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On the Convergence of an Implicit Iteration Scheme for a Finite Family of Asymptotically Nonexpansive Mappings in Banach Spaces

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ABSTRACT

In this paper we define a new implicit iterative process with errors in a real Banach space and establish a necessary and sufficient condition for the strong convergence of the iteration to a common fixed point for the case of a finite family of asymptotically nonexpansive mappings in a arbitrary real Banach space. Also we study the weak and strong convergence results of this implicit iterative scheme for the same family of mappings in the setting of a uniformly convex Banach space. Our results extend and generalize a number of existing results.

Keywords: *Implicit iteration process with errors, asymptotically nonexpansive mapping, uniformly convex Banach space, common fixed point, Condition* $(\overline{B})$*, Opial's condition, Kadec-Klee property.*

1. Introduction

Let X be a normed space, C be a nonempty subset of X and $T:C\to C$ be a given mapping. Then T is said to be asymptotically nonexpansive if there exists a sequence $\{h_n\}\subset[1,\infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\|T^n x - T^n y\| \le h_n \|x-y\|, \text{ for all } x, y\in C \text{ and each } n\ge 1.$$

If $F(T)\neq\phi$, then T is said to be asymptotically quasi-nonexpansive mapping if there exists a sequence $\{h_n\}\subset[1,\infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\|T^n x - p\| \le h_n \|x-p\|, \text{ for all } x\in C, p\in F(T) \text{ and each } n\ge 1.$$

T is said to be uniformly L-Lipschitzian if there exists a constant L>0 such that

$$\| T^n x - T^n y \| \le L \| x-y \|, \text{ for all } x, y\in C \text{ and each } n\ge 1.$$

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From the above definitions it is clear that an asymptotically nonexpansive mapping with a fixed point must be uniformly L-Lipschitzian as well as asymptotically quasi-nonexpansive but it is well-known that the converse does not hold in general. In 2001 Xu and Ori[14] introduced the following implicit iteration process for a finite family of N nonexpansive self mappings $\{T_i : i \in I_N\}$ of C (here $I_N = \{1, 2, .., N\}$) with $\{t_n\}$ a real sequence in (0, 1) and an initial point $x_0 \in C$, which is defined as follows:

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \; n \geq 1, \tag{1.1}$$

where $T_n = T_{n \bmod N}$ (here mod N function takes values in I_N) and they[14] proved the weak convergence of the process (1.1) to a common fixed point in the setting of a Hilbert space. In the rest of the paper we denote $\{1, 2, .., N\}$ by I_N. Zhou and Chang [16] studied the modified implicit iteration with errors for a finite family of asymptotically nonexpansive mappings which in compact form can be written as

$$x_n = \alpha_n x_{n-1} + \beta_n T^n_{n(\bmod N)} x_n + \gamma_n u_n, \qquad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequence in C. Chang et al.[1] defined an implicit iteration process with errors by

$$\begin{cases} x_1 = \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1 + u_1, \\ x_2 = \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2 + u_2, \\ . \\ . \\ x_N = \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N + u_N, \\ x_{N+1} = \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1^2 x_{N+1} + u_{N+1}, \\ . \\ . \\ x_{2N} = \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N}) T_N^2 x_{2N} + u_{2N}, \\ x_{2N+1} = \alpha_{2N+1} x_{2N} + (1 - \alpha_{2N+1}) T_1^3 x_{2N+1} + u_{2N+1}, \\ . \\ . \end{cases} \tag{1.3}$$

Since each $n \geq 1$ can be written as *n=(k-1)N+i,* where *i=I (n) ∈ {1,2,…,N}* , *k=k(n)* ≥ 1. is a positive integer and $k(n) \to \infty$, it follows that (1.3) can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^{k(n)} x_n + u_n, \qquad n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ and $\{u_n\}$ is a bounded sequence in C where C is a nonempty closed convex subset of X satisfying $C+C\subset C$. Chang et al.[1] defined and studied the implicit iterative process with errors (1.4) for a finite family of asymptotically nonexpansive mappings. Very recently Zhao et al. [15] introduced the following implicit iteration scheme for a finite family of nonexpansive mappings:

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \gamma_n T_n x_n, \qquad n \ge 1, \tag{1.5}$$

where $T_n = T_{n \bmod N}$. Motivated and inspired by these facts we introduce and study a new implicit iteration scheme with errors for a finite family of asymptotically nonexpansive mappings which is defined as follows:

$$x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_{n-1} + \gamma_n T_{i(n)}^{k(n)} x_n + \delta_n u_n, n \ge 1, \tag{1.6}$$

where $n=(k-1)N+i$, $i=i(n)\in\{1,2,...,N\}, k=k(n)\ge 1$ is a positive integer and $k(n)\to\infty$ as $n\to\infty$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ and $\{u_n\}$ is a bounded sequence in C.

The following argument shows that the sequence $\{x_n\}$ according to (1.6) can be actually constructed for the case of asymptotically nonexpansive mappings. For fixed $x,u\in C$, any asymptotically nonexpansive mapping $T: C\to C$ and nonnegative real numbers $\alpha,\beta,\gamma,\delta$ satisfying $\alpha+\beta+\gamma+\delta=1$, we construct the mapping $A:C\to C$ given by $Ay=\alpha x+\beta T^n x+\gamma T^n y+\delta u$. Then $\| Ay-Az\|\le \gamma h_n \| y-z\|$. If we take $\gamma<\frac{1}{L}$ where $L=\sup_{n\ge 1} h_n$, then A is a contraction mapping and has a unique fixed point. Hence the iteration procedure given by (1.6) is well defined for choices of $\gamma_n<\frac{1}{L}$ for all $n\ge 1$ where $L=\sup\{h_n^i : n\ge 1 \text{ and } i\in I_N\}$.

To prove our main results we need the following definitions and Lemmas.

