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Unsteady Viscous Flow Past a Flat Plate in a Rotating System

G. Manna, S. N. Maji, M. Guria* and R. N. Jana

Department of Applied Mathematics with Oceanology and Computer Programming,
Vidyasagar University, Midnapore 721 102, West Bengal, India
* Corresponding author: mrinmoy9832@yahoo.com

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ABSTRACT

An exact solution has been obtained for the flow of viscous incompressible fluid bounded by an infinite flat plate in a rotating fluid where both the plate and the fluid rotate in unison with uniform angular velocity Ω about an axis normal to the plate. Initially $(t = 0)$, the fluid at infinity moves with uniform velocity U_0 . At time $t > 0$, the plate suddenly moves with uniform velocity U_0 in the direction of the flow. An exact solution has been obtained using Laplace transform technique. It is found that an inertial oscillations occur at large time and do not occur in the absence of rotation. Also, the analytical solutions describing the flow at large and small times are obtained.

1. Introduction

The study of motion of viscous incompressible fluid has considerable interest in recent year due to its wide applications in cosmical, geophysical fluid dynamics and meteorology. The large scale and moderate motions of the atmosphere are greatly affected by vorticity of the earth's rotation. The motion in the earth's core is somehow responsible for the main geomagnetic field. It has been seen that, when the fluid is rotating near a flat plate, the pressure field of the flow far away from the plate also exists near the plate, but the Coriolis force near the plate is reduced owing to friction force. As a result, there exists a flow in the direction in which the pressure is falling until the Coriolis forces are compensated by viscous forces. Such a layer formed near the plate is known as Ekman layer. The steady flow near the plate, in a rotating fluid has been studied by Batchelor [1]. The effect of suction at the plate on the steady rotating flow was discussed by Gupta [2]. The unsteady flow past a flat plate in a rotating fluid have been studied by Greenspan and Howard [3], Soundalgekar and Pop [4], Gupta and Gupta [5], Datta and Jana [6] and many other researchers.

The aim of this paper is to study the flow of viscous incompressible fluid bounded by an infinite flat plate in a rotating system where both the plate and the fluid rotate in unison with uniform angular velocity Ω about an axis normal to the plate. Initially *(t* = 0), the fluid at infinity moves with uniform velocity U_0 . At time $t > 0$, the plate suddenly moves with uniform velocity U_0 in the direction of the flow. An exact solution has been obtained by using Laplace transform technique. Another solution is obtained which is valid for small times. The solution for large times has also been obtained .

2. Mathematical Formulation and its Solution

Consider the viscous incompressible fluid filling the semi infinite space $z \geq 0$ in a cartesian coordinates system. The plate and the fluid at infinity are rotating with angular velocity Ω about *z*— axis. Initially (t=0), the fluid flows past an infinite flat plate with free-stream velocity U_0 along *x*— axis. At time $t > 0$, the plate suddenly starts to move with same uniform velocity as that of the free stream velocity U_0. At time $t = 0$, we assume, for the velocity components $\hat{u}, \hat{v}, \hat{w}$ in *x*—, *y*— and *z*— directions respectively, the equations of motion in a rotating frame of reference are

$$-2\Omega\hat{v} = -\frac{1\partial p}{\rho\partial x} + v\frac{d^2\hat{u}}{dz^2}, \tag{1}$$

$$2\Omega\hat{u} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + v\frac{d^2\hat{v}}{dz^2}, \tag{2}$$

$$0 = \frac{1}{\rho}\frac{\partial p}{\partial z,} \tag{3}$$

where *p* is the modified pressure including centrifugal force, *ρ* is the density of the fluid and *v* is the kinematic coefficient of viscosity. The boundary conditions are

$$\hat{u} = 0,\ \hat{v} = 0,\ z = 0,\ \hat{u} \to U_0\ \ \hat{v} \to 0, \text{ as } z \to \alpha \tag{4}$$

