the ray $R = [k\xi, 0 \leq k \leq k_0]$. We now consider a set S defined by $S = \{k: k\xi \in C(t_2)\}$. Let $\ell = \sup\limits_{k \in S} k$ which will evidently exist (because $k \leq k_0$). Evidently $\ell \geq 1$. Now there exists a sequence $\{k_n\}$ such that $\lim k_n = \ell$ and $k_n\xi = x_n \in R \cap C(t_2)$. Again, since R is compact, there is a subsequence $\{x_{n_k}\}$ such that $\lim x_{n_k} = x_0 \in R$. Also, as $x_{n_k} \in C(t_2)$ and $C(t_2)$ is closed, therefore $x_0 = \ell\xi \in C(t_2)$. Now, $x_0 \notin \text{Int}\, C(t_2)$, because, if $x_0 \in \text{Int}\, C(t_2)$ then there will be an open sphere S_ε of radius ε which will be contained entirely within $C(t_2)$. Consider the point

$x = x_0 + \dfrac{\varepsilon}{2} \dfrac{\xi}{N_2\{(\xi, w) : \xi, w \in D\}}$. Then $x \in S_\varepsilon$ and $x \in R$. But then

$x = \left\{1 + \dfrac{\varepsilon}{2N_2\{(\xi, w) : \xi, w \in D\}}\right\}\xi$, which contradicts the fact that $\ell = \sup\limits_{k \in S} k$.

This completes the proof of the corollary.

Proof of Theorem 2.6: We shall prove the theorem for $t_2 \geq t_1$. Then we are required to show that $\max_{\psi} \frac{\langle \xi, \psi \rangle}{N_1^{'}\{(T_{t2}^{*}\psi, f) : T_{t2}^{*}\psi, f \in B_t^{*}\}}$ for a given $\xi \in C(t_1) \cap \delta C(t_1)$. Now, because $t_2 \geq t_1$, we have $B_{t1} \subset B_{t2}$ and $U_{t1} \subset U_{t2}$ (by assumption). The transformation T_{t2} is such that T_{t1} is the restriction of T_{t2} on U_{t1}. Hence $C(t_1) = T_{t1}U_{t1} = T_{t2}U_{t1} \subset T_{t2}U_{t2} = C(t_2)$ therefore $\xi \in C(t_2)$. Let $\psi \in D^*$, D^* is the conjugate space to the 2-Banach space D. Consequently by theorem 2.5, there exists a point $\xi' \in \delta C(t_2)$, such that ψ defines a supporting hyperplane to $C(t_2)$ at ξ'. Again, since $\xi' \in \delta C(t_2)$ and ψ defines a supporting hyperplane to $C(t_2)$ at ξ', hence we can write $\langle \xi', \psi \rangle = N_1^{'}\{(T_{t2}^{*}\psi, f) : T_{t2}^{*}\psi, f \in B_t^{*}\}$ (by Theo. 2.3).

But $\langle \xi, \psi \rangle \leq \langle \xi', \psi \rangle$ as ψ defines a supporting hyperplane at $\xi' \in \delta C(t_2)$. Therefore $\langle \xi, \psi \rangle \leq \langle \xi', \psi \rangle = N_1^{'}\{(T_{t2}^{*}\psi, f) : T_{t2}^{*}\psi, f \in B_t^{*}\}$ such that

$\frac{\langle \xi, \psi \rangle}{N_1^{'}\{(T_{t2}^{*}\psi, f) : T_{t2}^{*}\psi, f \in B_t^{*}\}} \leq 1$. Hence $\sup_{\psi} \frac{\langle \xi, \psi \rangle}{N_1^{'}\{(T_{t2}^{*}\psi, f) : T_{t2}^{*}\psi, f \in B_t^{*}\}} \leq 1$. Now $\xi \in \delta C(t_1)$ and let $\eta = \ell\xi \in \delta C(t_2)$ $(t_2 \geq t_1)$ (by the above corollary).
Let ψ denotes a supporting hyperplane to $\delta C(t_2)$ at η.