A Banach space X is said to satisfy Opial's condition $[8]$ if $x_n\to x$ weakly and $x\ne y$ imply

$$\limsup_{n\to\infty}\| x_n-x\| < \limsup_{n\to\infty}\| x_n-y\|$$

A Banach space X is said to satisfy Kadec-Klee property if for every sequence $\{x_n\}\subset X, x_n\to x$ weakly and $\| x_n\|\to\|x\|$ together imply that $x_n\to x$ as $n\to\infty$. There are uniformly convex Banach spaces which have neither a *Frechet* differentiable norm nor they satisfy Opial's property but their duals do have the Kadec-Klee property (see $[5]$, $[7]$).

Condition $(A)[10]$: A self mapping T of C with nonempty fixed point set $F(T)$ satisfies Condition (A) if there is a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ such that

$$f(d(x,F(T)))\le\|x-Tx\| \text{ for all } x\in C$$

A finite family $\{T_i : i\in I_N\}$ of N self-mappings of C with $F=\bigcap_{i=1}^{N}F(T_i)\neq\phi$ is said to satisfy

(i) **Condition** $(\overline{A})[2]$ if there is a nondecreasing function $f:[0,\infty]\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ such that

$$f(d(x,F))\le\frac{1}{N}\left(\sum_{i=1}^{N}\|x-T_ix\|\right) \text{ for all } x\in C$$

(ii) **Condition** $(\overline{B})[2]$ if there is a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ such that

$$f(d(x,F))\le\max_{1\le i\le N}\{\|x-T_ix\|\} \text{ for all } x\in C$$

(iii) **Condition** $(\overline{C})[2]$ if there is a nondecreasing function $f:[0,\infty)\to[0,\infty)$ *with* $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ such that at least one of the T_i's satisfies Condition (A).

Clearly if $T_i=T$, for all $i=1,2,...,N$, then Condition $(\overline{A})$ reduces to Condition (A). Also Condition $(\overline{B})$ reduces to Condition (A) if all but one of T_i's are identities. It is well known that every continuous and demicompact mapping must satisfy Condition (A) (See $[10]$). Since every completely continuous mapping is continuous and demicompact, it must satisfy Condition (A). Therefore the study of the strong convergence of the sequence $\{x_n\}$ defined by (1.6) under the assumption of Condition $(\overline{B})$ includes the same study in the case under complete continuity of the mappings $\{T_1,T_2,...,T_N\}$.

Lemma 1.1 ([12], *Lemma* 1) *Let* $\{a_n\},\{b_n\}$ *and* $\{\delta_n\}$ be *sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1}\le(1+\delta_n)a_n+b_n,\ \forall n\ge1.$$

If $\sum_{n=1}^{\infty}\delta_n<\infty$ *and* $\sum_{n=1}^{\infty}b_n<\infty$, then

(i) $\lim_{n\to\infty}a_n$ *exists*,

(ii) $\lim_{n\to\infty}a_n=0$ *whenever* $\liminf_{n\to\infty}a_n=0$.

Lemma 1.2 $([9])$ *Suppose that* X *is a uniformly convex Banach space and* $0 < p \leq t_n \leq q \leq 1$ *for all positive integers* n. *Also suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are two sequences in* X *such that* $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ *and* $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ *hold for some* $r \geq 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.3 $([4], Theorem 3.4)$ *Let* X *be a real uniformly convex Banach space,* C *be a nonempty closed convex subset of* X *and* $T: C \to X$ *be asymptotically nonexpansive mapping. Then* $I - T$ *is demiclosed at zero, i.e. if* $\{x_n\}$ *is a sequence in* C *which converges weakly to* x *and if the sequence* $\{x_n - Tx_n\}$ *converges strongly to zero, then* $x - Tx = 0$.

Lemma 1.4 $([7], Theorem 2)$ *Let* X *be a real reflexive Banach space such that* X^* *has the Kadec-Klee property. Let* $\{x_n\}$ *be a bounded sequence in* X *and* $x^*, y^* \in w_W(x_n)$ (*week* $w-$*limit set of* $\{x_n\}$). *Suppose* $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ *exists for all* $t \in [0,1]$. *Then* $x^* = y^*$.

Lemma 1.5 *Let* X *be a uniformly convex Banach space,* C *be a nonempty bounded closed convex subset of* X. *Then there exists a strictly increasing continuous convex function* $\phi: [0,\infty) \to [0,\infty)$ *with* $\phi(0) = 0$ *such that for any Lipschitzian mapping* $T: C \to X$ *with the Lipschitz constant* $L \geq 1$ *and for any* $x, y \in C$ *and* $t \in [0,1]$ *the following inequality holds:*

$$\|T(tx + (1-t)y) - (tTx + (1-t)Ty)\| \leq L\phi^{-1}\left(\|x-y\| - L^{-1}\|Tx - Ty\|\right)$$

The purpose of this paper is to establish a necessary and sufficient condition for the strong convergence of the implicit iterative process with errors to a common fixed point for a finite family of asymptotically nonexpansive mapping which we have defined in (1.6) in an arbitrary real Banach space. Also we establish some strong convergences of the implicit iterative process (1.6) satisfying some additional condition and some weak convergences of the implicit iterative process (1.6) to a common fixed point for a finite family of nonexpansive mappings in uniformly convex Banach space. The results presented in this paper extend and improve some well known results.

2. Main Results

In this section we begin with the following lemmas.

