Journal of Physical Sciences, Vol. 12, 2008, 83-96

On Asymptotic Solutions of Fourth Order Over-Damped Nonlinear Systems in Presence on Certain Damping Forces

M. Ali Akbar* and M. Sharif Uddin**
*Department of Applied Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh.
Email: ali_math74@yahoo.com
**Mathematics Discipline, Khulna University, Khulna-9208, Bangladesh.
Email: sharif_ku@yahoo.com

Received October 15, 2008; accepted October 27, 2008

ABSTRACT

The nature of the solutions of the over-damped systems depends on the nature of damping forces. In this article, the Krylov-Bogoliubov-Mitropolskii (KBM) method has been extended and modified for obtaining the solutions of fourth order over-damped nonlinear systems, when the damping force is such that, one of the eigenvalues of the linear systems is vanishes or tends to zero and the other eigenvalues are in integral multiple. The method is illustrated by an example. The solutions obtained by the presented KBM method for different set of initial conditions show good coincidence with those obtained by the numerical method.

Keywords: *Asymptotic Solutions, Over-damped Systems, Damping Forces.*

1. Introduction

The Krylov-Bogoliubov-Mitroploskii (KBM) [4, 6] method is a widely used tool to study nonlinear oscillatory and non-oscillatory differential systems with small nonlinearities. Originally, the method was developed by Krylov and Bogoliubov [6] for obtaining the periodic solutions of second order nonlinear differential systems with small nonlinearities. The method was then amplified and justified mathematically by Bogoliubov and Mitroposkii [4]. Popov [13] extended the method to damped oscillatory nonlinear systems. Owing to physical importance of the damped oscillatory systems, Popov's results were rediscovered by Bojadziev [5] and Mendelson [7]. Murty *et al*. [10] developed an asymptotic method base on the theory of Bogoliubov to obtain the response of over-damped nonlinear systems. Murty [11] presented a unified KBM method, which covers the undamped, damped and over-damped cases. Sattar [14] found an asymptotic solution of a second order critically damped nonlinear system. Shamsul [17] examined a new asymptotic solution for both over-damped and critically damped nonlinear systems.

First, Osiniskii [12] found an asymptotic solution of a third order nonlinear system by making use of the KBM method. But the solution was over simplified due to the restrictions on the parameters. Mulholland [8] removed this restrictions and found desired results. Sattar [15] studied a three-dimensional over-damped nonlinear system. Shamsul and Sattar [16] developed a perturbation technique based on the work of the KBM for obtaining the solutions of third order critically damped nonlinear systems. Shamsul [20] investigated approximate solutions of third order critically damped nonlinear systems whose unequal eigenvalues are in integral multiple. Shamsul [21] also presented a perturbation method for solving third order over-damped systems based on the KBM method when two eigenvalues of the linear equation are almost equal (rather than equal) and the other is very small.

In article [10], Murty *et al*. also extended the KBM method for solving fourth order over-damped nonlinear systems. But their technique is too much complex and laborious. Akbar *et al.* [1] presented an asymptotic method for fourth order over-damped nonlinear systems which is simple and easier than the method presented in [10], but the results obtained by [1] are same as the results obtained by [10]. Later, Akbar *et al.* [2] extended the method presented in [1] for fourth order damped oscillatory systems. Akbar *et al.* [3] have investigated a technique for obtaining over-damped solutions of *n*-th order nonlinear differential equation. But the solution presented in [3] breakdown (see **Appendix**) when one of the eigenvalues is near zero or vanishes.

In this article, we have filled up this gap and found desired results when one of the eigenvalues is near zero or vanishes and the others are integral multiple.

2. The Method

Consider a fourth order weakly nonlinear differential system

$$x^{(4)} + k_1 \dddot{x} + k_2 \ddot{x} + k_3 \dot{x} + k_4 x = -\varepsilon\, f(x, \dot{x}, \ddot{x}, \dddot{x}) \tag{1}$$

where $x^{(4)}$ stands for the fourth derivative of *x* with respect to t and over dots are used to denote first, second and third derivatives of x; k_1, k_2, k_3, k_4 are constants, ε is the small parameter and *f* is the given nonlinear function. As the equation is fourth order, so, there are four eigenvalues of the corresponding linear equation and all are non-positive, since the system is over-damped. Suppose the eigenvalues are $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$.

Therefore, the solution of the corresponding linear equation of (1) is

$$x(t,0) = a_{1,0}\, e^{-\lambda_1 t} + a_{2,0}\, e^{-\lambda_2 t} + a_{3,0}\, e^{-\lambda_3 t} + a_{4,0}\, e^{-\lambda_4 t} \tag{2}$$

where $a_{j,0}$, j =1, 2, 3, 4 are constants of integration.

When $\varepsilon \neq 0$, following [18] an asymptotic solution of (1) is sought in the form

$$x(t,\varepsilon) = \sum_{j=1}^{4} a_j(t)\, e^{-\lambda_j t} + \varepsilon\, u_1(a_1, a_2, a_3, a_4, t) + \cdots \tag{3}$$

where each $a_j(t)$ satisfies the first order differential equation

$$\dot{a}_j(t) = \varepsilon\, A_j(a_1, a_2, a_3, a_4, t) + \cdots \tag{4}$$

We only consider the first few terms $1, 2, \ldots, m$ in the series expansion of (3) and (4), we evaluate the functions u_j and A_j, j =1, 2, 3, 4 such that $a_j(t)$ appearing in (3) and (4) satisfy the given differential equation (1) with an accuracy of ε^{m+1}. In order to determine these unknown functions, it was assumed by Murty and Deekshatulu [9], that the functions, $u_1, u_2, \ldots$ do not contain the terms involving $e^{-\lambda_j t}$, j = 1, 2, 3, 4, since these are included in the series expansion (3) at order ε^0. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to the lower order, usually the first [11]. In order to obtain some special solutions of (1), Shamsul [17, 19] imposed the restriction that, $u_1, u_2, \ldots$ exclude terms involving $e^{-(i_1\lambda_1 + i_2\lambda_2 + i_3\lambda_3 + i_4\lambda_4)t}$

where $$i_1\lambda_1 + i_2\lambda_2 + i_3\lambda_3 + i_4\lambda_4 \le \frac{1}{4}(i_1 + i_2 + i_3 + i_4)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \qquad (5)$$

For the restriction (5); $u_1, u_2, \ldots$ include all terms of $e^{-(i_1\lambda_1 + i_2\lambda_2 + i_3\lambda_3 + i_4\lambda_4)t}$, where $i_4 = 1, 2, 3, \cdots$ etc. if $\lambda_1, \lambda_2, \lambda_3 < \lambda_4$. Akbar *et al*. [3] have refined this restriction and impose a new restriction that $u_1, u_2, \ldots$ include the terms involving $e^{-(i_1\lambda_1 + i_2\lambda_2 + i_3\lambda_3 + i_4\lambda_4)t}$, when $i_4 > 1$ (6)

In article [3] Akbar *et al*. have shown that, under the new restrictions, the results show good coincidence with numerical results and it is useful even if $\varepsilon = 1.0$. In this article, we have used the restriction presented in (6) and have investigated desired results when one of the eigenvalues tends to zero or vanishes and the others are integral multiple.