Hence by Theorem 2.3, we get $\frac{\langle \xi, \psi \rangle}{N_1^{'}\{(T_{t2}^{*}\psi, f) : T_{t2}^{*}\psi, f \in B_t^{*}\}} = 1$. Therefore

$\frac{\langle \xi, \psi \rangle}{N_1^{'}\{(T_{t2}^{*}\psi, f) : T_{t2}^{*}\psi, f \in B_t^{*}\}} = 1/\ell \leq 1$. Consequently sup is attained at a point $\Phi = \psi \in D^*$, where ψ denotes a supporting hyperplane at $\eta = \ell\xi \in C(t_2)$. Thus we have proved the theorem for $t_2 \geq t_1$. Similarly, we can show that if $t_2 \leq t_1$ then

$$\max_{\Phi} \frac{\langle \xi, \Phi \rangle}{N_1^{'}\{(T_{t2}^{*}\Phi, f) : T_{t2}^{*}\Phi, f \in B_t^{*}\}} \geq 1.$$ This completes the proof of the theorem.

Theorem 2.7: Let $t_1 < t_2$ and $T_{t1}:B_{t1} \to D$, $T_{t2}:B_{t2} \to D$ be bounded linear onto transformations. Then $C(t_1) \subseteq C(t_2)$ and $\delta C(t_1) \cap \delta C(t_1) = \Phi^{'}$ iff $N_1^{'}\{(T_{t2}^{*}\Phi, f) : T_{t2}^{*}\Phi, f \in B_t^{*}\} > N_1^{'}\{(T_{t1}^{*}\Phi, f) : T_{t1}^{*}\Phi, f \in B_t^{*}\}$, $\Phi \in D^*$ and $\Phi^{'}$ denotes the null set. (Here B_{t1} and B_{t2} are to be considered as in corollary 2.1).

Proof: We have already assumed that if $t_1 < t_2$, then $B_{t1} \subseteq B_{t2}$. U_{t1} and U_{t2} denote the unit balls in B_{t1} and B_{t2} respectively. Let $C(t_1)$ and $C(t_2)$ be reachable regions in D with respect to the transformations T_{t1} and T_{t2} corresponding to the time t_1 and t_2

respectively. Let $\xi \in C(t_1)$. Then there exists $u_1 \in U_{t1}$ such that $T_{t1}(u_1) = \xi$. As $u_1 \in B_{t1} \subset B_{t2}$, hence $T_{t2}(u_1) \in C(t_2)$. But $T_{t1}(u_1) = T_{t2}(u_1)$ as T_{t1} is the restriction of T_{t2} on B_{t1}. Hence $\xi \in C(t_2)$. $\therefore C(t_1) \subseteq C(t_2)$. To prove the next part, let us assume that $N_1^{'}\{(T_{t2}^{*}\Phi, f) : T_{t2}^{*}\Phi, f \in B_t^{*}\} > N_1^{'}\{(T_{t1}^{*}\Phi, f) : T_{t1}^{*}\Phi, f \in B_t^{*}\}\cdot$ We shall show that $\delta C(t_1) \cap \delta C(t_2) = \Phi$. Let $\Phi \in D^{*}$ be a functional over D. Then corresponding to Φ we can find by Theorem 2.5, a point $\xi \in \delta C(t_1)$ and a point $\eta \in \delta C(t_2)$, such that Φ is a supporting hyperplane at ξ to $C(t_1)$ and at η to $C(t_2)$ (Because according to the conditions of the theorems, $C(t_1)$ and $C(t_2)$ are weakly compact and convex sets). Also $\langle \eta, \Phi\rangle = N_1^{'}\{(T_{t2}^{*}\Phi, f) : T_{t2}^{*}\Phi, f \in B_t^{*}\}$ and $\langle \xi, \Phi\rangle = N_1^{'}\{(T_{t1}^{*}\Phi, f) : T_{t1}^{*}\Phi, f \in B_t^{*}\}$ (by Theorem 2.3). Again by the Corollary 2.2, corresponding to $\eta \in \delta C(t_2)$, we can find a point $\xi' \in \delta C(t_1)$ such that $\xi' = \ell\eta$ where $\ell \leq 1$. Now, we have, $N_1^{'}\{(T_{t2}^{*}\Phi, f) : T_{t2}^{*}\Phi, f \in B_t^{*}\} > N_1^{'}\{(T_{t1}^{*}\Phi, f) : T_{t1}^{*}\Phi, f \in B_t^{*}$ (by hypothesis). Hence $\langle \eta, \Phi\rangle > \langle \xi, \Phi\rangle$. Also $\langle \xi, \Phi\rangle \geq \langle \xi', \Phi\rangle$, since Φ is a supporting hyperplane to ξ and ξ' is any point in $\delta C(t_1)$. Consequently $\langle \eta,\Phi\rangle > \langle \xi',\Phi\rangle$ i.e. $\langle \eta,\Phi\rangle > \langle \ell\eta,\Phi\rangle = \ell\langle \eta, \Phi\rangle$ (Because $\xi' = \ell\eta$, $\ell \leq 1$). Hence $\delta C(t_1) \cap \delta C(t_2) = \Phi^{'}$.