Lemma 2.1 *Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I_N\}$ be a finite family of N asymptotically nonexpansive self-mappings of C with sequences $\{h_n^i\} \subseteq [1,\infty)$ such that $\sum_{n-1}^{\infty}(h_n^i - 1) < \infty$ for all $i \in I_N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence as defined in (1.6) with $0 < \tau_1 \le \gamma_n \le \tau_2 < \frac{1}{L} < 1, \alpha_n - \beta_n > \tau_3 > 0$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, where $L = \sup\{h_n^i : n \in N \text{ and } i \in I_N\}$. Then $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$.*

Proof: Let $p \in F$. *Since $\{u_n\}$ is a bounded sequence in $C, M = \sup_{n\ge1}\|u_n - p\|$ is finite. Let $h_n = \max\{h_n^i : i \in I_N\}$. Thus $\{h_n\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$. Now for all $n \ge 1$,*

$$\|x_n - p\| = \left\|\alpha_n(x_{n-1} - p) + \beta_n\left(T_{i(n)}^{k(n)}x_{n-1} - p\right) + \gamma_n\left(T_{i(n)}^{k(n)}x_n - p\right) + \delta_n(u_n - p)\right\|$$

$$\le \alpha_n\|x_{n-1} - p\| + \beta_n\left\|T_{i(n)}^{k(n)}x_{n-1} - p\right\| + \gamma_n\left\|T_{i(n)}^{k(n)}x_n - p\right\| + \delta_n\|u_n - p\|$$

$$\le \alpha_n\|x_{n-1} - p\| + \beta_n h_{k(n)}\|x_{n-1} - p\| + \gamma_n h_{k(n)}\|x_n - p\| + \delta_n M$$

$$= \alpha_n\|x_{n-1} - p\| + (1 - \alpha_n - \gamma_n - \delta_n)(1 + \mu_n)\|x_{n-1} - p\| +$$

$$(1 - \alpha_n - \beta_n - \delta_n)(1 + \mu_n)\|x_n - p\| + \delta_n M, \ \left(\text{where } \mu_n = (h_{k(n)} - 1)\right)$$

$$\le \alpha_n\|x_{n-1} - p\| + (1 - \alpha_n - \gamma_n + \mu_n)\|x_{n-1} - p\| +$$

$$(1 - \alpha_n - \beta_n - \delta_n + \mu_n)\|x_n - p\| + \delta_n M,$$

$$= (1 - \gamma_n + \mu_n)\|x_{n-1} - p\| + (1 - \alpha_n - \beta_n - \delta_n + \mu_n)\|x_n - p\| + \delta_n M$$

It follows from above that

$$(a_n + \beta_n + \delta_n)\|x_n - p\| \le (1 - \gamma_n + \mu_n)\|x_{n-1} - p\| + \mu_n\|x_n - p\| + \delta_n M$$

which implies that

$$\|x_n - p\| \le \left(1 + \frac{\mu_n}{1 - \gamma_n}\right)\|x_{n-1} - p\| + \frac{\mu_n}{1 - \gamma_n}\|x_n - p\| + \frac{\delta_n M}{1 - \gamma_n}$$

$$\le \left(1 + \frac{\mu_n}{1 - \tau_2}\right)\|x_{n-1} - p\| + \frac{\mu_n}{1 - \tau_2}\|x_n - p\| + \frac{\delta_n M}{1 - \tau_2}$$

By arranging both sides we get

$$\|x_n - p\| \le \frac{1 - \tau_2 + \mu_n}{1 - \tau_2 - \mu_n}\|x_{n-1} - p\| + \frac{\delta_n M}{1 - \tau_2 - \mu_n}$$

$$= \left(1+\frac{2\mu_n}{1-\tau_2-\mu_n}\right)\|x_{n-1}-p\| + \frac{\delta_n M}{1-\tau_2-\mu_n} \tag{2.1}$$

Since $\mu_n = h_{k(n)} - 1$ and $\sum_{n=1}^{\infty}(h_{k(n)}-1) < \infty$, we have $\sum_{n=1}^{\infty}\mu_n < \infty$. So there exists a positive integer n_1 such that $\mu_n < \frac{1-\tau_2}{2}$ for all $n \geq n_1$. Thus from (2.1) we have that for all $n \geq n_1$,

$$\begin{aligned}\|x_n - p\| &\leq \left(1+\frac{4\mu_n}{1-\tau_2}\right)\|x_{n-1}-p\| + \frac{2M}{1-\tau_2}\delta_n \\ &= (1+b_n)\|x_{n-1}-p\| + c_n \end{aligned} \tag{2.2}$$

where $b_n = \frac{4\mu_n}{1-\tau_2}$ and $c_n = \frac{2M}{1-\tau_2}\delta_n$. Since $\sum_{n=1}^{\infty}\mu_n < \infty$, it follows that $\sum_{n=1}^{\infty} b_n < \infty$. Again $\sum_{n=1}^{\infty}\delta_n < \infty$ implies that $\sum_{n=1}^{\infty} c_n < \infty$. Thus by Lemma 1.1 we have that $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$.

Theorem 2.1 *Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I_N\}$ be a finite family of N asymptotically nonexpansive self-mapping of C with sequences $\{h_n^i\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(h_n^i - 1) < \infty$ for all $i \in I_N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence as defined in (1.6) with $0 < \tau_1 \leq \gamma_n \leq \tau_2 < \frac{1}{L} < 1, \alpha_n - \beta_n > \tau_3 > 0$ and $\sum_{n=1}^{\infty}\delta_n < \infty$, where $L = \sup\{h_n^i : n \in N \text{ and } i \in I_N\}$. Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_N$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof: The necessary part is trivial. We only prove sufficiency part. From (2.2) we get that for all $n \geq n_1$,

$$\|x_n - p\| \leq (1+b_n)\|x_{n-1}-p\| + c_n$$

where $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Taking infimum over all $p \in F$ we have that for all $n \geq n_1$,

$$d(x_n, F) \leq (1+b_n)\, d(x_{n-1}, F) + c_n$$

Then by Lemma 1.1 we have that $\lim_{n\to\infty} d(x_n, F)$ exists. Since $\liminf_{n\to\infty} d(x_n, F) = 0$, we get that $\lim_{n\to\infty} d(x_n, F) = 0$. Now from (2.2) we get that for all $n \geq n_1$,

$$\|x_n - p\| \leq (1 + b_n)\|x_{n-1} - p\| + c_n \leq e^{b_n}\|x_{n-1} - p\| + c_n$$