Differentiating equation (3) four times with respect to t, substituting the value of x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(4)}$ in equation (1) and equating the coefficients of ε, we obtain the following equation (see also [18] for details)

$$\sum_{j=1}^{4}\left(\prod_{k=1, k\ne j}^{4}\left(\frac{d}{dt} + \lambda_k\right)\left(e^{-\lambda_j t} A_j\right)\right) + \prod_{j=1}^{4}\left(\frac{d}{dt} + \lambda_j\right)u_1 = f^{(0)}(a_1, a_2, a_3, a_4, t) \qquad (7)$$

where $f^{(0)} = f(x_0, \dot{x}_0, \ddot{x}_0, \dddot{x}_0)$ and $x_0 = \sum_{j=1}^{4} a_j(t) e^{-\lambda_j t}$

In general, the functional $f^{(0)}$ can be expanded in Taylor's series (see also [9] for details) of the form:

$$f^{(0)} = \sum_{i_1=0, \ldots, i_4=0}^{\infty, \ldots, \infty} F_{i_1, i_2, i_3, i_4}(a_1, a_2, a_3, a_4)\, e^{-(i_1\lambda_1 + i_2\lambda_2 + i_3\lambda_3 + i_4\lambda_4)t} \qquad (8)$$

Substituting the value of $f^{(0)}$ from (8) into equation (7), we obtain

$$\sum_{j=1}^{4}\left(\prod_{k=1,k\neq j}^{4}\left(\frac{d}{dt}+\lambda_k\right)\left(e^{-\lambda_j t}A_j\right)\right)+\prod_{j=1}^{4}\left(\frac{d}{dt}+\lambda_j\right)u_1$$
$$=\sum_{i_1=0,\dots,i_4=0}^{\infty,\dots,\infty}F_{i_1,i_2,i_3,i_4}(a_1,a_2,a_3,a_4)\;e^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4)t} \tag{9}$$

It is noted that the limits of i_1, i_2, i_3, i_4 are 0 to ∞, but for a particular problem they have some definite values. The eigenvalues are unequal and for the sake of definiteness, we may consider that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$. Since u_1 does not contain the term $e^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4)t}$ where $i_4 \le 1$ (by condition (6)), we shall be able to find the unknown functions u_j and A_j, $j = 1, 2, 3, 4$ subject to the condition that the coefficients of A_j, $j = 1, 2, 3, 4$ do not become large or undefined when the small root say λ_1 is vanishes or $\lambda_1 \to 0^+$ and the other eigenvalues are in integral multiple. This completes the determination of the solution of the equation (1).

3. Example

For illustration of the method, we have considered a Duffing equation type fourth order nonlinear differential system

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon x^3 \tag{10}$$

Here $f = x^3$. Therefore, we obtain

$$\begin{aligned} f^{(0)} =& a_4^{\,3}e^{-3\lambda_4 t} + 3a_3a_4^{\,2}e^{-(\lambda_3+2\lambda_4)t} + 3a_2a_4^{\,2}e^{-(\lambda_2+2\lambda_4)t} + 3a_1a_4^{\,2}e^{-(\lambda_1+2\lambda_4)t} + a_1^{\,3}e^{-3\lambda_1 t}\\ &+3a_1^{\,2}a_2e^{-(2\lambda_1+\lambda_2)t} + 3a_1^{\,2}a_3e^{-(2\lambda_1+\lambda_3)t} + a_1a_2^{\,2}e^{-(\lambda_1+2\lambda_2)t} + a_2^{\,3}e^{-3\lambda_2 t}\\ &+6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3a_2^{\,2}a_3e^{-(2\lambda_2+\lambda_3)t} + 3a_1a_3^{\,2}e^{-(\lambda_1+2\lambda_3)t} + 3a_2a_3^{\,2}e^{-(\lambda_2+2\lambda_3)t}\\ &+3a_2^{\,2}a_4e^{-(2\lambda_2+\lambda_4)t} + 6a_1a_3a_4e^{-(\lambda_1+\lambda_3+\lambda_4)t} + 3a_1^{\,2}a_4e^{-(2\lambda_1+\lambda_4)t}\\ &+6a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t} + 6a_2a_3a_4e^{-(\lambda_2+\lambda_3+\lambda_4)t} + a_3^{\,3}e^{-3\lambda_3 t} + 3a_3^{\,2}a_4e^{-(2\lambda_3+\lambda_4)t}. \end{aligned}$$

In accordance with our assumption (6), we obtain the following equations for determining the functions A_1, A_2, A_3, A_4 and u_1:

$$\begin{aligned} &\sum_{j=1}^{4}\left(\prod_{k=1,k\neq j}^{4}\left(\frac{d}{dt}+\lambda_k\right)\left(e^{-\lambda_j t}A_j\right)\right)\\ &= a_1^{\,3}e^{-3\lambda_1 t} + 3a_1^{\,2}a_2e^{-(2\lambda_1+\lambda_2)t} + 3a_1^{\,2}a_3e^{-(2\lambda_1+\lambda_3)t} + a_1a_2^{\,2}e^{-(\lambda_1+2\lambda_2)t}\\ &+a_2^{\,3}e^{-3\lambda_2 t} + 6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3a_2^{\,2}a_3e^{-(2\lambda_2+\lambda_3)t}\\ &+3a_1a_3^{\,2}e^{-(\lambda_1+2\lambda_3)t} + 3a_2a_3^{\,2}e^{-(\lambda_2+2\lambda_3)t} + 3a_2^{\,2}a_4e^{-(2\lambda_2+\lambda_4)t}\\ &+6a_1a_3a_4e^{-(\lambda_1+\lambda_3+\lambda_4)t} + 3a_1^{\,2}a_4e^{-(2\lambda_1+\lambda_4)t} + 6a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t}\\ &+6a_2a_3a_4e^{-(\lambda_2+\lambda_3+\lambda_4)t} + a_3^{\,3}e^{-3\lambda_3 t} + 3a_3^{\,2}a_4e^{-(2\lambda_3+\lambda_4)t}. \end{aligned} \tag{11}$$

