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Existence of Solutions for Fourth-Order Discrete Neumann Boundary Value Problem

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ABSTRACT

In this paper, we investigate the solutions of fourth-order discrete boundary value problem. By using critical point theory the existence of positive solutions and infinitely many solutions are obtained.

Keywords: *Fourth-order discrete boundary value problem; Variational methods; Mountain Pass Theorem.*

1. Introduction

In this paper, we study the existence of solutions for the following Fourth-order discrete boundary value problem (BVP).

$$\begin{cases}\Delta^4 y(k-2)+r(k)y(k)=f(k,y(k)),\ k\in[2,T],\\ \Delta y(0)=\Delta y(T+1)=0,\ \ \Delta^3 y(0)=\Delta^3 y(T-1)=0\end{cases}\tag{1.1}$$

where T is a positive integer, $[2,T]$ is the discrete interval $\{2,\dots,T\}$ and $\Delta^{i+1}y(k)=\Delta^i y(k+1)-\Delta^i y(k),(i=0,1,2,3)$ is the forward difference operator, $\Delta^0 y(k)=y(k), r:[2,T]\to(0,\infty), f:[2,T]\times R\to R$ is continuous,

$$F(k,x)=\int_0^x f(k,s)ds.$$

In recent years, a great deal of work has been done in the study of properties of solutions for discrete boundary value problems, by which a number of physical, biological phenomena are described. For the background and results, we refer the reader to the monograph by Agarwal et al. and some recent contributions as [1-11, 14-15].

There are two most common techniques to study the existence of solutions: (i) fixed point theorem in cones has been used see [2, 3, 5, 6, 7, 10, 11, 14, 15]; (ii) critical point theory has been used. About the basic knowledge for critical point theory, please refer to [13] [14]. For recent papers, we refer to [8], [10], [15].

In [10] 15], the existence of periodic solutions for second-order difference equations were obtained under various conditions. G. Anello [8] established a multiplicity theorem for Neumann boundary value problem for second order differential equation,

$$\begin{cases} -\Delta u + \lambda(x)u = \mu f(x,u) + h(x,u) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial v} = 0 \text{ in } \partial\Omega, \end{cases}$$

where v is the outward unit normal to $\partial\Omega, \mu \in R$ and $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian operator.

However, to the best of our knowledge, there are rare results about positive solutions and multiple solutions for higher-order discrete boundary value problem with critical point theory. In particular, problem (1.1) has not been found in the literature.

In order to apply critical point theory to problem (1.1), the following difficulties have to be overcome: constructing suitable space with suitable norm; transferring the solutions of (1.1) into the critical points of some functional, which is called the variational framework. Then we prove the existence of positive solutions and infinitely many solutions for BVP (1.1) by applying critical point theory.

The following two lemmas will be used in the paper.

Lemma 1.1. [9] Let E be a Banach space and $\varphi \in C^1(E,R)$ satisfy Palais-Smale condition. Assume there exist $x_0, x_1 \in E$, and a bounded open neighbourhood Ω of x_0 such that $x_1 \in E \setminus \bar{\Omega}$ and

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x \in \partial\Omega} \varphi(x)$$

Let

$$\Gamma = \{h \mid h : [0,1] \to E \text{ is continuous and } h(0) = x_0, h(1) = x_1\}$$

and

$$c = \inf_{h \in \Gamma} \max_{s \in [0,1]} \varphi(h(s)).$$

Then c is a critical value of φ, that is, there exists $x^* \in E$ such that $\varphi'(x^*) = \theta$ and $\varphi(x^*) = c$, where $c > \max\{\varphi(x_0), \varphi(x_1)\}$.

Lemma 1.2. [14] Let E be an infinite dimensional real Banach space and let $\varphi \in C^1(E,R)$ be even, satisfying Palais-Simale condition and $\varphi(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and φ satisfies

(J1) there exist constants $\rho, a > 0$ such that $\varphi \setminus \partial B_\rho \cap X \ge \alpha$ and

(J2) for each finite dimensional subspace $V_1 \subset E$, the set $\{x \in V_1 : \varphi(x) \ge 0\}$ is bounded. Then φ has an unbounded sequence of critical values.