Therefore for $n \geq n_1$ and for any positive integer m, we get that

$$\begin{aligned}
\|x_{n+m} - p\| &\leq e^{b_{n+m}}\|x_{n+m-1} - p\| + c_{n+m} \\
&\leq e^{b_{n+m}+b_{n+m-1}}\|x_{n+m-2} - p\| + c^{b_{n+m}} c_{n+m-1} + c_{n+m} \\
&\leq \ldots\ldots\ldots\ldots\ldots\ldots \\
&\leq e^{\sum_{i=n+1}^{n+m} b_i}\|x_n - p\| + \sum_{k=n+1}^{n+m-1} c_k e^{\sum_{i=k+1}^{n+m} b_i} + c_{n+m} \\
&\leq R\|x_n - p\| + R\sum_{k=n+1}^{\infty} c_k
\end{aligned}$$

where $R = e^{\sum_{n=1}^{\infty} b_n}$. Therefore for any $p \in F$ we have that for all $n \geq n_1$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq (R+1)\|x_n - p\| + R\sum_{k=n+1}^{\infty} c_k \qquad (2.3)$$

Since $\sum_{n=1}^{\infty} c_n < \infty$ and $\lim_{n\to\infty} d(x_n, F) = 0$, there exists $n_2 (\geq n_1) \in \mathrm{N}$ such that for all $n \geq n_2$ we have $d(x_n, F) < \dfrac{\varepsilon}{2(R+1)}$ and $\sum_{k=n+1}^{\infty} c_k < \dfrac{\varepsilon}{2R}$. So there exists $\overline{p} \in F$ such that $\|x_n - \overline{p}\| < \dfrac{\varepsilon}{2(R+1)}$ for all $n \geq n_2$. Therefore from (2.3) we have that for all $n \geq n_2$,

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - \overline{p}\| + \|x_n - \overline{p}\| \leq (R+1)\|x_n - \overline{p}\| + R\sum_{k=n+1}^{\infty} c_k \\
&< (R+1)\frac{\varepsilon}{2(R+1)} + R\frac{\varepsilon}{2R} = \varepsilon
\end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete and C is a closed subset of X, we have $x_n \to q (\in C)$ as $n \to \infty$. Now we will show that $q \in F$. Each T_i is asymptotically nonexpansive, so $F(T_i)$ is closed, which implies that F being the intersection of a finite number of closed sets is closed. Now

$$|d(q, F) - d(x_n, F)| \leq \|q - x_n\| \to 0 \text{ for all } n \geq 1. \qquad (2.4)$$

Since $\lim_{n\to\infty} x_n = q$ and $\lim_{n\to\infty} d(x_n, F) = 0$, from (2.4) it follows that $d(q, F) = 0$ that is $q \in F$. This completes the proof of the Theorem.

Lemma 2.2 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I_N\}$ be a finite family of N asymptotically nonexpansive self-mappings of C with sequences $\{h_n^i\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(h_n^i - 1) < \infty$ for all $i \in I_N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence as defined in (1.6) with $0 < \tau_1 \le \gamma_n \le \tau_2 < \frac{1}{L} < 1, \alpha_n - \beta_n > \tau_3 > 0$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, where $L = \sup\{h_n^i : n \in \mathrm{N} \text{ and } i \in I_N\}$. Then $\lim_{n\to\infty}\|x_n - T_l x_n\| = 0$ for all $l \in I_N$.*

Proof: Let $p \in F$. Since $\{u_n\}$ is a bounded sequence in $C, M = \sup_{n\ge1}\|u_n - p\|$ is finite. Let $h_n = \max\{h_n^i : i \in I_N\}$. Thus $\{h_n\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$. Now by Lemma 2.1 we have $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$. Therefore $\{x_n\}$ is bounded. Let $\lim_{n\to\infty}\|x_n - p\| = d$, for some $d \ge 0$. Now,

$$
\begin{aligned}
d &= \lim_{n\to\infty}\|x_n - p\| \\
&= \lim_{n\to\infty}\left\|\alpha_n(x_{n-1} - p) + \beta_n\left(T_{i(n)}^{k(n)}x_{n-1} - p\right) + \gamma_n\left(T_{i(n)}^{k(n)}x_n - p\right) + \delta_n(u_n - p)\right\| \\
&= \lim_{n\to\infty}\Bigg\|(1-\gamma_n)\left[\frac{\alpha_n}{1-\gamma_n}(x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n}\left(T_{i(n)}^{k(n)}x_{n-1} - p\right) + \delta_n(u_n - p)\right] + \\
&\qquad \gamma_n\left[T_{i(n)}^{k(n)}x_n - p + \delta_n(u_n - p)\right]\Bigg\| \qquad (2.5)
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\left\|\frac{\alpha_n}{1-\gamma_n}(x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n}\left(T_{i(n)}^{k(n)}x_{n-1} - p\right) + \delta_n(u_n - p)\right\| \\
&\le \frac{\alpha_n}{1-\gamma_n}\|x_{n-1} - p\| + \frac{\beta_n}{1-\gamma_n}h_{k(n)}\|x_{n-1} - p\| + \delta_n\|u_n - p\|.
\end{aligned}
$$

Taking limsup on the both sides of above we get

$$
\limsup_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}(x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n}\left(T_{i(n)}^{k(n)}x_{n-1} - p\right) + \delta_n(u_n - p)\right\| \le d. \qquad (2.6)
$$

Also

$$
\left\|T_{i(n)}^{k(n)}x_n - p + \delta_n(u_n - p)\right\| \le h_{k(n)}\|x_n - p\| + \delta_n\|u_n - p\|.
$$

Taking limsup on the both sides of above we get

$$\limsup_{n\to\infty}\left\|T_{i(n)}^{k(n)}x_n - p + \delta_n(u_n - p)\right\| \le d \tag{2.7}$$