And

$$\prod_{j=1}^{4}\left(\frac{d}{d\,t}+\lambda_j\right)u_1 \tag{12}$$
$$= a_4^{\,3}e^{-3\lambda_4 t} + 3a_3\,a_4^{\,2}e^{-(\lambda_3+2\lambda_4)\,t} + 3a_2\,a_4^{\,2}e^{-(\lambda_2+2\lambda_4)\,t} + 3a_1\,a_4^{\,2}e^{-(\lambda_1+2\lambda_4)\,t}$$

Since λ_1 is very small or vanishes and the other eigenvalues are in integral multiple, together with the condition that the coefficients of $A_j,\ j=1,2,3,4$ do not become large or undefined, so, equation (11) can be separated for the unknown functions $A_j,\ j=1,2,3,4$, in the following way:

$$\left(\frac{d}{d\,t}+\lambda_2\right)\left(\frac{d}{d\,t}+\lambda_3\right)\left(\frac{d}{d\,t}+\lambda_4\right)\left(e^{-\lambda_1 t}A_1\right) = -a_1^{\,3}e^{-3\lambda_1 t} \tag{13}$$

$$\left(\frac{d}{d\,t}+\lambda_1\right)\left(\frac{d}{d\,t}+\lambda_3\right)\left(\frac{d}{d\,t}+\lambda_4\right)\left(e^{-\lambda_2 t}A_2\right) \tag{14}$$
$$= -\left(3a_1^{\,2}\,a_2\,e^{-(2\lambda_1+\lambda_2)\,t} + 3a_1\,a_2^{\,2}e^{-(\lambda_1+2\lambda_2)\,t} + a_2^{\,3}\,e^{-3\lambda_2\,t}\right)$$

$$\left(\frac{d}{d\,t}+\lambda_1\right)\left(\frac{d}{d\,t}+\lambda_2\right)\left(\frac{d}{d\,t}+\lambda_4\right)\left(e^{-\lambda_3 t}A_3\right)$$
$$= -\{3\,a_1^{\,2}a_3e^{-(2\lambda_1+\lambda_3)t} + 6\,a_1\,a_2\,a_3e^{-(\lambda_1+\lambda_2+\lambda_3)\,t} \tag{15}$$
$$+ 3\,a_1\,a_3^{\,2}e^{-(\lambda_1+2\lambda_3)\,t} + 3\ a_2^{\,2}\,a_3e^{-(2\lambda_2+\lambda_3)\,t}$$
$$+ 3\,a_2\,a_3^{\,2}e^{-(\lambda_2+2\lambda_3)\,t} + a_3^{\,3}\,e^{-3\lambda_3\,t}\}$$

And

$$\left(\frac{d}{d\,t}+\lambda_1\right)\left(\frac{d}{d\,t}+\lambda_2\right)\left(\frac{d}{d\,t}+\lambda_3\right)\left(e^{-\lambda_4 t}A_4\right)$$
$$= -\{3a_2^{\,2}a_4e^{-(2\lambda_2+\lambda_4)\,t} + 6\,a_1\,a_3\,a_4\,e^{-(\lambda_1+\lambda_3+\lambda_4)\,t} + 3a_1^{\,2}\,a_4\,e^{-(2\lambda_1+\lambda_4)\,t} \tag{16}$$
$$+6a_1\,a_2\,a_4e^{(\lambda_1+\lambda_2+\lambda_4)\,t} + 6a_2\,a_3\,a_4e^{-(\lambda_2+\lambda_3+\lambda_4)\,t} + 3\,a_3^{\,2}a_4\,e^{-(2\lambda_3+\lambda_4)\,t}\}$$

Solving equations (12)-(16), we obtain

$$u_1 = d_1\,a_1\,a_4^{\,2}e^{-(\lambda_1+2\lambda_4)\,t} + d_2\,a_2\,a_4^{\,2}e^{-(\lambda_2+2\lambda_4)\,t} \tag{17}$$
$$+ d_3\,a_3\,a_4^{\,2}e^{-(\lambda_3+2\lambda_4)\ t} + d_4\,a_4^{\,3}\,e^{-3\lambda_4\,t}$$

And

$$
\begin{aligned}
A_1 &= l_1 a_1^{\,3} e^{-2\lambda_1 t} \\
A_2 &= p_1\, a_1^{\,2} a_2 e^{-2\lambda_1 t} + p_2\, a_1\, a_2^{\,2} e^{-(\lambda_1+\lambda_2)\,t} + p_3\, a_2^{\,3} e^{-2\,\lambda_2\,t}, \\
A_3 &= q_1\, a_1^{\,2} a_3 e^{-2\,\lambda_1 t} + q_2\, a_1\, a_2\, a_3\, e^{-(\lambda_1+\lambda_2)\,t} + q_3\, a_1\, a_3^{\,2} e^{-(\lambda_1+\lambda_3)\,t} \\
&\quad + q_4\, a_2^{\,2}\, a_3\, e^{-2\,\lambda_2\,t} + q_5\, a_2\, a_3^{\,2} e^{-(\lambda_2+\lambda_3)\,t} + q_6\, a_3^{\,3}\, e^{-2\,\lambda_3\,t}, \\
A_4 &= s_1\, a_2^{\,2} a_4 e^{-2\lambda_2\,t} + s_2\, a_1\, a_3\, a_4 e^{-(\lambda_1+\lambda_3)\,t} + s_3\, a_1^{\,2}\, a_4\, e^{-2\lambda_1\,t} \\
&\quad + s_4\, a_1\, a_2\, a_4 e^{-(\lambda_1+\lambda_2)\,t} + s_5\, a_2\, a_3\, a_4\, e^{-(\lambda_2+\lambda_3)\,t} + s_6\, a_3^{\,2}\, a_4\, e^{-2\lambda_3\,t}
\end{aligned}
\qquad (18)
$$

