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Fuzzy β-Open Mappings and Their Structural Properties in Fuzzy Topological Spaces

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ABSTRACT

This paper focuses on the study of fuzzy β-open maps within the framework of fuzzy topological spaces. Definitions and examples are presented to illustrate the behaviour of fuzzy β-open mappings. Several characterizations of fuzzy β-open maps are obtained using fuzzy β-interior and fuzzy β-closure operators. The relationships between fuzzy β-open maps and other well-known classes of fuzzy open-type mappings such as fuzzy open, fuzzy α-open, fuzzy semi-open, and fuzzy pre-open maps are also discussed. In addition, the notion of M-fuzzy β-open maps from existing literature is included to provide a broader perspective. The results contribute to the theoretical understanding of openness structures in fuzzy topology.

Keywords: Fuzzy topological spaces, fuzzy β-open sets, fuzzy β-open maps, M-fuzzy β-open maps.

AMS Mathematics Subject Classification (2010): 54A40, 54C35, 54H05, 03E72

Abstract in Bengali

এই প্রবন্ধে ফাজি টপোলজিক্যাল স্পেসের কাঠামোর মধ্যে ফাজি β-ওপেন ম্যাপ–এর অধ্যয়নের উপর গুরুত্ব দেওয়া হয়েছে। ফাজি β-ওপেন ম্যাপিং-এর আচরণ ব্যাখ্যা করার জন্য উপযুক্ত সংজ্ঞা ও উদাহরণ উপস্থাপন করা হয়েছে। ফাজি β-ইন্টেরিয়র এবং ফাজি β-ক্লোজার অপারেটর ব্যবহার করে ফাজি β-ওপেন ম্যাপের একাধিক চরিত্রায়ণ নির্ণয় করা হয়েছে।

এছাড়াও, ফাজি β-ওপেন ম্যাপ এবং অন্যান্য সুপরিচিত ফাজি ওপেন-ধরনের ম্যাপিং যেমন ফাজি ওপেন, ফাজি α-ওপেন, ফাজি সেমি-ওপেন এবং ফাজি প্রি-ওপেন ম্যাপের মধ্যে সম্পর্ক আলোচনা করা হয়েছে। বিদ্যমান সাহিত্য থেকে M-ফাজি β-ওপেন ম্যাপ–এর ধারণাও অন্তর্ভুক্ত করা হয়েছে, যাতে বিষয়টির একটি বিস্তৃত দৃষ্টিভঙ্গি প্রদান করা যায়। প্রাপ্ত ফলাফলসমূহ ফাজি টপোলজিতে ওপেননেস (openness) কাঠামোর তাত্ত্বিক বোঝাপড়াকে সমৃদ্ধ করে।

1. Introduction

There are approaches such as fuzzy sets [16], intuitionistic fuzzy sets [15], vague sets [08], and rough sets [12], which can be treated as mathematical tools to handle ambiguous data. But all these approaches have their inherent limitation in solving the problems involving indeterminate and inconsistent data due to inadequacy of parametrization tools. Molodtsov [10] introduced soft set theory. Fuzzy sets have applications in many fields such as information systems [13] and control theory [14]. The theory of fuzzy topological spaces was introduced and developed by Chang [7] and since then various notions in classical topology have been extended to fuzzy topological spaces. The concept of fuzzy β-open sets was introduced by Abd El-Monsef [1] and studied also by Allam and El-Hakeim [2]. Parallel to this, the concept of fuzzy β-open maps plays a significant role in understanding the openness behavior of functions between fuzzy topological spaces. A function $f : (X,\tau) \to (Y,\sigma)$ is called a fuzzy β-open map if the image of every fuzzy open set in X is a fuzzy β-open set in Y . This notion extends the classical fuzzy open map studied by Chang [7], and relates closely to fuzzy α-open maps [11], fuzzy semi-open maps [3], and fuzzy pre-open maps [6]. These relationships highlight the layered structure of openness notions in fuzzy topology, where fuzzy β-openness lies at a more inclusive level and has implications for applications in areas where uncertainty and gradation play critical roles.

2.Preliminaries

Definition 2.1. [16] If X is a collection of objects, denoted generically by x, then a fuzzy set A in X is a set of ordered pairs $A = \{(x,\mu_A(x)) : x \in X\}$. Here, $\mu_A(x)$ is the membership function (generalized characteristic function), which maps X into the membership space M. Its range is the subset of nonnegative real numbers whose supremum is finite.