From (2.5), (2.6), (2.7) and by Lemma 1.2 we have

$$\lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}(x_{n-1}-p)+\frac{\beta_n}{1-\gamma_n}\left(T_{i(n)}^{k(n)}x_{n-1}-p\right)+\delta_n(u_n-p)\right.$$
$$\left.-\left(T_{i(n)}^{k(n)}x_n - p + \delta_n(u_n-p)\right)\right\| = 0,$$

which implies that

$$\lim_{n\to\infty}\left\|\alpha_n(x_{n-1}-p)+\beta_n\left(T_{i(n)}^{k(n)}x_{n-1}-p\right)-(1-\gamma_n)\left(T_{i(n)}^{k(n)}x_n-p\right)\right\|=0$$

which further implies that

$$\lim_{n\to\infty}\left\|x_n - T_{i(n)}^{k(n)}x_n - \delta_n(u_n-p)\right\| = 0. \tag{2.8}$$

Since

$$\left\|x_n - T_{i(n)}^{k(n)}x_n\right\| \le \left\|x_n - T_{i(n)}^{k(n)}x_n - \delta_n(u_n-p)\right\| + \delta_n\|u_n - p\|,$$

by (2.8) and the given condition $\sum_{n=1}^{\infty}\delta_n < \infty$ it follows that

$$\lim_{n\to\infty}\left\|x_n - T_{i(n)}^{k(n)}x_n\right\| = 0. \tag{2.9}$$

Now,

$$\begin{aligned}\|x_n - x_{n-1}\| &\le \beta_n\left\|T_{i(n)}^{k(n)}x_{n-1}-x_{n-1}\right\|+\gamma_n\left\|T_{i(n)}^{k(n)}x_n - x_{n-1}\right\|+\delta_n\|u_n - x_{n-1}\| \\ &\le \beta_n[\left\|T_{i(n)}^{k(n)}x_{n-1}-T_{i(n)}^{k(n)}x_n\right\|+\left\|T_{i(n)}^{k(n)}x_n - x_n\right\|+\|x_n - x_{n-1}\|]+ \\ &\quad \gamma_n[\left\|T_{i(n)}^{k(n)}x_n - x_n\right\|+\|x_n - x_{n-1}\|]+\delta_n\|u_n - x_{n-1}\| \\ &\le \beta_n h_{k(n)}\|x_{n-1}-x_n\|+\beta_n\left\|T_{i(n)}^{k(n)}x_n - x_n\right\|+\beta_n\|x_n - x_{n-1}\|+ \\ &\quad \gamma_n[\left\|T_{i(n)}^{k(n)}x_n - x_n\right\|+\|x_n - x_{n-1}\|]+\delta_n\|u_n - x_{n-1}\|\end{aligned}$$

$$=[\beta_n(1+\mu_n)+\beta_n+\gamma_n]\|x_{n-1}-x_n\|+(\beta_n+\gamma_n)\left\|T_{i(n)}^{k(n)}x_n - x_n\right\|+\delta_n\|u_n - x_{n-1}\|$$

$$\le(1-\alpha_n+\beta_n+\mu_n)\|x_{n-1}-x_n\|+(\beta_n+\gamma_n)\left\|T_{i(n)}^{k(n)}x_n - x_n\right\|+\delta_n\|u_n - x_{n-1}\|$$

Simplyfying we get

$$(\alpha_n - \beta_n)\ \ \|x_n - x_{n-1}\|$$
$$\le \mu_n\|x_{n-1}-x_n\|+(\beta_n+\gamma_n)\|T_{i(n)}^{k(n)}x_n - x_n\|+\delta_n\|u_n - x_{n-1}\|$$

Since $\alpha_n - \beta_n > \tau_3 > 0$, from above it follows that

$$\| x_n - x_{n-1} \|$$
$$\leq \frac{\mu_n}{\tau_3} \| x_{n-1} - x_n \| + \frac{\beta_n + \gamma_n}{\tau_3} \| T_{i(n)}^{k(n)} x_n - x_n \| + \frac{\delta_n}{\tau_3} \| u_n - x_{n-1} \|$$

After rearrangement we get from the above that

$$\| x_n - x_{n-1} \| \leq \frac{\beta_n + \gamma_n}{\tau_3 - \mu_n} \| T_{i(n)}^{k(n)} x_n - x_n \| + \frac{\delta_n}{\tau_3 - \mu_n} \| u_n - x_{n-1} \| \tag{2.10}$$

Since $\sum_{n=1}^{\infty} \mu_n < \infty$, there exists a positive integer n_3 such that $\mu_n < \frac{\tau_3}{2}$ for all $n \geq n_3$. Thus from (2.10) we have that for all $n \geq n_2$,

$$\| x_n - x_{n-1} \| \leq \frac{2(\beta_n + \gamma_n)}{\tau_3} \| T_{i(n)}^{k(n)} x_n - x_n \| + \frac{2\delta_n}{\tau_3} \| u_n - x_{n-1} \| \tag{2.11}$$

Thus from (2.9) and (2.11) it follows that

$$\lim_{n\to\infty} \| x_n - x_{n-1} \| = 0. \tag{2.12}$$

Therefore for any $l \in I_N$,

$$\lim_{n\to\infty} \| x_n - x_{n+l} \| = 0. \tag{2.13}$$

Let $\sigma_n = \| T_{i(n)}^{k(n)} x_n - x_n \|$. From (2.9), we have $\sigma_n \to 0$ as $n \to \infty$. For each $n > N$, $n = (n-N) \pmod N$, and for any $n > N$, $n = (k(n)-1)N + i(n), i(n) \in \{1, 2........, N\}$, so $k(n-N) = k(n)-1$ and $i(n-N) = i(n)$. Hence,

$$\| x_n - T_n x_n \| =$$
$$\| x_n - T_{i(n)}^{k(n)} x_n \| + \| T_{i(n)}^{k(n)} x_n - T_n x_n \| \leq \sigma_n + L \| T_{i(n)}^{k(n)-1} x_n - x_n \|$$
$$\leq \sigma_n + L[\| T_{i(n)}^{k(n)-1} x_n - T_{i(n)}^{k(n)-1} x_{n-N} \| + \| T_{i(n)}^{k(n)-1} x_{n-N} - x_n \|]$$
$$= \sigma_n + L[\| T_{i(n-N)}^{k(n-N)} x_n - T_{i(n-N))}^{k(n-N)} x_{n-N} \| + \| T_{i(n-N)}^{k(n-N)} x_{n-N} - x_n \|]$$
$$\leq \sigma_n + L[L \| x_n - x_{n-N} \| + \| T_{i(n-N)}^{k(n-N)} x_{n-N} - x_{n-N} \| + \| x_{n-N} - x_n \|]$$
$$= \sigma_n + L[(L+1)\| x_n - x_{n-N} \| + \sigma_{n-N}] \tag{2.14}$$