where

$d_1 = -3/[2\lambda_4(\lambda_1+\lambda_4)(\lambda_1-\lambda_2+2\lambda_4)(\lambda_1-\lambda_3+2\lambda_4)]$,
$d_2 = -3/[2\lambda_4(\lambda_2+\lambda_4)(\lambda_2-\lambda_1+2\lambda_4)(\lambda_2-\lambda_3+2\lambda_4)]$,
$d_3 = -3/[2\lambda_4(\lambda_3+\lambda_4)(\lambda_3-\lambda_1+2\lambda_4)(\lambda_3-\lambda_2+2\lambda_4)]$,
$d_4 = 1/[2\lambda_4(\lambda_1-3\lambda_4)(\lambda_2-3\lambda_4)(\lambda_3-3\lambda_4)]$,
$l_1 = 1/[(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)]$,
$p_1 = 3/[(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_4)]$,
$p_2 = 3/[2\lambda_2(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_4)]$,
$p_3 = 3/[(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(3\lambda_2-\lambda_4)]$,
$q_1 = 3/[(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_1+\lambda_3-\lambda_4)]$,
$q_2 = 6/[(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)]$,
$q_3 = 3/[2\lambda_3(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+2\lambda_3-\lambda_4)]$,
$q_4 = 3/[(\lambda_2+\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)(2\lambda_2+\lambda_3-\lambda_4)]$,
$q_5 = 3/[2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_3+\lambda_2-\lambda_4)]$,
$q_6 = 1/[(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(3\lambda_3-\lambda_4)]$,
$s_1 = 3/[(\lambda_2+\lambda_4)(\lambda_4+2\lambda_2-\lambda_1)(\lambda_4+2\lambda_2-\lambda_3)]$,
$s_2 = 6/[(\lambda_3+\lambda_4)(\lambda_1-\lambda_2+\lambda_3+\lambda_4)(\lambda_1+\lambda_4)]$,
$s_3 = 3/[(\lambda_1+\lambda_4)(2\lambda_1-\lambda_2+\lambda_4)(2\lambda_1-\lambda_3+\lambda_4)]$,
$s_4 = 6/[(\lambda_1+\lambda_4)(\lambda_1+\lambda_2-\lambda_3+\lambda_4)(\lambda_2+\lambda_4)]$,
$s_5 = 6/[(\lambda_2+\lambda_4)(\lambda_3+\lambda_4)(\lambda_2-\lambda_1+\lambda_3+\lambda_4)]$,
$s_6 = 3/[(\lambda_3+\lambda_4)(2\lambda_3+\lambda_4-\lambda_1)(2\lambda_3+\lambda_4-\lambda_2)]$.

Substituting the values of A_1, A_2, A_3 and A_4 from equation (18) into the equation (4), we obtain

$$\begin{aligned}
\dot{a}_1 &= \varepsilon\, l_1 a_1^{\,3} e^{-2\lambda_1 t},\\
\dot{a}_2 &= \varepsilon \left(p_1\, a_1^{\,2} a_2 e^{-2\lambda_1 t} + p_2\, a_1\, a_2^{\,2} e^{-(\lambda_1+\lambda_2)\,t} + p_3\, a_2^{\,3} e^{-2\,\lambda_2\,t}\right),\\
\dot{a}_3 &= \varepsilon \Big(q_1\, a_1^{\,2} a_3 e^{-2\,\lambda_1 t} + q_2\, a_1\, a_2\, a_3\, e^{-(\lambda_1+\lambda_2)\,t} + q_3\, a_1\, a_3^{\,2} e^{-(\lambda_1+\lambda_3)\,t}\\
&\quad + q_4\, a_2^{\,2}\, a_3\, e^{-2\,\lambda_2\,t} + q_5\, a_2\, a_3^{\,2} e^{-(\lambda_2+\lambda_3)\,t} + q_6\, a_3^{\,3}\, e^{-2\,\lambda_3\,t}\Big),\\
\dot{a}_4 &= \varepsilon \Big(s_1\, a_2^{\,2}\, a_4\, e^{-2\lambda_2 t} + s_2\, a_1\, a_3\, a_4\, e^{(\lambda_1+\lambda_3)\,t} + s_3\, a_1^{\,2} a_4\, e^{-2\lambda_1\,t}\\
&\quad + s_4\, a_1\, a_2\, a_4 e^{-(\lambda_1+\lambda_2)\,t} + s_5\, a_2\, a_3\, a_4\, e^{-(\lambda_2+\lambda_3)\,t} + s_6\, a_3^{\,2} a_4\, e^{-2\lambda_3\,t}\Big).
\end{aligned} \tag{19}$$

Now, the second, third and fourth equations in (19) have no exact solutions. So, we have solved equation (19) by assuming that a_1, a_2, a_3 and a_4 are constants in the right hand sides of (19), since ε is a small parameter. This assumption was first made by Murty and Deekshatulu [9] and Murty *et al.* [10] to solve the similar type of nonlinear equations. Thus, the solution of (19) is

$$\begin{aligned}
a_1 &= a_{1,0} + \varepsilon\, l_1 a_{1,0}^3\, \frac{\left(1-e^{-2\,\lambda_1 t}\right)}{2\lambda_1},\\
a_2 &= a_{2,0} + \varepsilon\{p_1 a_{1,0}^2\, a_{2,0}\, \frac{\left(1-e^{-2\,\lambda_1 t}\right)}{2\lambda_1}\\
&\quad + p_2\, a_{1,0}\, a_{2,0}^2\, \frac{\left(1-e^{-(\lambda_1+\lambda_2)\,t}\right)}{(\lambda_1+\lambda_2)} + p_3\, a_{2,0}^3\, \frac{\left(1-e^{-2\,\lambda_2\,t}\right)}{2\lambda_2}\},\\
a_3 &= a_{3,0} + \varepsilon\{q_1 a_{1,0}^2\, a_{3,0}\, \frac{\left(1-e^{-2\,\lambda_1 t}\right)}{2\lambda_1} + q_2\, a_{1,0}\, a_{2,0}\, a_{3,0}\, \frac{\left(1-e^{-(\lambda_1+\lambda_2)\,t}\right)}{(\lambda_1+\lambda_2)}\\
&\quad + q_3\, a_{1,0}\, a_{3,0}^2\, \frac{\left(1-e^{-(\lambda_1+\lambda_3)\,t}\right)}{(\lambda_1+\lambda_3)} + q_4\, a_{2,0}^2\, a_{3,0}\, \frac{\left(1-e^{-2\lambda_2\,t}\right)}{2\lambda_2}\\
&\quad + q_5\, a_{2,0}\, a_{3,0}^2\, \frac{\left(1-e^{-(\lambda_2+\lambda_3)\,t}\right)}{(\lambda_2+\lambda_3)} + q_6\, a_{3,0}^3\, \frac{\left(1-e^{-2\,\lambda_3 t}\right)}{2\,\lambda_3}\},\\
a_4 &= a_{4,0} + \varepsilon\{\, s_1\, a_{2,0}^2\, a_{4,0}\, \frac{\left(1-e^{-2\,\lambda_2\,t}\right)}{2\lambda_2} + s_2\, a_{1,0}\, a_{3,0}\, a_{4,0}\, \frac{\left(1-e^{-(\lambda_1+\lambda_3)\,t}\right)}{\lambda_1+\lambda_3}\\
&\quad + s_3\, a_{1,0}^2\, a_{4,0}\, \frac{\left(1-e^{-2\,\lambda_1 t}\right)}{2\lambda_1} + s_4\, a_{1,0}\, a_{2,0}\, a_{4,0}\, \frac{\left(1-e^{-(\lambda_1+\lambda_2)\,t}\right)}{\lambda_1+\lambda_2}\\
&\quad + s_5\, a_{2,0}\, a_{3,0}\, a_{4,0}\, \frac{\left(1-e^{-(\lambda_2+\lambda_3)\,t}\right)}{\lambda_2+\lambda_3} + s_6\, a_{3,0}^2\, a_{4,0}\, \frac{\left(1-e^{-2\,\lambda_3 t}\right)}{2\lambda_3}\},
\end{aligned} \tag{20}$$