Under usual boundary layer approximations Eqs. (1) and (2) become

$$-2\Omega\hat{v} = v\frac{d^2\hat{u}}{dz^2}, \tag{5}$$

$$2\Omega(\hat{u} - U_0) = v\frac{d^2\hat{v}}{dz^2} \tag{6}$$

Introducing non-dimensional variables

$$\eta = U_0 z / \nu,\ K^2 = \Omega\nu / U_0^2,\ \hat{F} = (\hat{u} + i\hat{v}) / U_0,\ i = \sqrt{-1}, \tag{7}$$

the solutions of (5) and (6) satisfying the boundary conditions (4) can be obtained as

$$\hat{u} / U_0 = 1 - e^{-K\eta} \cos K\eta, \tag{8}$$

$$\hat{v} / U_0 = e^{-K\eta} \sin K\eta \tag{9}$$

At time $t > 0$, the plate suddenly moves with uniform velocity U_0 along x— axis. Assuming the velocity components *(u,v,w)* along the coordinate axes, we have the equations of motion in a rotating frame of reference as

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \tag{10}$$

$$\frac{\partial u}{\partial t} - 2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 v}{\partial z^2}, \tag{11}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{12}$$

The initial and boundary conditions are

$$u = \hat{u},\ v = \hat{v} \text{ at } t = 0 \text{ for } z \ge 0, \tag{13}$$

$$u = U_0,\ v = 0 \text{ at } z = 0, \text{ and } u \to U_0,\ v \to 0 \text{ as } z \to \infty,\quad t > 0 \tag{14}$$

Using infinity conditions (14) Eqs. (10) and (11) become

$$\frac{\partial u}{\partial t} - 2\Omega v = \nu\frac{\partial^2 u}{\partial z^2}, \tag{15}$$

$$\frac{\partial v}{\partial t} + 2\Omega(u - U_0) = \nu\frac{\partial^2 v}{\partial z^2}. \tag{16}$$

Equations (15) and (16) can be written in combined form as

$$\frac{\partial F}{\partial t} + 2i\Omega F = \nu\frac{\partial^2 F}{\partial z^2} \tag{17}$$

where

$$F = (u + iv) / U_0 - 1. \tag{18}$$

Introducing the non-dimensional time $\tau = U_0^2 t / \nu$ and using (7), Eq. (17) becomes

$$\frac{\partial F}{\partial \tau} + 2iK^2 F = \frac{\partial^2 F}{\partial \eta^2}. \tag{19}$$

The corresponding initial and the boundary conditions (13) and (14) become

$$F(\eta, 0) = \hat{F}(\eta) \quad \eta \geq 0 \tag{20}$$

$$F(0, \tau) = 0 \quad F(\infty, \tau) = 0 \quad \text{for} \quad \tau > 0, \tag{21}$$

where $\hat{F}(\eta, \tau)$ is given by (7). To solve the Eq. (19), we assume

$$F(\eta, \tau) = H(\eta, \tau) e^{-2iK^2\tau}. \tag{22}$$

On the use of (22), Eq. (19) becomes

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 H}{\partial \eta^2} \tag{23}$$

with initial and boundary conditions

$$H(\eta, 0) = \hat{F}(\eta) \quad for \ \eta \geq 0, \tag{24}$$

$$H(0, \tau) = 0, \quad H(\infty, \tau) \tag{25}$$

Taking the Laplace transform of the Eq .(23) and solving subject to the initial and boundary conditions (24) and (25), we get

$$\overline{H}(\eta, s) = \frac{1}{2\sqrt{s}} \left[\frac{1}{\sqrt{s} + \sqrt{2iK^2}} + \frac{1}{\sqrt{s} - \sqrt{2iK^2}} \right] e^{-\sqrt{s}\eta} - \frac{e^{(1+i)K\eta}}{s - 2iK^2}. \tag{26}$$