From (2.9), (2.13) and (2.14) it follows that

$$\lim_{n\to\infty} \| x_n - T_n x_n \| = 0 \tag{2.15}$$

Now for all $l \in I_N$ by using (2.15) and (2.13) we get that

$$\| x_n - T_{n+l} x_n \| \leq \| x_n - x_{n+l} \| + \| x_{n+l} - T_{n+l} x_{n+l} \| + \| T_{n+l} x_{n+l} - T_{n+l} x_n \|$$
$$\leq \| x_n - x_{n+l} \| + \| x_{n+l} - T_{n+l} x_{n+l} \| + L \| x_{n+l} - x_n \|$$
$$\leq (1+L) \| x_n - x_{n+l} \| + \| x_{n+l} - T_{n+l} x_{n+l} \|$$
$$\to 0 \text{ as } n \to \infty.$$

Consequently we have

$$\lim_{n\to\infty} \| x_n - T_l x_n \| = 0 \text{, for all } l \in I_N . \tag{2.16}$$

Theorem 2.2 Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I_N\}$ be a finite family of N asymptotically nonexpansive self-mappings of C with sequence $\{h_n^i\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty} (h_n^i - 1) < \infty$ for all $i \in I_N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$.. Let $\{x_n\}$ be the sequence as defined in (1.6) with $0 < \tau_1 \le \gamma_n \le \tau_2 \le \frac{1}{L} < 1$, $\alpha_n - \beta_n > \tau_3 > 0$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, where $L = \sup\{h_n^i : n \in N$ and $i \in I_N\}$. If $\{T_i : i \in I_n\}$ satisfy condition $(\overline{B})$, then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots\ldots, T_N$.

Proof : As in the proof of Theorem 2.1 we have that $\lim_{n\to\infty} d(x_n, F)$ exists. Now by Lemma 2.2 we get $\lim_{n\to\infty} \| x_n - T_l x_n \| = 0,$ for $l \in I_N$. Then by Condition $(\overline{B})$ it follows that $\lim_{n\to\infty} f(d(x_n, F)) = 0.$ Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function with $f(0) = 0$ and $\lim_{n\to\infty} d(x_n, F)$ exists, we have $\lim_{n\to\infty} d(x_n, F) = 0.$ Then the Theorem follows by an application of Theorem 2.1.

Theorem 2.3 Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I_N\}$ be a finite family of N asymptotically nonexpansive self-mapping of C with sequence $\{h_n^i\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty} (h_n^i - 1) < \infty$ for all $i \in I_N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$.. Let $\{x_n\}$ be the sequence as defined in (1.6) with

$$0 < \tau_1 \le \gamma_n \le \tau_2 \le \frac{1}{L} < 1,\ \alpha_n - \beta_n > \tau_3 > 0 \text{ and } \sum_{n=1}^{\infty} \delta_n < \infty \text{,where}$$

$L = \sup\{h_n^i : n \in N$ and $i \in I_N\}$. If one member of the family $\{T_i : i \in I_N\}$ is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots\ldots, T_N$.

Proof : From Lemma 2.2 we get that $\lim_{n\to\infty} \| x_n - T_l x_n \| = 0,$ for all $l \in I_N$. Let us assume that T_1 is semi-compact operator. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p.$ Now

$\| p - T_l p \| = \lim_{k\to\infty} \| x_n - T_l x_n \| = 0$ for all $l \in I_N$, which implies that $p \in F$. Since $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to p$, we have that lim $\inf_{n\to\infty} d(x_n, F) = 0$. Hence the result follows by an application of Theorem 2.1.

Theorem 2.4 Let X be a uniformly convex Banach space satisfying Opial's condition, C be a nonempty closed convex subset of X. Let $\{T_i : i \in I_N\}$ be a finite family of N asymptotically nonexpansive self-mappings of C with sequences $\{h_n^i\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty} (h_n^i - 1) < \infty$ for all $i \in I_N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi.$. Let $\{x_n\}$ be the sequence as defined in (1.6) with $0 < \tau_1 \leq \gamma_n \leq \tau_2 \leq \frac{1}{L} < 1$, $\alpha_n - \beta_n > \tau_3 > 0$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, where $L = \sup\{h_n^i : n \in N$ and $i \in I_N\}$. Then $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \ldots\ldots., T_N$.

Proof : Since $F \neq \phi$, let $q \in F$. Then by by Lemma 2.1 $\lim_{n\to\infty} \|x_n - q\|$ exists, so $\{x_n\}$ is bounded. As E be a uniformly convex Banach space, it is reflexive, hence $\{x_n\}$ has a subsequence $\{x_n\}$ which is weakly convergent to $p \in C$ (say). From Lemma 2.2 we get $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$, for all $l \in I_N$. By Lemma 1.3 we have T_l is demiclosed at 0 so that $p \in F(T_l)$ for all $l \in I_N$ and hence $p \in F$. If possible let $\{x_n\}$ have another subsequence $\{x_{n_k}\}$ which converges weakly to another point $q \in C$. Then by similar argument as above we have that $q \in F$. Then by Opial's property we have

$$\begin{aligned} \| x_n - p \| &= \limsup_{j\to\infty} \|x_{n_j} - p\| < \limsup_{j\to\infty} \|x_{n_j} - q\| = \lim_{n\to\infty} \|x_n - q\| \\ &= \limsup_{k\to\infty} \|x_{n_k} - q\| < \limsup_{k\to\infty} \|x_{n_k} - q\| = \lim_{n\to\infty} \|x_n - p\| \end{aligned}$$

a contradiction. So $p = q$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \ldots\ldots., T_N$.