when λ_1 is small but not equal to zero.
And

$$a_1 = a_{1,0} + \varepsilon\, l_1 a_{1,0}^3\, t,$$

$$a_2 = a_{2,0} + \varepsilon\{p_1 a_{1,0}^2\, a_{2,0}\; t + p_2\, a_{1,0}\; a_{2,0}^2\; \frac{\left(1 - e^{-(\lambda_1 + \lambda_2)\,t}\right)}{(\lambda_1 + \lambda_2)}$$
$$+ p_3\; a_{2,0}^3\; \frac{\left(1 - e^{-2\lambda_2 t}\right)}{2\lambda_2}\},$$

$$a_3 = a_{3,0} + \varepsilon\{q_1 a_{1,0}^2\; a_{3,0}\; t + q_2\, a_{1,0}\; a_{2,0}\; a_{3,0}\; \frac{\left(1 - e^{-(\lambda_1 + \lambda_2)\,t}\right)}{(\lambda_1 + \lambda_2)} \tag{21}$$
$$+ q_3\, a_{1,0}\; a_{3,0}^2\; \frac{\left(1 - e^{-(\lambda_1 + \lambda_3)\,t}\right)}{(\lambda_1 + \lambda_3)} + q_4\; a_{2,0}^2\; a_{3,0}\; \frac{\left(1 - e^{-2\lambda_2 t}\right)}{2\lambda_2}$$
$$+ q_5\, a_{2,0}\; a_{3,0}^2\; \frac{\left(1 - e^{-(\lambda_2 + \lambda_3)\,t}\right)}{(\lambda_2 + \lambda_3)} + q_6\; a_{3,0}^3\; \frac{\left(1 - e^{-2\lambda_3 t}\right)}{2\lambda_3}\},$$

$$a_4 = a_{4,0} + \varepsilon\{\, s_1\, a_{2,0}^2\; a_{4,0}\; \frac{\left(1 - e^{-2\lambda_2 t}\right)}{2\lambda_2} + s_2\, a_{1,0}\; a_{3,0}\; a_{4,0}\; \frac{\left(1 - e^{-(\lambda_1 + \lambda_3)\,t}\right)}{\lambda_1 + \lambda_3}$$
$$+ s_3\, a_{1,0}^2\; a_{4,0}\; t + s_4\, a_{1,0}\; a_{2,0}\; a_{4,0}\; \frac{\left(1 - e^{-(\lambda_1 + \lambda_2)\,t}\right)}{\lambda_1 + \lambda_2}$$
$$+ s_5\, a_{2,0}\; a_{3,0}\; a_{4,0}\; \frac{\left(1 - e^{-(\lambda_2 + \lambda_3)\,t}\right)}{\lambda_2 + \lambda_3} + s_6\, a_{3,0}^2\; a_{4,0}\; \frac{\left(1 - e^{-2\lambda_3 t}\right)}{2\lambda_3}\},$$

when λ_1 is equal to zero.

Therefore, we have obtained the first approximate solution of the equation (10) as

$$x = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} + \varepsilon\, u_1 \tag{22}$$

where a_1, a_2, a_3, a_4 are calculated by the equation (20) and u_1 is calculated by the equation (17) when λ_1 is small or $\lambda_1 \to 0^+$ and a_1, a_2, a_3, a_4 are calculated by the equation (21) when $\lambda_1 = 0$ and u_1 is calculated by the equation (17) in this case also.

4. Results and Discussion

It is usual to compare the perturbation results obtained by a certain perturbation method to the numerical results to test the accuracy of the approximate solution. Firstly, we have considered the eigenvalues $\lambda_1 = 0.01$, $\lambda_2 = 1.0$, $\lambda_3 = 3.1$, $\lambda_4 = 9.5$ and $x(t,\varepsilon)$ is computed by (22) in which a_1, a_2, a_3, a_4 are computed by (20) and u_1 is computed by (17) when $\varepsilon = 0.1$ together with initial conditions $a_{1,0} = 0.5$, $a_{2,0} = 0.5$, $a_{3,0} = 0.5$, $a_{4,0} = 0.5$ [or $x(0) = 1.999998$, $\dot{x}(0) = -6.785315$, $\ddot{x}(0) = 50.364510$,

$\dddot{x}(0) = -443.935760$] for various values of t and the results are presented in the second column of the Table 1. A second solution (designated by $x^{\#}$) of the equation (10) has also been calculated by fourth order Runge-Kutta method and the results are presented in the third column of the Table 1. Percentage errors have also been calculated and are shown in the fourth column of the Table 1. From Table 1, we see that the Percentage errors are much smaller than 1%.

Table 1: Comparison of the approximate solution x to the numerical solution $x^{\#}$ when the eigenvalues are $\lambda_1 = 0.01$, $\lambda_2 = 1.0$, $\lambda_3 = 3.1$, and $\lambda_4 = 9.5$

t	x	$x^{\#}$	$E\%$
0.0	1.999998	1.999998	0.0000
1.0	0.705366	0.704637	-0.1035
2.0	0.560031	0.558837	-0.2137
3.0	0.509801	0.508443	-0.2671
4.0	0.488283	0.486876	-0.2890
5.0	0.477125	0.475713	-0.2968
6.0	0.469820	0.468419	-0.2991
7.0	0.463977	0.462594	-0.2990
8.0	0.458710	0.457348	-0.2978
9.0	0.453698	0.452354	-0.3100
10.0	0.448808	0.447890	-0.3099

Secondly, we have considered the eigenvalues $\lambda_1 = 0.0$, $\lambda_2 = 0.5$, $\lambda_3 = 1.6$ and $\lambda_4 = 5.0$. Therefore, the product of the eigenvalues is equal to zero. *i. e.* the coefficient of the linear restoring force is equal to zero. In this case, only the nonlinear restoring force exists in the system. For this situation, the perturbation solution $x(t, \varepsilon)$ is computed by (22) in which a_1, a_2, a_3, a_4 are computed by (21) and u_1 is computed by (17) when $\varepsilon = 0.1$ together with initial conditions $a_{1,0} = 0.25$, $a_{2,0} = 0.25$, $a_{3,0} = 0.25$, $a_{4,0} = 0.25$ [or $x(0) = 0.999997$, $\dot{x}(0) = -1.765399$, $\ddot{x}(0) = 6.937829$, $\dddot{x}(0) = -32.288620$] for various values of t and the results are presented in the second column of the Table 2. A second solution (designated by $x^{\#}$) of the equation (10) has also been calculated by fourth order Runge-Kutta method and the results are presented in the third column of the Table 2. Percentage errors have also been calculated and are shown in the fourth column of the Table 2.