Conversely, let $\delta C(t_1) \cap \delta C(t_2) = \Phi^{'}$.
We are to show $N_1^{'}\{(T_{t2}^{*}\Phi, f) : T_{t2}^{*}\Phi, f \in B_t^{*}\} > N_1^{'}\{(T_{t1}^{*}\Phi, f) : T_{t1}^{*}\Phi, f \in B_t^{*}$ for $t_1 < t_2$ and for all $\Phi \in D^{*}$. Let $\Phi \in D^{*}$, be any functional over D. Hence corresponding to Φ we can find by Theorem 2.5, a point $\xi \in \delta C(t_1)$ and a point $\eta \in \delta C(t_2)$ such that Φ is a supporting hyperplane at $\xi \in \delta C(t_1)$ and at $\eta \in \delta C(t_2)$. Then by Theorem 2.3,
$\langle \xi , \Phi\rangle = N_1^{'}\{(T_{t1}^{*}\Phi, f) : T_{t1}^{*}\Phi, f \in B_t^{*}\}$, $\langle \eta, \Phi\rangle = N_1^{'}\{(T_{t2}^{*}\Phi, f) : T_{t2}^{*}\Phi, f \in B_t^{*}\}$. Now because $C(t_1) \subseteq C(t_2)$ and by hypothesis $\delta C(t_1) \cap \delta C(t_2) = \Phi^{'}$, hence $\xi \in \text{Int } C(t_2)$. Thus $\langle \xi,\Phi\rangle < \langle \eta, \Phi\rangle$ where Φ is a supporting hyperplane at η to $C(t_2)$. Therefore
$N_1^{'}\{(T_{t1}^{*}\Phi, f) : T_{t1}^{*}\Phi, f \in B_t^{*}\} < N_1^{'}\{(T_{t2}^{*}\Phi, f) : T_{t2}^{*}\Phi, f \in B_t^{*}\}$.
The following corollary can be proved.

Corollary 2.3: If $\delta C(t_1) \cap \delta C(t_2) = \Phi^{'}$ then

$$\sup \frac{N_1^{'}\{(T_{t1}^{*}\Phi, f) : T_{t1}^{*}\Phi, f \in B_t^{*}\}}{N_2^{'}\{(\Phi, \Phi_1) : \Phi, \Phi_1 \in D^{*}\}, N_2^{'}(.,.) \neq o} < \sup \frac{N_1^{'}\{(T_{t2}^{*}\Phi, f) : T_{t2}^{*}\Phi, f \in B_t^{*}\}}{N_2^{'}\{\Phi, \Phi_1) : \Phi, \Phi_1 \in D^{*}\}, N_2^{'}(.,.) \neq o}.$$

Proof: We have $N_1'\{(T_{t1}^*\Phi, f) : T_{t1}^*\Phi, f \in B_t^*\} < N_1'\{(T_{t2}^*\Phi, f) : T_{t2}^*\Phi, f \in B_t^*\} \leq$ $N_1'\{(T_{t2}^*\Phi, f) : T_{t2}^*\Phi, f \in B_t^*\} \cdot \sup \dfrac{N_2'\{(\Phi(\Phi_1) : \Phi, \Phi_1 \in D^*\}}{N_2'\{(\Phi, \Phi_1) : \Phi, \Phi_1 \in D^*\}, N_2'(.,.) \neq o}$. This completes the proof of the corollary.

Theorem 2.10: Let $\xi \in \delta C(t_f) \cap C(t_f)$ and $t \geq t_f$.

Then $\max\limits_{\Phi} \dfrac{\langle \xi, \Phi \rangle}{N_1'\{(T_t^*\Phi, f) : T_t^*\Phi, f \in B_t^*\}}$ is a non-increasing function of t, $t \geq t_f$.

(Here B_t is to be considered as in Corollary of Theorem 2.1).