The inverse transformation of the Eq.(26) gives

$$H(\eta, \tau) = \frac{1}{2} e^{2iK^2} \left[e^{(1+i)K\eta} er\,fc\left(\frac{\eta}{2\sqrt{\tau}} + (1+i)K\sqrt{\tau} \right) \right.$$
$$\left. + e^{-(1+i)K\eta} er\,fc\left(\frac{\eta}{2\sqrt{\tau}} + (1+i)K\sqrt{\tau} \right) - 2e^{-(1+i)K\eta} \right]. \tag{27}$$

Substituting the above value of $H(\eta, \tau)$ in Eq. (22), we have

$$F(\eta, \tau) = \frac{1}{2} \left[e^{(1+i)K\eta} er\,fc\left(\frac{\eta}{2\sqrt{\tau}} + (1+i)K\sqrt{\tau} \right) \right.$$

$$+e^{-(1+i)K\eta} erfc\left(\frac{\eta}{2\sqrt{\tau}}-(1+i)K\sqrt{\tau}\right)\Bigg]-e^{-(1+i)K\eta}, \quad (28)$$

and on using (18), we obtain

$$\frac{u+iv}{U_0}=1-e^{(1+i)K\eta}+\frac{1}{2}\Bigg[e^{(1+i)K\eta} erfc\left(\frac{\eta}{2\sqrt{\tau}}+(1+i)K\sqrt{\tau}\right)$$

$$e^{-(1+i)K\eta} erfc\left(\frac{\eta}{2\sqrt{\tau}}-(1+i)K\sqrt{\tau}\right)\Bigg]. \quad (29)$$

On separating into real and imaginary parts one can easily obtain the velocity components $\frac{u}{U_0}$ and $\frac{v}{U_0}$.

Solution at small times

Now, we investigate another solution which is valid for small values of time. Thus our purpose is to compare the results obtained in two different cases. For small times, the Eq. (26) can written as

$$\overline{H}(\eta,\tau)=\sum_{n=0}^{\infty}\frac{\left(2iK^2\right)^n}{s^{n+1}}\left[e^{-\sqrt{s}\eta}-e^{-(1+i)K\eta}\right]. \quad (30)$$

The inverse transformation of the above Eq. (30) is

$$H(\eta,\tau)=\sum_{n=0}^{\infty}\left(2iK^2\right)^n(4\tau)^n i^{2n} erfc\left(\eta/2\sqrt{\tau}\right)-e^{-(1+i)K\eta}e^{2iK^2\tau} \quad (31)$$

Substituting the above value of *H(η,τ)* in the Eq.(22), we get

$$F(\eta,\tau)=\sum_{n=0}^{\infty}\left(2iK^2\right)^n(4\tau)^n i^{2n} erfc\left(\eta/2\sqrt{\tau}\right)e^{-2iK^2\tau}-e^{-(1+i)K\eta}, \quad (32)$$

where $i^n erfc$ (.) denotes the repeated integrals of the complementary error function given by

$$i^n erfc(x)=\int_0^{\infty} i^{n-1} erfc(\xi)d\xi, n=0,1,2,...$$

$$i^0 erfc(x)=erfc(x), \quad (33)$$

$$i^{-1}erfc(x)=\frac{2}{\sqrt{\pi}}e^{-x^2}$$

On the use of (18) and on separating into real and imaginary parts, we have

$$u/U_0=1-e^{-K\eta}\cos K\eta+\left[A(\eta,\tau)\cos 2K^2\tau+B(\eta,\tau)\sin 2K^2\tau\right], \tag{34}$$

$$u/U_0=e^{-K\eta}\sin K\eta+\left[B(\eta,\tau)\cos 2K^2\tau-A(\eta,\tau)\sin 2K^2\tau\right], \tag{35}$$