Definition 2.2. [7] A fuzzy topology on a non-empty set X is a family τ of fuzzy sets in X satisfying the following axioms:

$(A1)$ $0_X,1_X \in \tau$,

$(A2)$ $G_1 \wedge G_2 \in \tau$ for any $G_1,G_2 \in \tau$,

$(A3)$ $\vee_i G_i \in \tau$ for every $\{G_i : i \in J\} \leq \tau$.

In this case, the pair (X,τ) is called a fuzzy topological space. The elements of τ are called fuzzy open sets. A fuzzy set A is fuzzy closed if A^c is fuzzy open.

Definition 2.3. [7] Let (X,τ) be a fuzzy topological space and A be a fuzzy set in X. Then the fuzzy closure and fuzzy interior of A are defined by

$cl(A) = \wedge\{K : K$ is a fuzzy closed set in X and $A \leq K \}$,

$int(A) = \vee\{G : G$ is a fuzzy open set in X and $G \leq A \}$.

It can be also shown that $cl(A)$ is fuzzy closed set and $int(A)$ is a fuzzy open set in X. Moreover, the following hold:

a) A is fuzzy open set if and only if $A = int(A)$.

b) A is fuzzy closed set if and only if $A = cl(A)$.

Definition 2.4. Let (X,τ) be a fuzzy topological space and A is a fuzzy set. Then A is called a

(i) fuzzy α-open set [9] if $A \leq int(cl(int(A)))$,
(ii) fuzzy semi-open set [3] if $A \leq cl(int(A))$,
(iii) fuzzy pre-open set [6] if $A \leq int(cl(A))$, *and*
(iv) fuzzy β-open set [1] if $A \leq cl(int(cl(\mathrm{A})))$.

Definition 2.5. [1] A fuzzy set A in a fuzzy topological space X is called a

(i) fuzzy β-open set if $A \leq cl(int(cl(\mathrm{A})))$, and
(ii) fuzzy β-closed set if $A \geq int(cl(int(A)))$.

From the literature, we see the following relation between the fuzzy open sets:

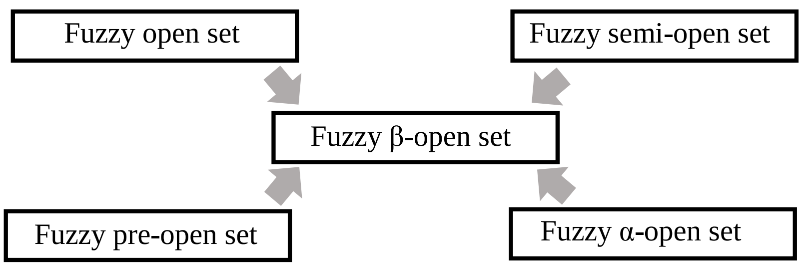

Figure 1: Relationship between the fuzzy β-open set and other open sets in fuzzy topological space

Example 2.1. *Let* $X = \{a,b,c\}$ *and let A,B and C be fuzzy sets in X defined by*
$A = \{\langle a,0.1\rangle,\langle b,0.1\rangle,\langle c,0.5\rangle\}$, $B = \{\langle a,1.0\rangle,\langle b,0.7\rangle,\langle c,0.1\rangle\}$, *and*
$C = \{\langle a,0.1\rangle,\langle b,0.4\rangle,\langle c,0.5\rangle\}$.
Then the family $\tau = \{0_X,A,B,A \vee B,1_X\}$ *is a fuzzy topology on X. We see that* $C \leq cl(int(cl(C)))$,
so C is a fuzzy β-open set in X. Here, $0_X = \{(x,0) : x \in X\}$ *and* $1_X = \{(x,1) : x \in X\}$.

Definition 2.6. [4] Let (X,τ) be a fuzzy topological space and U be a fuzzy set in X. Then:

(i) The fuzzy β-interior of U is the union of all fuzzy β-open sets of X contained in U. That is, $\beta int(U) = \vee\{G : G$ is a fuzzy β-open set in X and $G \leq U\}$.
(ii) The fuzzy β-closure of U is the intersection of all fuzzy β-closed sets of X contained in U. That is, $\beta cl(U) = \wedge\{G : G$ is a fuzzy β-open set in X and$G \geq U\}$.

Definition 2.7. [4] A map $f : (X,\tau) \to (Y,\sigma)$ is said to be fuzzy β-open if $f(U)$ is fuzzy β-open set in Y for each fuzzy open set U in X.