In this paper, we assume that the following conditions hold:

(C1) $f(k,x) = 0(|x|)$ as $x \to 0$ uniformly in $k \in [2,T]$,

(C2) there exists some constants $l > 0, \mu > 2$, such that $0 < \mu F(k,x) \le x f(k,x)$ for any $|x| \ge l, k \in [2,T]$.

Remark 1.1. Assumptions (C1) (C2) imply

(1) $F(k,x) = o(|x|)$ as $x \to 0$ uniformly in $k \in [1,T]$;

(2) there exist $a_1 > 0$ and $a_2 > 0$ such that $F(k,x) \ge a_1|x|^\mu - a_2$ for each $x \in R$.

2 Related Lemmas

In this section, we are going to establish the corresponding variational framework for (1.1).

Here, and in the sequel, we define the space

$$Y=\left\{y:(0,T+2)\to R\left|\Delta y(0)=0=\Delta y(T+1),\Delta^3 y(0)=0=\Delta^3 y(T-1)\right.\right\}$$

with the norm

$$\|y\|\left(\sum_{k=2}^{T}r(k)|y(k)|^2+\sum_{k=2}^{T+2}\left|\Delta^2 y(k-2)\right|^2\right)^{\frac{1}{2}}$$

Clearly Y is a finite dimensional space.

Lemma 2.1. $(Y,\|\cdot\|)$ is a Banach space.

Proof. Obviously, the space $X=\{y:[0,T+2]\to R\}$ with the norm

$$\|x\|X=\left(\sum_{k=2}^{T}r(k)|x(k)|^2+|x(0)|^2+|x(1)|^2+|x(T+1)|^2+|x(T+2)|^2+\sum_{k=2}^{T+2}\left|\Delta^2 x(k-2)\right|^2\right)^{\frac{1}{2}}$$

is a Banach space. We claim that Y is a closed subspace of X. In fact let $\{y_n\}\subseteq Y$, $y_n\to y^*$ as $n\to\infty$. Then

$$\lim_{n\to\infty}\sum_{k=2}^{T}\left\{r(k)|y_n(k)-y^*(k)|^2+|y_n(0)-y^*(0)|^2+|y_n(1)-y^*(1)|^2+\right.$$

$$\left.|y_n(T+1)-y^*(T+1)|^2+|y_n(T+2)-y^*(T+2)|^2+\sum_{k=2}^{T+2}\left|\Delta^2 y_n(k-2)-\Delta^2 y^*(k-2)\right|^2\right\}=0,$$

which yields

$$\begin{cases}\Delta^2 y_n(i)\to\Delta^2 y^*(i), & i=0,\dots,T,\\ y_n(i)\to y^*(i), & i=0,\dots,T+2,\end{cases}$$

that is

$$\begin{cases}\Delta^2 y_n(i)\to\Delta^3 y^*(i), i=0,\dots,T-1,\\ \Delta y_n(i)\to\Delta y^*(i), i=0,\dots,T+1,\end{cases}$$

as $n\to\infty$. So $\Delta y^*(0)=\Delta y^*(T+1)=0,\Delta^3 y^*(0)=\Delta^3 y^*(T-1)=0,$ which means $y^*\in Y.$ Thus Y is a closed subspace in X. Following we will show that the norm $\|\cdot\|$ is equivalent to $\|\cdot\|x$. Since $y\in Y,\Delta y(0)=\Delta y(T+1)=0,\Delta^3 y(0)=\Delta^3 y(T-1)=0$. Thus the norm $\|\cdot\|$ is equivalent to $\|\cdot\|x$. Therefore, $(Y,\|\cdot\|)$ is a Banach space.