where

$$A(\eta,\tau)=erfc\left(\eta/2\sqrt{\tau}\right)-\left(2K^2\right)^2(4\tau)^2 i^4 erfc\left(\eta/2\sqrt{\tau}\right)+\left(2K^2\right)^4(4\tau)^2 i^8 erfc\left(\eta/2\sqrt{\tau}\right)+\ldots$$

$$B(\eta,\tau)=2K^2(4\tau)i^2 erfc\left(\eta/2\sqrt{\tau}\right)-\left(2K^2\right)^3(4\tau)^3 i^6 er\ fc\left(\eta/2\sqrt{\tau}\right)$$
$$+\left(2K^2\right)^5(4\tau)^5 i^{10} erfc\left(\eta/2\sqrt{\tau}\right)+\ldots \tag{36}$$

Solution at large times :

For large time τ, the expression (29) can be written as

$$\frac{u+iv}{U_0}=1+\frac{1}{2}\left[e^{(1+i)K\eta}erfc\left((1+i)K\sqrt{\tau}+\frac{\eta}{2\sqrt{\tau}}\right)\right.$$
$$\left.-e^{-(1+i)K\eta}erfc\left((1+i)K\sqrt{\tau}-\frac{\eta}{2\sqrt{\tau}}\right)\right] \tag{37}$$

Additionally, if $\eta\langle\langle 2\sqrt{\tau},\tau\rangle\rangle 1$ then the above Eq. (37) gives

$$u/U_0=1+\frac{1}{2K\sqrt{\pi\tau}}\left[\left(\cos 2K^2\tau-\sin 2K^2\tau\right)\sinh K\eta\cos K\eta\right.$$
$$\left.+\left(\cos 2K^2\tau+\sin 2K^2\tau\right)\cosh K\eta\sin K\eta\right], \tag{38}$$

$$v/U_0=\frac{1}{2K\sqrt{\pi\tau}}\left[\left(\cos 2K^2\tau-\sin 2K^2\tau\right)\cosh K\eta\ \sin K\eta\right.$$
$$\left.-\left(\cos 2K^2\tau+\sin 2K^2\tau\right)\sinh K\eta\cos K\eta\right], \tag{39}$$

It is observed from above Eqs. (38) and (39) that the rotation not only causes the cross flow but also occurs inertial oscillations of the fluid velocity which decreases with time τ and does not occurs in the absence of rotation. The frequency of these oscillation is $2K^2$.

3. Discussions

To study the flow situations due to the impulsive start of the plate in a rotating system for different values of rotation parameter K and time τ, the velocity profiles are examined numerically and plotted against η in Figs. 1-4. The primary velocity $\frac{u}{U_0}$ and the secondary velocity $\frac{v}{U_0}$ are shown in Fig.1 and 2 against η for different values of K with τ = 0.2. Fig.1 shows that the primary velocity $\frac{u}{U_0}$ decreases for $K \leq 3.5$ and then oscillatory in nature for $K \geq 4.0$. On the other hand it is observed from Fig.2 that the secondary velocity $\frac{v}{U_0}$ decreases for $K \leq 2.5$ and then oscillatory in nature for $K \geq 3.0$. Figs. 3 and 4 show the velocity components $\frac{u}{U_0}$ and $\frac{v}{U_0}$ for various values of time τ with $K = 2.0$. It is seen from Fig. 3 that the primary velocity first increases reaches a maximum and then decreases with increase in time τ. It is observed from Fig.4 that an increase in τ leads to fall in the secondary velocity .

The numerical values of u/U_0 and v/U_0 are obtained from equations (29) , (38) and (39) are shown in Figs. 5 and 6 for different values of time τ. It is observed that for small times both the velocity components U/U_0 and V/U_0 obtained from equations (38) and (39) converge more rapidly than that obtained from equation (29). Hence , for small times, the numerical values of u/U_0 and v/U_0 should be calculated from equations (38) and (39) instead of equation (29). The shear stresses at the plate $\eta = 0$ due to the primary and secondary flows are given by [from Eq. (28)]

$$\tau_x + i\tau_y = \frac{u' + iv'}{U_0}$$

$$= (1+i)K\, erfc\,(1+i)K\sqrt{\tau} - \frac{1}{\sqrt{\pi\tau}} e^{-(1+i)^2\tau} \Big] \quad (40)$$

The numerical results of τ_x and τ_y are shown in Fig.7 against time τ for different values of K. It is observed that the shear stresses are positive and negative alternatively and becomes zero for large times.