Example 2.2. *Let* $X = Y = \{a,b,c\}$ *and define the fuzzy sets A,B and C as follows:*

$A = \{\langle a,0.6\rangle,\langle b,0.4\rangle,\langle c,0.6\rangle\}$, $B = \{\langle a,0.4\rangle,\langle b,0.6\rangle,\langle c,0.5\rangle\}$, and $C=\{\langle a,0.4\rangle,\langle b,0.4\rangle,\langle c,0.4\rangle\}$. Then $\tau = \{0_X,A,1_X\}$, and $\sigma = \{0_X,B,B \wedge C,B \vee C,1_X\}$ are fuzzy topologies on X and Y respectively. Consider a map $f : (X,\tau) \to (Y,\sigma)$ defined by $f(a) = a$, $f(b) = b$, $f(c) = c$. It is clear that f is a fuzzy β-open map.

Definition 2.8. A map $f : (X,\tau) \to (Y,\sigma)$ is said to be:

(i) fuzzy open [7] if $f(A)$ is fuzzy open set in Y for each fuzzy open set A in X,

(ii) fuzzy α-open [11] if $f(A)$ is fuzzy α-open set in Y for each fuzzy open set A in X,

(iii) fuzzy semi-open [3] if $f(A)$ is fuzzy semi-open set in Y for each fuzzy open set A in X, and

(iv) fuzzy pre-open [6] if $f(A)$ is fuzzy pre-open set in Y for each fuzzy open set A in X.

Definition 2.9. [4] A map $f : (X,\tau) \to (Y,\sigma)$ is said to be M-fuzzy β-open if $f(U)$ is a fuzzy β-open set in Y for each fuzzy β-open set U in X.

3. Main results

Proposition 3.1. *Let (X,τ) and (Y,σ) be two fuzzy topological spaces and let $f : (X,\tau) \to (Y,\sigma)$ be a map. Then*

(i) every fuzzy open map is a fuzzy β-open map,

(ii) every fuzzy α-open map is a fuzzy β-open map,

(iii) every fuzzy semi-open map is a fuzzy β-open map,

(iv) every fuzzy pre-open map is a fuzzy β-open map.

Proof. (i) Let f be a fuzzy open map and let U be a fuzzy open set in X. Then by definition, $f(U)$ is a fuzzy open set in Y. Since every fuzzy open set is a fuzzy β-open set, $f(U)$ is a fuzzy β-open set in Y. Hence f is a fuzzy β-open map.

(ii) Let f be a fuzzy α-open map and let U be a fuzzy open set in X. Then by definition, $f(U)$ is a fuzzy α-open set in Y. Since every fuzzy α-open set is a fuzzy β-open set, $f(U)$ is a fuzzy β-open set in Y. Hence, f is a fuzzy β-open map.

(iii) Let f be a fuzzy semi-open map and let U be a fuzzy open set in X. Then by definition, $f(U)$ is a fuzzy semi-open set in Y. Since every fuzzy semi-open set is a fuzzy β-open set, $f(U)$ is a fuzzy β-open set in Y. Hence, f is fuzzy β-open map.

(iv) Let f be a fuzzy pre-open map and U be a fuzzy open set in X. Then by definition, $f(U)$ is a fuzzy pre-open set in Y. Since every fuzzy pre-open set is a fuzzy β-open set, $f(U)$ is a fuzzy β-open set in Y. Hence, f is fuzzy β-open map. ■

Remark 3.1. A map $f : (X,\tau) \to (Y,\sigma)$ is said to be fuzzy β-open if and only if the image of every fuzzy closed set in X is a fuzzy β-closed set in Y.

Theorem 3.1. *A map $f : (X,\tau) \to (Y,\sigma)$ is fuzzy β-open if and only if for every fuzzy set A in X, $f(int(A)) \leq \beta int(f(A))$.*

Proof. Suppose that $f : (X,\tau) \rightarrow (Y,\sigma)$ is a fuzzy β-open map and let A be a fuzzy set in X. Then $int(A) \leq A$ and $int(A)$ is a fuzzy open set in X. Since f is a fuzzy β-open map, $f(int(A))$ is fuzzy β-open set in Y. Hence, $f(int(A)) = \beta int(f(int(A)))$. Since $int(A) \leq A$, we have $f(int(A)) \leq f(A)$, which implies that $\beta int(f(int(A))) \leq \beta int(f(A))$. Hence, $f(int(A)) \leq \beta int(f(A))$.