Lemma 2.2. Let $y^{\pm}=\max\{\pm y,0\}$, then the following results hold

(i) $y=y^+-y^-$;

(ii) $y^+(k)y^-(k)=0,k\in[0,T+2],\Delta^i y^+(k)\Delta^i y^-(k)=0;$

(iii) $|y^+(k)|=|y(k)|,k\in[0,T+2].$

Now we consider the modified BVP

$$\begin{cases}\Delta^4 y(k-2)+r(k)y(k)=f\left(k,y^+(k)\right),\ k\in[2,T],\\ \Delta y(0)=\Delta y(T+1)=0,\quad \Delta^3 y(0)=\Delta^3 y(T-1)=0\end{cases}\tag{2.1}$$

Lemma 2.3. Assume that $f:[2,T]\times[0,+\infty)\to[0,+\infty)$ and y is a solution of (2.1). Then $y(i)\geq 0, i\in[0,T+2], y(i)\not\equiv 0$ is a solution of BVP (1.1).

Proof. If y is a solution of (2.1), by partial sum formula we have

$$\begin{aligned}
0 &= \sum_{k=2}^{T}\left[\Delta^4 y(k-2)+r(k)y(k)-f\left(k,y^+(k)\right)\right]y^-(k)\\
&= y^-(T)\Delta^3 y(T-1)-y^-(2)\Delta^3 y(0)-\sum_{k=2}^{T-1}\Delta^3 y(k-1)\Delta y^-(k)\\
&\quad +\sum_{k=2}^{T}\left[r(k)y(k)-f\left(k,y^+(k)\right)y^-(k)\right.\\
&= -\sum_{k=2}^{T-1}\Delta^3 y(k-1)\Delta y^-(k)+\sum_{k=2}^{T}\left[r(k)y(k)-f\left(k,y^+(k)\right)\right]y^-(k)\\
&= -\left[\Delta y^-(T-1)\Delta^2 y(T-1)-\Delta y^-(2)\Delta^2 y(1)-\sum_{k=2}^{T-2}\Delta^2 y(k)\Delta^2 y^-(k)\right]\\
&\quad +\sum_{k=2}^{T}\left[r(k)y(k)-f\left(k,y^+(k)\right)y^-(k)\right..
\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
0 &= -\left\{\sum_{k=2}^{T-2}\left(\Delta^2 y-(k)\right)^2+\sum_{k=2}^{T}\left[r(k)\left(y^-(k)\right)^2+f\left(k,y^+(k)\right)y^-(k)\right]\right.\\
&\quad \left.+\Delta y^-(T-1)\Delta^2 y(T-1)-\Delta y^-(2)\Delta^2 y(1)\right\}
\end{aligned}\tag{2.2}$$

Since y satisfies the boundary condition of (1.1), we have

$$\Delta y(2)=2\Delta y(1), \Delta y(1)=\Delta^2 y(0), \Delta^2 y(0)=\Delta^2 y(1),$$

$$\Delta y(2)=2\Delta y(1), \Delta y(1)=\Delta^2 y(0), \Delta^2 y(0)=\Delta^2 y(1),$$

Thus

$$\begin{aligned}
-\Delta y^-(2)\Delta^2 y(1) &= -2\Delta y^-(1)\Delta^2 y(1)=-2\Delta^2 y^-(1)\Delta^2 y(1)\\
&= -\Delta^2 y^-(0)\Delta^2 y(0)-\Delta^2 y^-(1)\Delta^2 y(1)\\
&= \left[\Delta^2 y^-(0)\right]^2+\left[\Delta^2 y^-(1)\right]^2
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}
\Delta y^-(T-1)\Delta^2 y(T-1) &= 2\Delta y^-(T)\Delta^2 y(T)=-2\Delta^2 y^-(T)\Delta^2 y(T)\\
&= -\Delta^2 y^-(T)\Delta^2 y(T)-\Delta^2 y^-(T-1)\Delta^2 y(T-1)\\
&= \left[\Delta^2 y(T)\right]^2+\left[\Delta^2 y^-(T-1)\right]^2
\end{aligned}\tag{2.4}$$

Therefore, (2.2) means that

$$0=-\left\{\sum_{k=0}^{T}\left(\Delta^2 y^-(k)\right)^2+\sum_{k=2}^{T}\left[r(k)\left(y^-(k)\right)^2+f\left(k,y^+(k)\right)y^-(k)\right]\right\}.\tag{2.5}$$

Thus $y^-(k)=0$ for all $k\in[0,T+2]$, that is, for all $y(k)\geq 0$. If $y(k)=0$ for all $k\in[0,T+2]$, y is not a solution of BVP (1.1) since $f(k,0)\not\equiv 0$.