For small times the shear stresses at the plate $\eta = 0$ due to primary and the secondary flows can be obtained as

$$\tau_x = \frac{u'(0,\tau)}{U_0} = K - \frac{1}{2\sqrt{\tau}}\left[P(0,\tau)\cos 2K^2\tau + Q(0,\tau)\cos 2K^2\tau\right], \tag{41}$$

$$\tau_y = \frac{u'(0,\tau)}{U_0} = K - \frac{1}{2\sqrt{\tau}}\left[Q(0,\tau)\cos 2K^2\tau - P(0,\tau)\sin 2K^2\tau\right], \tag{42}$$

where

$$P(0,\tau) = Y_{-1} - \left(2K^2\right)^2 (4\tau)^2 Y_3 + \left(2K^2\right)^4 (4\tau)^2 Y_7 + ..., \tag{43}$$

$$Q(0,\tau) = (2K)^2 (4\tau) Y_1 - \left(2K^2\right)^3 (4\tau)^3 Y_5 + \left(2K^2\right)^5 (4\tau)^5 Y_9 ..., \tag{44}$$

with

$$\frac{dT_{2n}}{d\eta} = -\frac{Y_{2n-1}}{2\sqrt{\tau}}, \tag{45}$$

where

$$T_{2n} = i^{2n} erfc\left(\eta / 2\sqrt{\tau}\right), Y_{2n-1} = i^{2n-1} erfc\left(\eta / 2\sqrt{\tau}\right) \tag{46}$$

For small times, a comparison of the numerical values for the dimensionless shear stress components τ_x and τ_y calculated from equations (40), (41)and (42) against K for several values of τ are presented in the Figs.8 and 9. It is seen from Fig.8 that the shear stress component τ_x due to the primary flow increases with increase in either τ or K. Fig.9 shows that the shear stress components τ_y due to the secondary flow decreases with increase in τ while it increases with increase in K. It is observed from Figs.8 and 9 that for small times, the shear stress τ_x calculated from equations (41) and (42) is greater than that of calculated from equation (40). On the other hand the shear stress τ_y obtained from equations (41) and (42) is less than that of obtained from equation (40). Hence for small times shear stresses should be calculated from equations (41) and (42) instead of equation (40).

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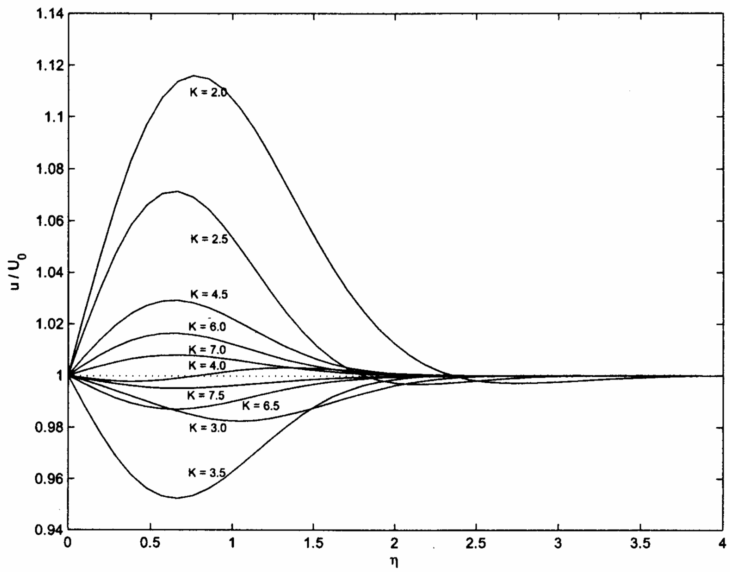

Figure 1: Variations of u/U_0 for $\tau = 0.2$.