The KBM method was developed for the systems in which the linear restoring forces must present and the case where only nonlinear restoring force exists and linear restoring forces vanish was not discussed. But from Table 2, we see that the results obtained by the solution equation (22) (an extension of the KBM method) show good agreement with those obtained by numerical method in the case when the linear restoring force is absent. This is the achievement of the technique presented in this article.

Table 2: Comparison of the approximate solution x to the numerical solution $x^{\#}$ when the eigenvalues are $\lambda_1 = 0.0$, $\lambda_2 = 0.5$, $\lambda_3 = 1.6$, and $\lambda_4 = 5.0$

t	x	$x^{\#}$	$E\%$
0.0	0.999997	0.999997	0.0000
1.0	0.458471	0.457905	-0.1236
2.0	0.356388	0.355072	-0.3706
3.0	0.310468	0.308630	-0.5955
4.0	0.285346	0.283182	-0.7642
5.0	0.270449	0.268087	-0.8811
6.0	0.261313	0.258834	-0.9978
7.0	0.255602	0.253058	-1.0053
8.0	0.251966	0.249989	-0.7908
9.0	0.249593	0.247006	-1.0473
10.0	0.247992	0.245408	-1.0529

In general, the KBM method is useful only when $\varepsilon << 1$. Sometimes the solution obtained by the KBM method is fit to be used even if $\varepsilon = 1.0$ (see also [7] for details). We have again computed $x(t,\varepsilon)$ by (22) when the eigenvalues are $\lambda_1 = 0.01$, $\lambda_2 = 1.0$, $\lambda_3 = 3.1$, $\lambda_4 = 9.5$ and $\varepsilon = 1.0$ together with initial conditions $a_{1,0} = 0.25$, $a_{2,0} = 0.25$, $a_{3,0} = 0.25$, $a_{4,0} = 0.25$ [or $x(0) = 0.999998$, $\dot{x}(0) = -3.399525$, $\ddot{x}(0) = 25.131584$, $\dddot{x}(0) = -221.847198$] for various values of t and the results are presented in the second column of the Table 3. The numerical solution (designated by $x^{\#}$) of the equation (10) has also been calculated by fourth order Runge-Kutta method and the results are presented in the third column of the Table 3. Percentage errors have also been calculated and are shown in the fourth column of the Table 3. From Table 3, we see that, the results are acceptable and the percentage errors are smaller than 1%.

Table 3: Comparison of the approximate solution x to the numerical solution $x^{\#}$ when the eigenvalues are $\lambda_1 = 0.01$, $\lambda_2 = 1.0$, $\lambda_3 = 3.1$, and $\lambda_4 = 9.5$

t	x	$x^{\#}$	$E\%$
0.0	0.999998	0.999998	0.0000
1.0	0.334205	0.333388	-0.2451
2.0	0.240449	0.239173	-0.5335
3.0	0.197285	0.195925	-0.6941
4.0	0.170977	0.169673	-0.7685
5.0	0.151817	0.150615	-0.7981
6.0	0.136222	0.135131	-0.8074
7.0	0.122777	0.121791	-0.8096
8.0	0.110871	0.109982	-0.8083
9.0	0.100206	0.099405	-0.8058
10.0	0.090604	0.089882	-0.8033

5. Conclusion

A technique is developed in this article for obtaining the solution of quasi-linear over-damped systems when the damping forces are such that one of the eigenvalues is nearly zero and the others are integral multiple. The results obtained by the technique presented in this article are not only fit to be used in the case of small nonlinearities but also fit to be used in the case of high nonlinearities. *i. e.* the results are useful even if $\varepsilon = 1.0$.

6. Appendix: Discussion on Akbar *et al.* [3]

Akbar *et al*. [3] also considered the Duffing equation type fourth order nonlinear differential equation, as we have considered in the equation (10). *i. e*. They considered

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon\, x^3 \tag{10}$$

In article [3] Akbar *et al*. imposed the restrictions that, $\lambda_2 \approx J\ \lambda_1,\quad J = 2,3,4,\cdots,\quad 2\lambda_1 + \lambda_2 < \lambda_3$ and $\lambda_1 + \lambda_2 + \lambda_3 < \lambda_4$. Therefore, they respectively obtained the variational equations and the correction terms in the form:

$$\begin{aligned}
\dot{a}_1 &= 0\\
\dot{a}_2 &= \varepsilon\, m_1 a_1^{\,3} e^{(\lambda_2 - 3\lambda_1)\, t},\\
\dot{a}_3 &= \varepsilon\left\{ n_1 a_1^{\,2} a_3 e^{-2\lambda_1\, t} + n_2 a_1 a_2^{\,2} e^{(\lambda_3 - \lambda_1 - 2\lambda_2)\, t} + n_3 a_2^{\,3} e^{(\lambda_3 - 3\lambda_2)\, t} \right\}\\
\dot{a}_4 &= \varepsilon\begin{Bmatrix} r_1 a_1^{\,2} a_2 e^{(\lambda_4 - 2\lambda_1 - \lambda_2)\, t} + r_2 a_1 a_2 a_3 e^{(\lambda_4 - \lambda_1 - \lambda_2 - \lambda_3)\, t} + r_3 a_2^{\,2} a_3 e^{(\lambda_4 - 2\lambda_2 - \lambda_3)\, t} \\ + r_4 a_1 a_3^{\,2} e^{(\lambda_4 - \lambda_1 - 2\lambda_3)\, t} + r_5 a_2 a_3^{\,2} e^{(\lambda_4 - \lambda_{12} - 2\lambda_3)\, t} + r_6 a_2^{\,2} a_4 e^{-2\lambda_2\, t} \\ + r_7 a_1 a_3 a_4 e^{(\lambda_4 + \lambda_3)\, t} + r_8 a_1^{\,2} a_4 e^{-2\lambda_1\, t} + r_9 a_1 a_2 a_4 e^{-(\lambda_1 + \lambda_2)\, t} \\ + r_{10} a_2 a_3 a_4 e^{-(\lambda_2 + \lambda_3)\, t} + r_{11} a_3^{\,3} e^{(\lambda_4 - 3\lambda_3)\, t} + r_{12} a_3^{\,2} a_4 e^{-2\lambda_3\, t} \end{Bmatrix}
\end{aligned} \tag{23}$$