For each $y\in Y$, put

$$\varphi(y):=\frac{1}{2}\sum_{k=2}^{T+2}\left|\Delta^2 y(k-2)\right|^2+\frac{1}{2}\sum_{k=2}^{T}r(k)\left|y(k)\right|^2-\sum_{k=2}^{T}F(k,y(k)) \tag{2.6}$$

and

$$\varphi(y):=\frac{1}{2}\sum_{k=2}^{T+2}\left|\Delta^2 y(k-2)\right|^2+\frac{1}{2}\sum_{k=2}^{T}r(k)\left|y(k)\right|^2-\sum_{k=2}^{T}\left[F(k,y^+(k))-f(k,0)y^-(k)\right] \tag{2.7}$$

Clearly, the functional $\varphi,\varphi+$ are C^1 with

$$(\varphi'(y),v)=\sum_{k=2}^{T+2}\Delta^2 y(k-2)\Delta^2 v(k-2)+\sum_{k=2}^{T}r(k)y(k)v(k)-\sum_{k=2}^{T}f(k,y(k))v(k), \tag{2.8}$$

and

$$(\varphi'_+(y),v)=\sum_{k=2}^{T+2}\Delta^2 y(k-2)\Delta^2 v(k-2)+\sum_{k=2}^{T}r(k)y(k)v(k)-\sum_{k=2}^{T}f(k,y^+(k))v(k) \tag{2.9}$$

for every $v\in Y$.

Lemma 2.4. The function $y\in Y$ is a critical point of the functional φ if and only if y is a solution of BVP (1.1).

Proof. If $y\in Y$ is a critical point of the functional φ, then by (2.8),

$$0=\sum_{k=2}^{T+2}\Delta^2 y(k-2)\Delta^2 v(k-2)+\sum_{k=2}^{T}r(k)y(k)v(k)-\sum_{k=2}^{T}f(k,f(k))v(k) \tag{2.10}$$

holds for all $v\in Y$. Applying partial sum formula twice

$$\sum_{k=1}^{T+1}y(k)\Delta z(k)=y(T+1)z(T+2)-y(1)z(1)-\sum_{k=1}^{T}z(k+1)\Delta y(k)$$

we have

$$\begin{aligned}
&\sum_{k=2}^{T+2}\Delta^2 y(k-2)\Delta^2 v(k-2)\\
&=\Delta^2 y(T)\Delta v(T+1)-\Delta^2 y(0)\Delta v(0)-\sum_{k=2}^{T+1}\Delta^3 y(k-2)\Delta v(k-1)\\
&=-\sum_{k=2}^{T+1}\Delta^3 y(k-2)\Delta v(k-1)\\
&=-\left[\Delta^3 y(T-1)v(T+1)-\Delta^3 y(0)v(1)-\sum_{k=2}^{T}\Delta^4 y(k-2)v(k)\right]\\
&=\sum_{k=2}^{T}\Delta^4 y(k-2)\Delta(k).
\end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10).

$$\sum_{k=2}^{T}\left[\Delta^4 y(k-2)+r(k)y(k)-f(k,y(k))\right]v(k)=0 \tag{2.12}$$

holds for all $v\in Y$. Therefore,

$$\Delta^4 y(k-2)+r(k)y(k)=f(k,y(k)),\quad k\in[2,T]. \tag{2.13}$$

Besides, $y \in Y$ means that the boundary condition in (1.1) holds. So y is a solution of BVP (1.1).

On the other hand, if y is a solution of BVP (1.1), multiplying by $v \in Y$ on the both sides of the equation in BVP (1.1), and summing on $[2,T]$, then y satisfies $(\varphi'(y), v) = 0$.