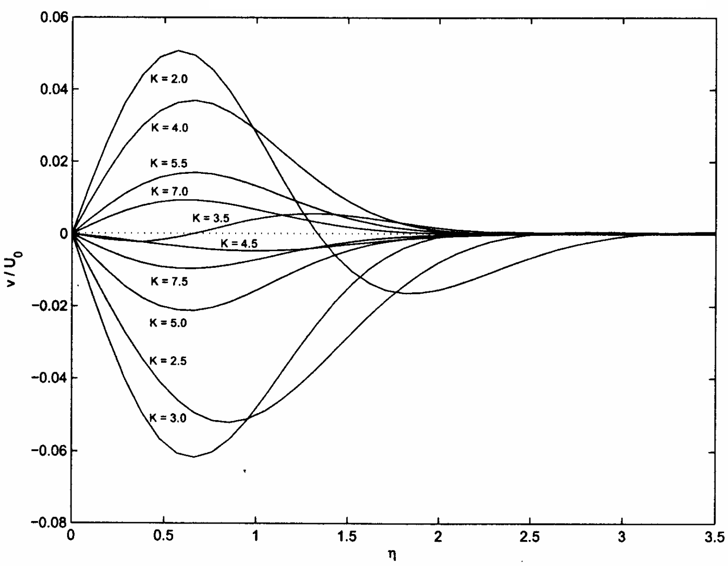

Figure 2: Variations of v/U_0 for $\tau = 0.2$.

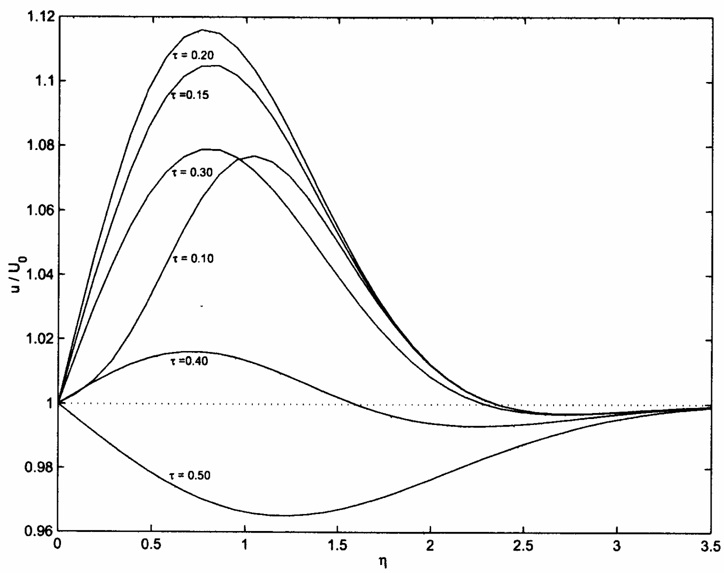

Figure 3: Variations of u/U_0 for $K = 2$.