Lemma 2.3 Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I_N\}$ be a finite family of N asymptotically nonexpansive self-mappings of C with sequences $\{h_n^i\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty} (h_n^i - 1) < \infty$ for all $i \in I_N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi.$. Let $\{x_n\}$ be the sequence as defined in (1.6) with $0 < \tau_1 \leq \gamma_n \leq \tau_2 \leq \frac{1}{L} < 1$, $\alpha_n - \beta_n > \tau_3 > 0$ and

$\sum_{n=1}^{\infty} \delta_n < \infty$, where $L = \sup\{h_n^i : n \in N$ and $i \in I_N\}$. Then $\lim_{n\to\infty} \| tx_n + (1-t)p_1 - p_2 \|$ exists for all $p_1, p_2 \in F$ and for all $t \in [0,1]$.

Proof : Let $h_n =$ max $\{h_n^i : i \in I_N\}$. Thus $\{h_n\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty} (h_n - 1) < \infty$. By Lemma 2.1 we have that $\{x_n\}$ is a bounded sequence, so there exists $r > 0$ such that $\{x_n\} \subset B_r(0) \bigcap C$. Then $B_r(0) \bigcap C$ is a nonempty closed bounded subset of X. Let $d_n(t) = \| tx_n + (1-t)p_1 - p_2 \|$ for all $t \in [0,1]$ and $p_1, p_2 \in F$. Then $\lim_{n\to\infty} d_n(0) = \| p_1 - p_2 \|$ and $\lim_{n\to\infty} d_n(1) = \lim_{n\to\infty} \| x_n - p_2 \|$ exist by Lemma 2.1. Let $t \in (0,1)$. Let $x \in C$ be given. Let us define the mapping S_{xj-1} from $C \to C$ by

$$S_{xj-1} = \alpha_j x + \beta_j T_{i(j)}^{k(j)} x + \gamma_j T_{i(j)}^{k(j)} + \delta_j u_j$$

Let $u, v \in C$. Now $\| S_{x,j-1} u - S_{x,j-1} v \| = \gamma_j \| T_{i(j)}^{k(j)} u - T_{i(j)}^{k(j)} v \| \le \gamma_j h_{k(j)} \| u - v \|$. Again by the condition of the theorem we have $\gamma_j h_{k(j)} < 1$. Hence $S_{x,j-1}$ is a contraction mapping. Therefore it has a unique fixed point which we denote by $G_{j-1}x$. Hence it is possible to define the function G_{j-1} from C to C for all values of $j \ge 1$. In view of the above argument we define the following sequence of functions $G_n : C \to C$ by

$$\begin{cases} G_0 = \alpha_1 I + \beta_1 T_1 I + \gamma_1 T_1 G_0 + \delta_1 u_1 \\ G_1 = \alpha_2 I + \beta_2 T_2 I + \gamma_2 T_2 G_1 + \delta_2 u_2 \\ . \\ . \\ G_{N-1} = \alpha_N I + \beta_N T_N I + \gamma_N T_N G_{N-1} + \delta_N u_N \\ G_N = \alpha_{N+1} I + \beta_{N+1} T_1^2 I + \gamma_{N+1} T_1^2 G_N + \delta_{N+1} u_{N+1} \\ . \\ . \\ G_{2N-1} = \alpha_{2N} I + \beta_{2N} T_N^2 I + \gamma_{2N} T_N^2 G_{2N-1} + \delta_{2N} u_{2N} \\ G_{2N} = \alpha_{2N+1} I + \beta_{2N+1} T_1^3 I + \gamma_{2N+1} T_1^3 G_{2N} + \delta_{2N+1} u_{2N+1} \\ . \\ . \end{cases}$$

which can be written in the compact form as

$$G_n = \alpha_{n+1} I + \beta_{n+1} T_{i(n+1)}^{k(n+1)} I + \gamma_{n+1} T_{i(n+1)}^{k(n+1)} G_n + \delta_{n+1} u_{n+1} \qquad (2.17)$$

$$
\begin{aligned}
&\| G_n x - G_n y \| \\
&\leq \alpha_{n+1} \|x-y\| + \beta_{n+1} h_{k(n+1)} \|x-y\| + \gamma_{n+1} h_{k(n+1)} \| G_n x - G_n y \| \\
&= \alpha_{n+1} \|x-y\| + \beta_{n+1} (1+\mu_{n+1}) \|x-y\| + \gamma_{n+1} (1+\mu_{n+1}) \| G_n x - G_n y \| \\
&= \alpha_{n+1} \|x-y\| + (1-\alpha_{n+1}-\gamma_{n+1}-\delta_{n+1})(1+\mu_{n+1}) \|x-y\| \\
&\quad + (1-\alpha_{n+1}-\beta_{n+1}-\delta_{n+1})(1+\mu_{n+1}) \| G_n x - G_n y \| \\
&\leq \alpha_{n+1} \|x-y\| + (1-\alpha_{n+1}-\gamma_{n+1}+\mu_{n+1}) \|x-y\| + \\
&\quad (1-\alpha_{n+1}-\beta_{n+1}-\delta_{n+1}+\mu_{n+1}) \| G_n x - G_n y \|
\end{aligned}
$$

which implies that

$$(1-\gamma_{n+1}) \| G_n x - G_n y \| \leq (1-\gamma_{n+1}+\mu_{n+1}) \|x-y\| + \mu_{n+1} \| G_n x - G_n y \|$$

which further implies that

$$\| G_n x - G_n y \| \leq \left(1 + \frac{\mu_{n+1}}{1-\gamma_{n+1}}\right) \|x-y\| + \frac{\mu_{n+1}}{1-\gamma_{n+1}} \| G_n x - G_n y \|$$