Lemma 2.5. Assume that $f:[2,T]\times[0,+\infty]\to[0,+\infty]$, the function $y \in Y$ is a critical point of the functional $\varphi+$ if and only if y is a positive solution of BVP (1.1), that is $y(k) \ge 0$ for $k \in [0, T+2]$.

Proof. By Lemma 2.3 and Lemma 2.4, it is easy to see that the result follows, we omit it here.

Lemma 2.6. Suppose that (C2) holds. Then the functional $\varphi+$ satisfies Palais-Smale condition, i.e., every sequence $\{y_n\}$ in Y satisfies $\varphi+(y_n)$ is bounded and $\varphi'+(y_n)\to 0$ has a convergent subsequence.

Proof. Since Y is a finite dimensional Banach space, we only need to show that (y_n) is a bounded sequence in Y.

By (2.9) we have

$$\begin{aligned}(\varphi'+(y_n), y_n^-) &= \sum_{k=2}^{T+2}\Delta^2 y_n(k-2)\Delta^2 y_n^-(k-2)+\sum_{k=2}^{T} r(k)y_n(k)y_n^-(k)-\\ &\qquad \sum_{k=2}^{T} f\left(k, y_n^+(k)\right)y_n^-(k)\\ &= \sum_{k=2}^{T}\left[\Delta^4 y_n(k-2)+r(k)y_n(k)-f\left(k,y_n^+(k)\right)\right]y_n^-(k).\end{aligned}$$

By Lemma 2.3 one has

$$\begin{aligned}&\left\langle \varphi'_+(y_n), y_n^-\right\rangle\\ &= -\left\{\sum_{k=0}^{T}\left(\Delta^2 y^-(k)\right)^2+\sum_{k=0}^{T}\left[r(k)\left(y^-(k)\right)^2+f\left(k,y^+(k)\right)y^-(k)\right]\right\} \qquad (2.14)\\ &\le -\left\|y_n^-\right\|^2.\end{aligned}$$

Let $w_n^- = \dfrac{y_n^-}{\left\|y_n^-\right\|}$. Dividing $\left\|y_n^-\right\|$ on both sides of (2.14) and let $n\to\infty$, one has

$$\left\|y_n^-\right\| \le -\left\langle \varphi'_+(y_n), w_n^-\right\rangle \to 0 \text{ as } n\to\infty.$$

So $y_n^- \to 0$ in Y.

Now we will show that (y_n^+) is bounded in Y. By (2.7) (2.9) (C2) we have

$$\left(\frac{\mu}{2}-1\right)\sum_{k=2}^{T+2}\left|\Delta^2 y_n^+(k-2)^2\right|+\left(\frac{\mu}{2}-1\right)\sum_{k=2}^{T} r(k)\left|y_n^+(k)\right|^2$$

$$
\begin{aligned}
&= \mu\varphi_+(y_n) - \left(\varphi'_+(y_n), y_n^+\right) + \\
&\qquad \sum_{K=2}^{T}\left[\mu F\left(k, y_n^+(k)\right) - \mu f(k,0)\, y_n^-(k) - f\left(k, y_n^+(k)\right) y_n^-(k)\right] \qquad (2.15)\\
&\le \mu\varphi_+(y_n) - \left\langle \varphi'_+(y_n), y_n^+\right\rangle + \max_{|y|\le 1}\sum_{k=2}^{T}\left|\mu F(k,y) - f(k,y)\,y\right|
\end{aligned}
$$

We assume $\|y_n^+\| \to \infty$ as $n \to \infty$. For convenience, we denote $w_n^+ = \dfrac{y_n^+}{\|y_n^+\|}$, then $\|w_n^+\| = 1$. Dividing $\|y_n^+\|^2$ on the both sides of (2.15), then (2.15) means that

$$
\frac{\mu-2}{2} \le \frac{\mu\varphi_+(y_n)}{\|y_n^+\|^2} - \frac{\left\langle \varphi'_+(y_n), w_n^+\right\rangle}{\|y_n^+\|} + \frac{\max\limits_{|y|\le 1}\sum\limits_{k=2}^{T}\left|\mu F(k,y) - f(k,y)\,y\right|}{\|y_n^+\|^2}.
$$

Since $\varphi(y_n^+)$ is bounded and $\varphi'_+(y_n) \to 0$ as $n \to \infty$, let $n \to \infty$, we get $\dfrac{\mu-2}{2} \le 0$, a contradiction. The result follows.