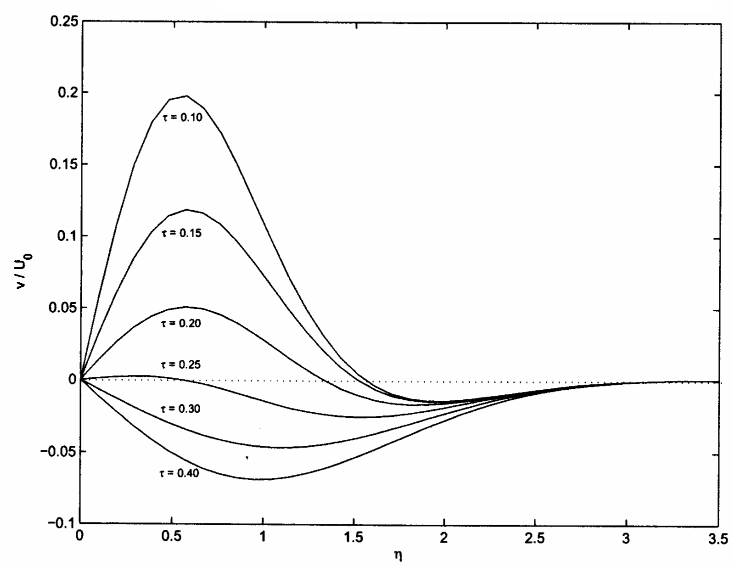

Figure 4: Variations of v/U_0 for $K = 2$.

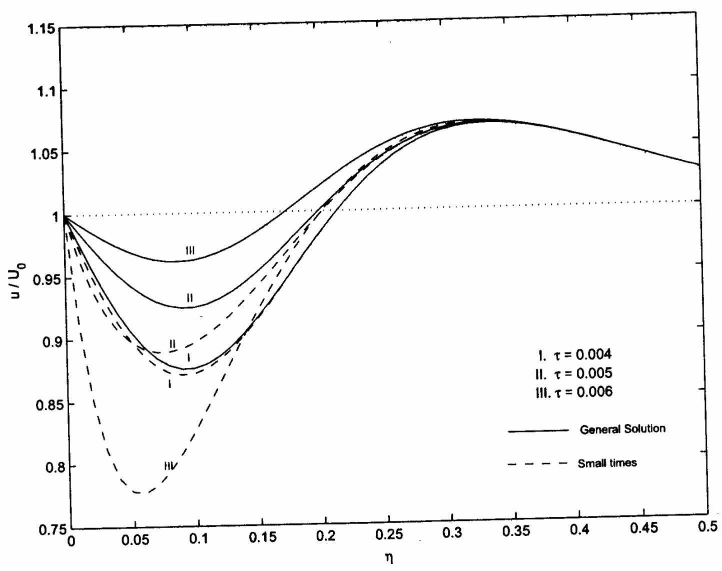

Figure 5: Variations of u/U_0 for $K = 7$.

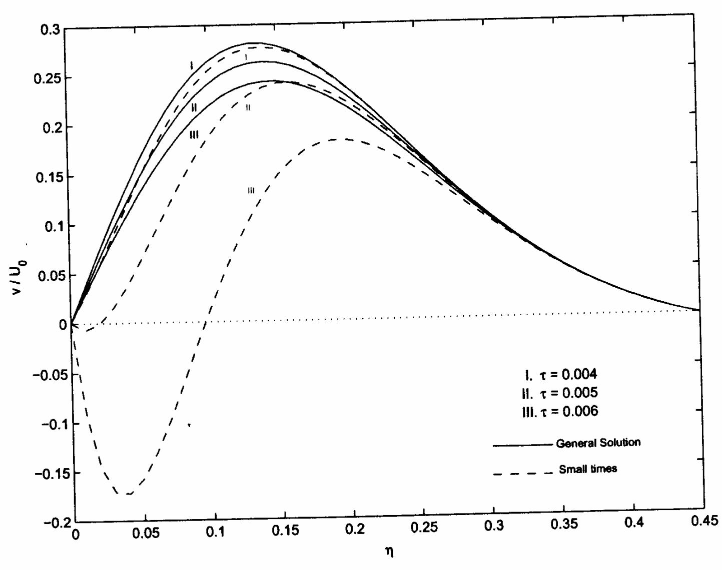

Figure 6: Variations of v/U_0 for $K = 7$.