And

$$\begin{aligned}
u_1 &= d_1\, a_1\, a_4^{\,2}\, e^{-(\lambda_1 + 2\lambda_4)\, t} + d_2\, a_2\, a_4^{\,2} e^{-(\lambda_2 + 2\lambda_4)\, t}\\
&\quad + d_3\, a_3\, a_4^{\,2}\, e^{-(\lambda_3 + 2\lambda_4)\, t} + d_4\, a_4^{\,3} e^{-3\lambda_4\, t}
\end{aligned} \tag{24}$$

where

$m_1 = 1/\left[2\lambda_1(\lambda_3 - 3\lambda_1)(\lambda_4 - 3\lambda_1)\right],$

$n_1 = 3/\left[(\lambda_1 + \lambda_3)(\lambda_3 - \lambda_2 + 2\lambda_1)(\lambda_3 - \lambda_4 + 2\lambda_1)\right],$

$$n_2 = 3/[2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_4)], \quad n_3 = 3/[2\lambda_2(\lambda_1-3\lambda_2)(\lambda_4-3\lambda_2)],$$
$$r_1 = 3/[2\lambda_1(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)], \quad r_2 = 6/[(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)],$$
$$r_3 = 3/[2\lambda_2(\lambda_2+\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)], \quad r_4 = 3/[2\lambda_3(\lambda_1+\lambda_3)(\lambda_1-\lambda_2+2\lambda_3)],$$
$$r_5 = 3/[2\lambda_3(\lambda_2+\lambda_3)(\lambda_2-\lambda_1+2\lambda_3)],$$
$$r_6 = 3/[(\lambda_2+\lambda_4)(\lambda_4+2\lambda_2-\lambda_1)(\lambda_4+2\lambda_2-\lambda_3)],$$
$$r_7 = 6/[(\lambda_3+\lambda_4)(\lambda_1-\lambda_2+\lambda_3+\lambda_4)(\lambda_1+\lambda_4)],$$
$$r_8 = 3/[(\lambda_1+\lambda_4)(2\lambda_1-\lambda_2+\lambda_4)(2\lambda_1-\lambda_3+\lambda_4)],$$
$$r_9 = 6/[(\lambda_1+\lambda_4)(\lambda_2+\lambda_4)(\lambda_1+\lambda_2-\lambda_3+\lambda_4)],$$
$$r_{10} = 6/[(\lambda_2+\lambda_4)(\lambda_3+\lambda_4)(\lambda_2-\lambda_1+\lambda_3+\lambda_4)],$$
$$r_{11} = 1/[2\lambda_3(\lambda_1-3\lambda_3)(\lambda_2-3\lambda_3)],$$
$$r_{12} = 3/[(\lambda_3+\lambda_4)(2\lambda_3+\lambda_4-\lambda_1)(\lambda_4+2\lambda_3-\lambda_2)],$$
$$d_1 = -3/[2\lambda_4(\lambda_1+\lambda_4)(\lambda_1-\lambda_2+2\lambda_4)(\lambda_1-\lambda_3+2\lambda_4)],$$
$$d_2 = -3/[2\lambda_4(\lambda_2+\lambda_4)(\lambda_2-\lambda_1+2\lambda_4)(\lambda_2-\lambda_3+2\lambda_4)],$$
$$d_3 = -3/[2\lambda_4(\lambda_3+\lambda_4)(\lambda_3-\lambda_1+2\lambda_4)(\lambda_3-\lambda_2+2\lambda_4)],$$
$$d_4 = 1/[2\lambda_4(\lambda_1-3\lambda_4)(\lambda_2-3\lambda_4)(\lambda_3-3\lambda_4)].$$

Equation (23) is solved by assuming that a_1, a_2, a_3 and a_4 are constants in the right side of the equation (23), since ε is a small quantity. This assumption was first made by Murty and Deekshatulu [9], Murty *et al*. [10] to solve similar type nonlinear equation (23). Thus the solution of (23) is

$$a_1 = a_{1,0}$$
$$a_2 = a_{2,0} + \varepsilon m_1 a_{1,0}^3 t, \qquad 3\lambda_1 = \lambda_2$$
$$a_{2,0} + \varepsilon m_1 a_{1,0}^3 \left(1 - e^{(\lambda_2 - 3\lambda_1)t}\right)/(3\lambda_1 - \lambda_2), \qquad 3\lambda_1 \neq \lambda_2$$
$$a_3 = a_{3,0} + \varepsilon \{ n_1 a_{1,0}^2 a_{3,0}\left(1 - e^{-2\lambda_1 t}\right)/(2\lambda_1)\}$$
$$+ n_2 a_{1,0} a_{2,0}^2 \left(e^{(\lambda_3-\lambda_1-2\lambda_2)t} - 1\right)/(\lambda_3 - \lambda_1 - 2\lambda_2)$$
$$+ n_3 a_{2,0}^3 t\}, \quad 3\lambda_2 = \lambda_3,$$
$$= a_{3,0} + \varepsilon \{ n_1 a_{1,0}^2 a_{3,0}\left(1 - e^{-2\lambda_1 t}\right)/(2\lambda_1)$$
$$+ n_2 a_{1,0} a_{2,0}^2 \left(e^{(\lambda_3-\lambda_1-2\lambda_2)t} - 1\right)/(\lambda_3 - \lambda_1 - 2\lambda_2)$$
$$+ n_3 a_{2,0}^3 \left(e^{(\lambda_3-3\lambda_2)t} - 1\right)/(\lambda_3 - 3\lambda_2)\}, \qquad 3\lambda_2 \neq \lambda_3,$$