Since $0 < \tau_1 \leq \gamma_n \leq \tau_2 < 1,$ from above it follows that

$$\| G_n x - G_n y \| \leq \left(1 + \frac{\mu_{n+1}}{1-\tau_2}\right) \|x-y\| + \frac{\mu_{n+1}}{1-\tau_2} \| G_n x - G_n y \|$$

After rearranging both sides we have that

$$\| G_n x - G_n y \| \leq \left(\frac{1-\tau_2+\mu_{n+1}}{1-\tau_2-\mu_{n+1}}\right) \|x-y\| = \left(1 + \frac{2\mu_{n+1}}{1-\tau_2-\mu_{n+1}}\right) \|x-y\|$$

Since $\sum_{n=1}^{\infty} \mu_n < \infty,$ there exists a positive integer n_4 such that for all $n \geq n_4$ we have $\mu_n < \frac{1-\tau_2}{2}.$

Thus for all $n \geq n_4$, it follows from the above that

$$\| G_n x - G_n y \| \leq \left(1 + \frac{4\mu_{n+1}}{1-\tau_2}\right) \|x-y\| = (1 + b_{n+1}) \|x-y\| \tag{2.18}$$

where $b_{n+1} = \frac{4\mu_{n+1}}{1-\tau_2}.$ Again from (2.17) and (1.6) we have

$$\| G_n x_n - x_{n+1} \| = \gamma_{n+1} \left\| T_{i(n+1)}^{k(n+1)} G_n x_n - T_{i(n+1)}^{k(n+1)} x_{n+1} \right\| \leq \gamma_{n+1} h_{k(n+1)} \| G_n x_n - x_{n+1} \|$$

Since $\gamma_{n+1} h_{k(n+1)} < 1,$ we have

$$G_n x_n = x_{n+1}. \tag{2.19}$$

Similarly we can show that for any $q \in F$,

$$G_n q = q. \tag{2.20}$$

Set

$$S_{n,m} = G_{n+m-1}G_{n+m-2}..........G_n, m \geq 1 \text{ and}$$
$$b_{n,m} = \left\| S_{n,m}(tx_n + (1-t)q) - (tx_{n+m} + (1-t)q) \right\|.$$

Then

$$\|S_{n,m}x - S_{n.m}y\| \leq (1+b_{n+m})(1+b_{n+m-1})....(1+b_{n+1})\|x-y\|$$

Denote the sequence $\{(1+b_{n+m})(1+b_{n+m-1}).....(1+b_{n+1})\}$ by $\{H_{n,m}\}$. Then $\lim_{n,m\to\infty} H_{n.m} = 1$.

Now by (2.19) and (2.20) respectively it follows that $S_{n,m}x_n = x_{n+m}$ and $S_{n,m}p = p$ for all $p \in F$.

We have that $S_{n,m}$ is Lipschitzian with the lipschitz constant $H_{n,m}$. Now By Lemma 1.5 we have

$$b_{n,m} \leq H_{n,m}\phi^{-1}\left(\|x_n - q\| - H_{n,m}^{-1}\|x_{n+m} - q\|\right)$$

By Lemma 2.1 we have $\lim_{n,m\to\infty} b_{n,m} = 0.$

$$d_{n+m}(t) = \|tx_{n+m} + (1-t)p - q\| \leq b_{n,m} + H_n\|tx_n + (1-t)p - q\| = b_{n,m} + H_n d_n(t)$$

Hence,

$$\limsup_{n\to\infty} d_n(t) \leq \phi^{-1}(0) + \liminf_{n\to\infty} d_n(t) = \liminf_{n\to\infty} d_n(t).$$

This completes the proof of the Lemma.

Theorem 2.5 *Let X be a uniformly convex Banach space such that it's dual X^* has the Kadec-Klee property and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I_N\}$ be a finite family of N asymptotically nonexpansive self-mappings of C with sequences $\{h_n^i\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(h_n^i - 1) < \infty$ for all $i \in I_N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence as defined in (1.6) with $0 < \tau_1 \leq \gamma_n \leq \tau_2 < \frac{1}{L} < 1, \alpha_n - \beta_n > \tau_3 > 0$ and $\sum_{n=1}^{\infty}\delta_n < \infty,$ where $L = \sup\{h_n^i : n \in \mathrm{N} \text{ and } i \in I_N\}$. Then $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, ...T_N$.*

Proof: Since $F \neq \phi$, let $q \in F$. Then by Lemma 2.1 $\lim_{n\to\infty}\|x_n - q\|$ exists so $\{x_n\}$ is bounded. Since E be a uniformly convex Banach space, it is reflexive. Hence $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which is weakly convergent to $p \in C$ (say). From Lemma 2.2 we get $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for $i \in I_N$. By Lemma 1.3 we have T_i is demiclosed at 0, so $p \in F(T_i)$ for all $i \in I_N$. Hence $p \in F$.
If possible let $\{x_n\}$ have another subsequence $\{x_{n_k}\}$ which converges weakly to another point $q \in C$. Then by similar argument as above we have that $q \in F(T)$. Now from Lemma 2.3 we get $\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$ exists, so by Lemma 1.4 we have that $p = q$. So $\{x_n\}$ converges weakly to some common fixed point of $T_1, T_2, \ldots T_N$. This completes the proof of the Theorem.

Remark 2.1 (1) *Furthermore Condition $(\overline{C})$ and Condition $(\overline{B})$ are equivalent (See [2]). If $\{T_i : i \in I_N\}$ satisfy Condition $(\overline{C})$, then theorem 2.2 still holds.*
(2) Results in this paper extend and improve the corresponding results of [3], [14], [15], [16].

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