Note 1. Any complete 2-normed space is said to be 2-Banach space. Every 2-normed space of dimension 2 is a 2-Banach space when the underlying field is complete. Examples 6 and 8 are 2-Banach spaces, while example 7 does not. For details see A.White [17]. A linear 2-normed space of dimension 3 is not a 2-Banach space. For details see A.White [17]. Every 2-normed space is a locally convex topological vector space. But convers is not true. In fact for a fixed $b \in X$, $P_b(x) = N_1(x,b)$ $\forall x \in X$, is a seminorm and the family $P = \{P_b : b \in X\}$ generates a locally convex topology on X. Such a topology is called the natural topology induced by 2-norm $N_1(.,.)$.

Example 9: We shall consider the system governed by the following first order differential equation: $\dfrac{dx_1}{dt} = x_2 + u_1$, $\dfrac{dx_2}{dt} = u_2$ ------------→(1), where $x_1(t)$, $x_2(t)$ represent the instantaneous state of the system in the phase plane at time t and u_1 and u_2 are the control functions. $x_1(t)$ and $x_2(t)$ can be considered as deviations of the actual trajectory from the nominal trajectory. The problem is, given any initial value of the deviation $[x_1(0), x_2(0)]$ – what will be the control function required to reduce the error to the value zero in minimum time. $u_1(t)$ and $u_2(t)$ may be considered to be fuel flow rates which emanate from independent fuel supplies, for which saturation will set in at the same value. Without any loss of generality we can set this value at 1. Thus the constraint imposed on $u_1(t)$ and $u_2(t)$ can be expressed as

$J(u) = \sup\limits_{t\in\xi} \max \{|u_1(t), u_2(t)| \cdot \sup\limits_{t\in\xi} \max \{|u_1(t), u_2(t)| \leq 1$ ---→(2),

where $\xi = [0, t_0]$, t_0 being the time for which the system is allowed to run. To solve this problem $u_t = [u_1(t), u_2(t)]$ may be considered to belong to $L_\infty(l_\infty(2), \tau) \times L_\infty(l_\infty(2), \tau)$. We shall denote $L_\infty(l_\infty(2), \tau) \times L_\infty(l_\infty(2), \tau)$ by the symbol $B_{\infty,\infty} \times B_{\infty,\infty}$. We observe that $N_1(u,v)$ in this space coincide exactly with J(u) i.e., $N_1(u,v) = J(u) =$

$$\sup_{t\in\xi}\max\{|u_1(t),u_2(t)|\cdot\sup_{t\in\xi}\max\{|u_1(t),u_2(t)|\le 1 \text{-------}\to(3).$$

The solution of the system (1) is given as $e^{-At}x(t)-x(0)=\int_0^t e^{-As}Bu(s)ds$,→(4) where $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $u=\begin{bmatrix}u_1\\ u_2\end{bmatrix}$. If the system reaches the null state at time t, equation (4) reduces to $-x(0)=\int_0^t e^{-As}Bu(s)ds$ or putting $-x(0)=\xi$, $\xi=\int_0^t e^{-As}Bu(s)ds = S_t u$, where T_t is linear and onto, as can be easily verified. Thus the problem becomes one of a linear transformation of the 2-Banach space $B_{\infty,\infty}\times B_{\infty,\infty}$ to R^2. Since $N_1(u,v)\le 1$, the above problem readily becomes one of mapping unit sphere U in $B_{\infty,\infty}\times B_{\infty,\infty}$ to R^2. The optimal control for the problem is given by

$$u_1(t)=\begin{cases}\text{sign}\lfloor\Phi_1\rfloor, \Phi_1\ne 0\\ u_1(t)\le 1, \Phi_1\ne 0\end{cases} t\in[0,t];\ u_2(t)=\text{sign}\lfloor\Phi_2-\Phi_1\rfloor, t\in[0,t].$$

Another example is given in U.Adak & H.K.Samanta [12] to show the technique of application of the control theory in generalized 2-normed spaces.

Conclusion: In this paper, we introduced generalized 2–normed spaces and 2-normed spaces. There are appropriate connections between: (i) normed spaces and generalized 2–normed spaces, (ii) 2-normed spaces and generalized 2–normed spaces, (iii) 2-normed spaces and 2-Banach spaces, (iv) 2-normed spaces and locally convex topological vector spaces, (v) generalized 2-normed spaces and generalized symmetric 2-normed spaces.

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