3. Main Results

Theorem 3.1. Suppose that $f:[2,T]\times[0,+\infty]\to[0,+\infty]$, and (C1) (C2) hold. Then problem (1.1) has at least one nontrivial positive solution.

Proof. We use Lemma 1.1 to prove the existence of a nontrivial critical point of φ_+. We already known that $\varphi_+(0) = 0$ and φ_+ satisfies Palais-Smale condition. Hence, it suffices to prove that φ_+ satisfies the following conditions:

(a) there are constants $\bar{\alpha}$ and $\rho > 0$ such that $\varphi_+\big|_{\partial B_\rho} \ge \bar{\alpha}$, where $B_\rho = \{y \in Y : \|y\| < \rho\}$

(b) there is $q_0 \in Y \setminus \bar{B}_\rho$ such that $\varphi_+(q_0) \le 0$.

By (C1), for all $\varepsilon > 0$, there is a $\delta > 0$ such that $|F(k,y)| \le \varepsilon |y|^2$ whenever $|y| \le \delta$. Let $\rho = \delta\left(\min\limits_{k} r(k)\right)^{\frac{1}{2}}$ and $\|y\| \le \rho$, we have $\max\limits_{k\varepsilon[0,T+2]} |y(k)| \le \delta$. Hence $|F(k, y(k))| \le \varepsilon |y(k)|^2$ for all $k \in [2,T]$. Summing on [2,T], we get

$$
\sum_{k=2}^{T} F\left(k, y^+(k)\right) \le \varepsilon \sum_{k=2}^{T} \left|y^+(k)\right|^2 \le \varepsilon \frac{\|y\|^2}{\min\limits_{k\varepsilon[2,T]} r(k)}.
$$

So if $\|y\| = \rho$, then

$$
\varphi_+(y) = \frac{1}{2}\sum_{k=2}^{T+2}\left|\Delta^2 y(k-2)\right|^2 + \frac{1}{2}\sum_{k=2}^{T} r(k)\left|y(k)\right|^2 - \sum_{k=2}^{T}\left[F\left(k, y^+(k)\right) - f(k,0)\, y^-(k)\right]
$$

$$\geq \frac{1}{2}\|y\|^2 - \varepsilon \frac{\|y\|^2}{\min\limits_{k\varepsilon[2,T]} r(k)} = \left[\frac{1}{2} - \frac{\varepsilon}{\min\limits_{k\varepsilon[2,T]} r(k)}\right]\rho^2 .$$

And it suffices to choose $\varepsilon = \frac{\min\limits_{k\varepsilon[2,T]} r(k)}{4}$ to get

$$\varphi + (y) \geq \frac{\rho^2}{4} > 0 .$$

Consider

$$\varphi_+(\lambda y) = \frac{\lambda^2}{2}\sum_{k=0}^{T+2}\left|\Delta^2 y(k-2)\right|^2 + \frac{\lambda^2}{2}\sum_{k=0}^{T} r(k)\left|y(k)\right|^2 - \sum_{k=2}^{T}\left[F\left(k,\lambda y^+(k)\right) - \lambda f(k,0)\lambda f(k,0) y^-(k)\right]$$

for all $\lambda > 0$. Since condition (C2) implies Remark 1.1(2), there are $a_1, a_2 > 0$ such that

$$F\left(k, y^+\right) \geq a_1\left(y^+\right)^\mu - a_2 \text{ for all } y^+ \in R^+ .$$