Conversely assume that $f(int(A)) \leq \beta int(f(A))$ for every fuzzy set A in X, and let U be a fuzzy open set in X. Then the hypothesis implies $f(int(U)) \leq \beta int(f(U))$. Since $\text{int}(U) = U$, we have $f(U) \leq \beta int(f(U))$. Therefore, $f(U)$ is fuzzy β-open set in Y. Hence f is a fuzzy β-open map. ■

Theorem 3.2. *A map $f : (X,\tau) \rightarrow (Y,\sigma)$ is fuzzy β-open if and only if for every fuzzy set A in X, $\beta cl(f(A)) \leq f(cl(A))$.*

Proof. Suppose that f is a fuzzy β-open map and let A be a fuzzy set in X. By Remark 3.1, $f(cl(A))$ is fuzzy β-closed in Y. Hence, $f(cl(A)) = \beta cl(f(cl(A)))$. Also, $f(A) \leq f(cl(A))$, which implies $\beta cl(f(A)) \leq \beta cl(f(cl(A))) = f(cl(A))$. Therefore $\beta cl(f(A)) \leq f(cl(A))$.

Conversely assume that $\beta cl(f(A)) \leq f(cl(A))$ for every fuzzy set A in X, and let U be a fuzzy closed set in X. Then $\beta cl(f(U)) \leq f(cl(U)) = (f(U))$, which show that $\beta cl(f(U)) = f(U)$. Hence, $f(U)$ is fuzzy β-closed set in Y, and thus f is a fuzzy β-open map. ■

Theorem 3.3. *Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a bijective map. Then f is fuzzy β-open map if and only if for every fuzzy set B in Y, $int(f^{-1}(B)) \leq f^{-1}(\beta int(B))$.*

Proof. Suppose that f is a fuzzy β-open map and let B be a fuzzy set in Y. Then $f^{-1}(B) \leq X$ and $int(f^{-1}(B))$ is a fuzzy open set in X. Hence, $int(f^{-1}(B)) \leq f^{-1}(B)$. Since f is fuzzy β-open map, $f(int(f^{-1}(B)))$ is a fuzzy β-open set in Y. Therefore, $f(int(f^{-1}(B))) = \beta int(f(int(f^{-1}(B)))$. Since f is onto, $f(int(f^{-1}(B))) = \beta int f(int(f^{-1}(B))) \leq \beta int(f(f^{-1}(B))) = \beta int(B)$, which implies $f(int(f^{-1}(B))) \leq \beta int(B)$. As f is one to one, we get $int(f^{-1}(B)) \leq f^{-1}(\beta int(B))$.

Conversely assume that $int(f^{-1}(B)) \leq f^{-1}(\beta int(B))$ for every fuzzy set B in Y and let U be a fuzzy open set in X. Then $f(U) \leq Y$ and $int(U) = U$. By hypothesis, $int(f^{-1}(f(U))) \leq f^{-1}(\beta int(f(U)))$. Since f is one to one, $int(U) \leq f^{-1}(\beta int(f(U)))$, which implies that $U \leq f^{-1}(\beta int(f(U)))$. Since f is onto, $f(U) \leq \beta int(f(U))$. Therefore $f(U) = \beta int(f(U))$, $f(U)$ is a fuzzy β-open set in Y. Hence f is a fuzzy β-open map. ■

Theorem 3.4. *Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a bijective map. Then f is a fuzzy β-open map if and only if, for every fuzzy set B in Y, $f^{-1}(\beta cl(B)) \leq cl(f^{-1}(B))$.*

Proof. Suppose that f is a fuzzy β-open map and let B be a fuzzy set in Y. Then $f^{-1}(B) \leq X$ and $cl(f^{-1}(B))$ is a fuzzy closed set in X. Therefore, $f^{-1}(B) \leq cl(f^{-1}(B))$. Since f is a fuzzy β-open map, and by the Remark 3.1, $f(cl(f^{-1}(B)))$ is a fuzzy β-closed set in Y. Hence, $f(cl(f^{-1}(B)))= \beta cl(f(cl(f^{-1}(B))))$. Since f is onto, we have $\beta cl(B) = \beta cl(f(f^{-1}(B))) \leq$

$\beta cl(f(cl(f^{-1}(B)))) = f(cl(f^{-1}(B)))$. Thus, $\beta cl(B) \leq f(cl(f^{-1}(B)))$. Since f is one to one, $f^{-1}(\beta cl(B)) \leq clf^{-1}(B)$.

Conversely, assume that $f^{-1}(\beta cl(B)) \leq cl(f^{-1}(B))$ for every fuzzy set B in Y, and let U be a fuzzy closed set in X. Then $f(U) \leq Y$ and $cl(U) = U$. By hypothesis, $f^{-1}(\beta cl(f(U))) \leq cl(f^{-1}(f(U)))$. Since f is one to one, this implies $f^{-1}(\beta cl(f(U))) \leq U$. Since f is onto, $\beta cl(f(U)) \leq f(U)$, and hence $\beta cl(f(U)) = f(U)$. Therefore, $f(U)$ is fuzzy β-closed set in Y. Hence, by Remark 3.1, f is a fuzzy β-open map. ■

Proposition 3.2. *Let (X,τ) and (Y,σ) be two fuzzy topological spaces and let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a map. Then every M-fuzzy β-open map is a fuzzy β-open map.*
Proof. Let f be a M-fuzzy β-open map and let U be a fuzzy open set in X. Since every fuzzy open set is a fuzzy β-open set, U is a fuzzy β-open set in X. As f is an M-fuzzy β-open map, $f(U)$ is a fuzzy β-open set in Y. Hence, f is a fuzzy β-open map. ■

The converse of the above proposition need not be true in general, as shown in the following example.