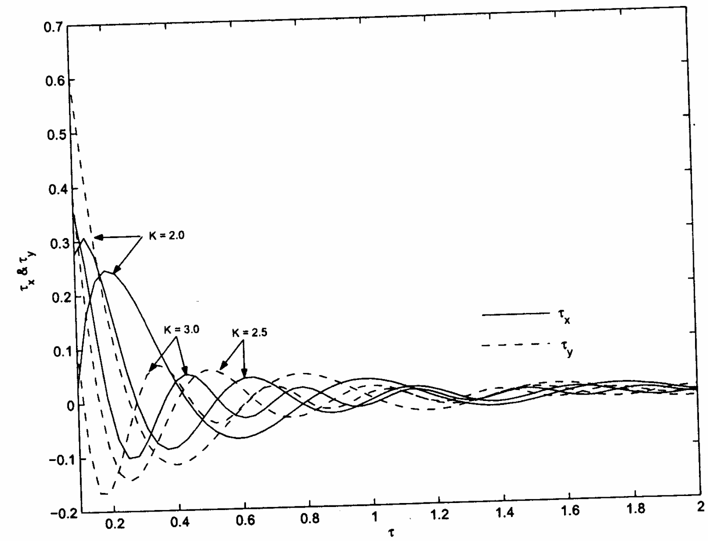

Figure 7: Variations of τ_x and τ_y.

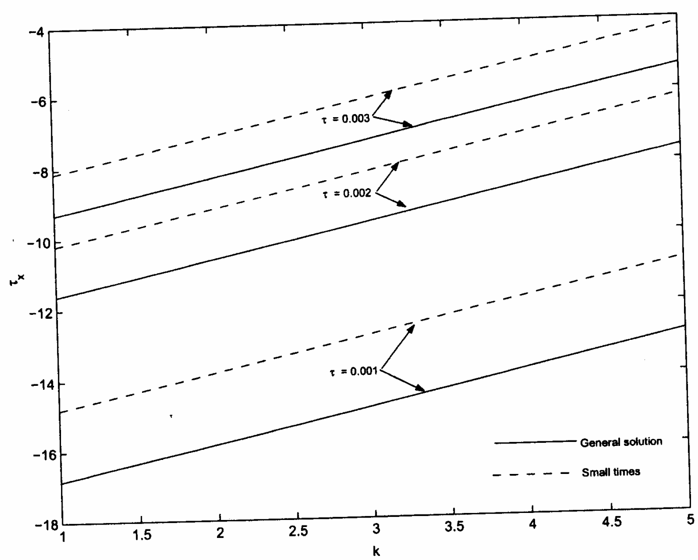

Figure 8: Variations of τ_x.

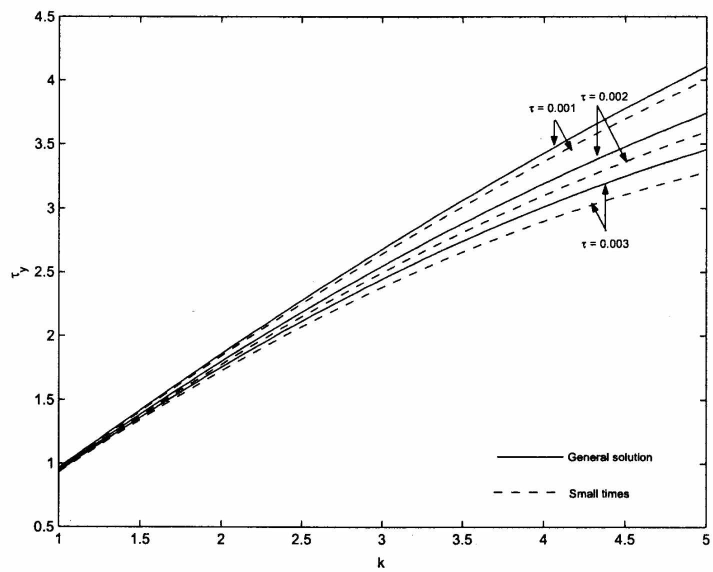

Figure 9: Variations of τ_y