Let $y_0 \in Y, \|y_0\| = 1$. For any $\lambda \geq 1$, we have

$$\varphi_+(\lambda y_0) = \frac{\lambda^2}{2}\sum_{k=2}^{T+2}\left|\Delta^2 y_0(k-2)\right|^2 + \frac{\lambda^2}{2}\sum_{k=2}^{T} r(k)\left|y_0(k)\right|^2 - \sum_{k=2}^{T}\left[F\left(k,\lambda y_0^+(k)\right) - \lambda f(k,0) y_0^-(k)\right]$$

$$\leq \frac{\lambda^2}{2}\sum_{k=2}^{T+2}\left|\Delta y_0(k-2)\right|^2 + \frac{\lambda^2}{2}\sum_{k=2}^{T} r(k)\left|y_0(k)\right|^2 - \sum_{k=2}^{T} a_1\lambda^\mu\left|y_0^+(k)\right|^\mu + a_2 T$$

So $\varphi_+(\lambda y_0) \to -\infty$ as $\lambda \to +\infty$. Since $\mu > 2$, let λ_0 sufficiently large such that $\left\|\lambda_{0y0}\right\| > \rho$, then

$$\varphi_+\left(\lambda_{0y0}\right) \leq 0 = \varphi_+(0) .$$

Therefore, φ_+ satisfies (a) (b). Applying Lemma 1.1 to φ_+, there exists y^* such that $\varphi_+'\left(y^*\right) = \Theta, \varphi_+\left(y^*\right) = c > \max\left\{\varphi_+(0), \varphi_+\left(\lambda_{0y0}\right)\right\} = 0$ and y^0 is not zero. Lemma 2.5 means that BVP (1.1) has at least one positive nontrivial solution.

Theorem 3.2. Suppose that $f(k,-y) = -f(k,y)$ for any $(k,y) \in [2,T] \times R$ and (C1) (C2) hold. Then problem (1.1) has infinitely many solutions.

Proof. Since $f(k,-y) = -f(k,y) \in [2,T] \times R$, the functional φ is even satisfying $\varphi(0) = 0$. Let $Y = R \oplus W$, where $W = \left\{y \in Y \,\middle|\, \left|y(1)\right| = 0\right\}$. Similar to the process of Lemma 2.6, φ satisfies Palais-Smale condition. From the proof of Theorem 3.1, there exist $\bar{\alpha}, \rho > 0$ such that $\varphi(y) \geq \bar{\alpha}$ for $y \in Y \cap \partial B_\rho$, and so $\varphi(y) \geq \bar{\alpha}$ for

$y \in Y \cap \partial B_\rho$. Now we will verify condition (J2) in Lemma 1.2. By Remark 1.1 (2) we have

$$\begin{aligned}\varphi(y) &= \frac{1}{2}\sum_{k=2}^{T+2}\left|\Delta^2 y(k-2)\right|^2 + \frac{1}{2}\sum_{k=2}^{T} r(k)\left|y(k)\right|^2 - \sum_{k=2}^{T} F(k, y(k)) \\ &\le \frac{1}{2}\|y\|^2 - \sum_{k=2}^{T} a_1\left|y(k)\right|^{\mu} + a_2 \times (T-1).\end{aligned} \tag{3.1}$$

Since Y is identified with finite dimensional space R^{T-1}, $(Y, \|\cdot\|)$ is equivalent to $(Y, \|\cdot\|_\mu)$. Hence there exist $k_3, k_4 > 0$ such that

$$k_3\|y\|_\mu \le \|y\| \le k_4\|y\|_\mu, \forall y \in Y, \text{ where } \|y\|_\mu = \left(\sum_{k=2}^{T}\left|y(k)\right|^{\mu}\right)^{\frac{1}{\mu}}$$

Therefore,

$$\varphi(y) \le \frac{1}{2}\|y\|^2 - a_1\left(\frac{1}{k_4}\right)^{\mu}\|y\|^{\mu} + a_2(T-1),$$

which implies that $\{y \in V_1 : \varphi(y) \ge 0\}$ is bounded. Then Lemma 1.2 can be applied to the functional φ. The proof is completed.

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