Example 3.5. *Let $X = Y = \{a,b,c\}$ and define the fuzzy sets A,B and C as follows: $A = \{\langle a,0.6\rangle,\langle b,0.4\rangle,\langle c,0.6\rangle\}$, $B = \{\langle a,0.4\rangle,\langle b,0.6\rangle,\langle c,0.5\rangle\}$, and $C = \{\langle a,0.4\rangle,\langle b,0.4\rangle,\langle c,0.4\rangle\}$. Then $\tau = \{0_X,A,1_X\}$, and $\sigma = \{0_X,B,B \wedge C,B \vee C,1_X\}$ are fuzzy topologies on X and Y, respectively. Consider the map $f : (X,\tau) \rightarrow (Y,\sigma)$ defined by $f(a) = a$, $f(b) = b$, $f(c) = c$. It is clear that f is a fuzzy β-open map but not a M-fuzzy β-open map.*

Theorem 3.6. *A map $f : (X,\tau) \rightarrow (Y,\sigma)$ is a M-fuzzy β-open map if and only if $f(\beta int(U)) \leq \beta int(f(U))$ for every fuzzy set U in X.*

Proof. Suppose that f be a M-fuzzy β-open map and let U be a fuzzy set in X. Then $\beta int(U)$ is fuzzy β-open set in X and $\beta int(U) \leq U$. Therefore, $f(\beta int(U)) \leq f(U)$, which implies that $\beta int(f(\beta int(U))) \leq \beta int(f(U))$. Since f is a M-fuzzy β-open map, $f(\beta int(U))$ is a fuzzy β-open set in Y. This implies $f(\beta int(U)) = \beta int(f(\beta int(U)))$. Hence, $f(\beta int(U)) \leq \beta int(f(U))$.

Conversely, assume that $f(\beta int(U)) \leq \beta int(f(U))$ for every fuzzy set U in X, and let H be a fuzzy β-open set in X. Then $H = \beta int(H)$, which implies that $f(H) = f(\beta int(H))$. By hypothesis, $f(H) \leq \beta int(f(H))$. Hence, $f(H) = \beta int(f(H))$, and so $f(H)$ is fuzzy β-open set in Y. Therefore, f is a M-fuzzy β-open map. ■

Theorem 3.7. *If a map $f : (X,\tau) \rightarrow (Y,\sigma)$ is bijective and $\beta int(f^{-1}(B)) \leq f^{-1}(\beta int(B))$ for every fuzzy set B in Y. Then f is a M-fuzzy β-open map.*
Proof. Suppose that $\beta int(f^{-1}(B)) \leq f^{-1}(\beta int(B))$ for every fuzzy set B in Y, and let H be a fuzzy β-open set in X. Then $H = \beta int(H)$ and $f(H)$ is a fuzzy set in Y. By hypothesis, $\beta int(f^{-1}(f(H))) \leq f^{-1}(\beta int(f(H)))$. Since f is one to one map, we have $\beta int(H) \leq f^{-1}(\beta int(f(H)))$ which implies $H \leq f^{-1}(\beta int(f(H)))$. Therefore, $f(H) \leq f(f^{-1}(\beta int(f(H))))$. Since f is onto, we obtain $f(H) \leq \beta int(f(H))$. Hence, $f(H) = \beta int(f(H))$. Thus, $f(H)$ is a fuzzy β-open set in X, and consequently, f is a M-fuzzy β-open map.■

4. Conclusion

This study establishes several fundamental characterizations of fuzzy β-open maps by linking their behaviour with β-interior and β-closure operations. These characterizations provide criteria that can be applied directly to verify β-openness of mappings between fuzzy topological spaces. The relationships proved in this work also clarify how fuzzy β-open maps fit within the hierarchy of open-type mappings in fuzzy topology. The introduction of results concerning M-fuzzy β-open maps further highlights the structural role of β-openness. Overall, the findings contribute useful tools for analysing the openness properties of fuzzy mappings and offer a basis for future extensions of β-open structures.

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