$$
\begin{aligned}
a_4 = a_{4,0} + \varepsilon \Bigg[& r_1 a_{1,0}^2 a_{2,0} \frac{\left(e^{(\lambda_4 - 2\lambda_1 - \lambda_2)t} - 1\right)}{(\lambda_4 - 2\lambda_1 - \lambda_2)} + r_2 a_{1,0} a_{2,0} a_{3,0} \frac{\left(1 - e^{(\lambda_4 - \lambda_1 - \lambda_2 - \lambda_3)t}\right)}{(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4)} \\
& + r_3 a_{2,0}^2 a_{3,0} \frac{\left(1 - e^{(\lambda_4 - 2\lambda_2 - \lambda_3)t}\right)}{(\lambda_3 - 2\lambda_2 + \lambda_4)} + r_4 a_{1,0} a_{3,0}^2 \frac{\left(e^{(\lambda_4 - \lambda_1 - 2\lambda_3)t} - 1\right)}{(\lambda_1 + 2\lambda_3 - \lambda_4)} \\
& + r_5 a_{3,0}^2 a_{2,0} \frac{\left(1 - e^{(\lambda_4 - \lambda_2 - 2\lambda_3)t}\right)}{(\lambda_2 + 2\lambda_3 - \lambda_4)} + r_6 a_{2,0}^2 a_{4,0} \frac{\left(1 - e^{-2\lambda_2 t}\right)}{2\lambda_2} \\
& + r_7 a_{1,0} a_{3,0} a_{4,0} \frac{\left(1 - e^{-(\lambda_1 + \lambda_3)t}\right)}{(\lambda_1 + \lambda_3)} + r_8 a_{1,0}^2 a_{4,0} \frac{\left(1 - e^{-2\lambda_1 t}\right)}{2\lambda_1} \\
& + r_9 a_{1,0} a_{2,0} a_{4,0} \frac{\left(1 - e^{-(\lambda_1 + \lambda_2)t}\right)}{\lambda_1 + \lambda_2} + r_{10} a_{2,0} a_{3,0} a_{4,0} \frac{\left(1 - e^{-(\lambda_2 + \lambda_3)t}\right)}{\lambda_2 + \lambda_3} \\
& + r_{11} a_{3,0}^3 \frac{\left(e^{(\lambda_4 3\lambda_3)t} - 1\right)}{\lambda_4 - 3\lambda_3} + r_{12} a_{3,0}^2 a_{4,0} \frac{\left(1 - e^{-2\lambda_3 t}\right)}{2\lambda_3} \Bigg],
\end{aligned}
\tag{25}
$$

The third term of a_4 will be $r_3 a_{2,0}^2 a_{3,0} t$ when $\lambda_4 = 2\lambda_2 + \lambda_3$. Therefore, they obtained the first approximate solution of the equation (10) as

$$x = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} + \varepsilon u_1 \tag{26}$$

where a_1, a_2, a_3 and a_4 are given by (25) and u_1 is given by (24).

When $\lambda_1 = 0$, we see that, the terms m_1 and r_1 become undefined and as a result the solution (26) breakdown, whereas our solution (22) remain valid in this case also.

REFERENCES

1. M. A. Akbar, A. C. Paul and M. A. Sattar, *An Asymptotic Method of Krylov-Bogoliubov for Fourth Order Over-damped Nonlinear Systems*, Ganit, J. Bangladesh Math. Soc., 22(2002), 83-96.
2. M. A. Akbar, M. A. Shamsul and M. A. Sattar, *Asymptotic Method for Fourth Order Damped Nonlinear Systems*, Ganit, J. Bangladesh Math. Soc., 23(2003), 41-49
3. M. A. Akbar, M. A. Shamsul and M. A. *Sattar, A Simple Technique for Obtaining Certain Over-damped Solutions of an n-th Order Nonlinear Differential Equation*, Soochow Journal of Mathematics, 31(2005), 291-299
4. N. N. Bogoliubov, and Yu. Mitropolskii, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Gordan and Breach, New York, 1961.
5. G. N. Bojadziev, *Damped Nonlinear Oscillations Modeled by a 3-dimensional Differential System*, Acta Mechanica, 48(1983), 193-201
6. N. N. Krylov, and N. N. Bogoliubov, *Introduction to Nonlinear Mechanics*, Princeton University Press, New Jersey, 1947.

7. K. S. Mendelson, *Perturbation Theory for Damped Nonlinear Oscillations*, J. Math. Physics, 2(1970), 3413-3415
8. R. J. Mulholland, *Nonlinear Oscillations of Third Order Differential Equation*, Int. J. Nonlinear Mechanics, 6(1971), 279-294
9. I. S. N. Murty, and B. L. Deekshatulu, *Method of Variation of Parameters for Over-Damped Nonlinear Systems*, J. Control, 9(1969), 259-266
10. I. S. N. Murty, B. L. Deekshatulu and G. Krishna, *On an Asymptotic Method of Krylov-Bogoliubov for Over-damped Nonlinear Systems*, J. Frank. Inst., 288(1969), 49-65
11. I. S. N. Murty**,** *A Unified Krylov-Bogoliubov Method for Solving Second Order Nonlinear Systems*, Int. J. Nonlinear Mech., 6(1971), 45-53.
12. Z. Osiniskii, *Longitudinal, Torsional and Bending Vibrations of a Uniform Bar with Nonlinear Internal Friction and Relaxation*, Nonlinear Vibration Problems, 4(1962), 159-166
13. I. P. Popov, *A Generalization of the Bogoliubov Asymptotic Method in the Theory of Nonlinear Oscillations (in Russian)*, Dokl. Akad. USSR, 3(1956), 308-310
14. M. A. Sattar, *An asymptotic Method for Second Order Critically Damped Nonlinear Equations*, J. Frank. Inst., 321(1986), 109-113
15. M. A. Sattar, *An Asymptotic Method for Three-dimensional Over-damped Nonlinear Systems*, Ganit, J. Bangladesh Math. Soc., 13(1993), 1-8
16. M. A. Shamsul and M. A. Sattar, *An Asymptotic Method for Third Order Critically Damped Nonlinear Systems*, J. Mathematical and Physical Sciences, 30(1996), 291-298
17. M. A. Shamsul, *Asymptotic Methods for Second Order Over-damped and Critically Damped Nonlinear Systems*, Soochow Journal of Math., 27(2001), 187-200
18. M. A. Shamsul, *A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving n-th Order Nonlinear Systems*, J. Frank. Inst., 339(2002), 239-248
19. M. A. Shamsul, *Method of Solution to the n-th Order Over-damped Nonlinear Systems Under Some Special Conditions*, Bull. Cal. Math. Soc., 94(2002), 437-440
20. M. A. Shamsul, *Bogoliubov's Method for Third Order Critically Damped Nonlinear Systems*, Soochow J. Math., 28(2002), 65-80
21. M. A. Shamsul, *Asymptotic Method for Non-oscillatory Nonlinear Systems*, Far East J. Appl. Math., 7(2002), 119-128