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Estimation of the Reliability Indexes for a Cold Standby System Under Type II Censoring Data

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ABSTRACT

Under type −II censoring data, the approximate confidence limits of the reliability indexes for a cold standby series system are investigated by using empirical Bayes approach. The formulae to calculate empirical Bayes approximate confidence limits of the failure rate, and the reliability function and average life are given. Finally, an illustrative example is examined numerically by means of the Monte-Carlo simulation. Also, the accuracy of confidence limits is discussed.

Keywords **:** *Cold standby series system; Reliability indexes; Type : II censoring data; Empirical Bayes confidence limits*

1. Introduction

Estimation of the reliability indexes of some system is one of the main problems in reliability theory. Suppose that the life units in the system have identical exponential distribution and the failure rate is a known constant, Mei [1] discussed the point estimation of the reliability indexes. Su and Bai [3][4] assume that the failure rate is a random variable, and they investigated the point estimation of the reliability indexes for a cold standby system by using Bayesian and multiple Bayesian methods respectively. Pham and Turkkan [5] studied the reliability of a standby system with Beta distribution Component live. But the confidence limits of the reliability indexes were not addressed under type : II censoring life test. In this paper, we study the approximate confidence limits of the reliability indexes for a cold standby series system under type −II censoring life test.

2 Approximate Confidence Limits of the Reliability Indexes

Firstly, we consider the Bayesian approximate upper confidence limit of failure rate λ.

Suppose that the system of our interest has (n+k-1) identical units, where k units work while the rest (n-1) units are in cold standby case, and the alternation switch is completely reliable. When a working unit fails, a cold standby unit replaces it immediately and the system can work as before. If the standby units are used up and one of k series units becomes unusable, then the system is said to be invalidation. Let the life of all units in the system be independent and have identical exponential distribution exp(λ), where λ is the failure rate of unit, and λ is a random variable with a prior density function

$$p(l \mid b) = b \exp(-b l),\ b, l > 0 .$$

From [1] we can find that this series system is equivalent to a cold standby system of n independent units, and every unit has the failure rate kλ with density function:

$$f(x \mid l) = l k \exp(-l kx),\ l, x, k > 0 .$$

We now perform type II censoring life test on those above units. Suppose that n units are tested. When r units failing, the test is stopped. We denote the failure moment in turn by $X_{(1)}, \cdots, X_{(r)}$.

Thus $(X_{(1)}, \cdots, X_{(r)})$ is a type II censored samples. Let the sample values satisfy $x_1 \le x_2 \le .... \le x_r$, so the associated density function of $(X_{(1)}, \cdots, X_{(r)})$ is

$$L = \frac{n!}{(n-r)!}[\prod_{i=1}^{r} h(x_i)][1 - F(x_r)]^{n-r} . \tag{1}$$

where *h(x)* and *F(t)* are regarded as respectively density function and distribution function of the unit X. We let

$$h(x_i) = f(x_i \mid l), \quad i = 1, 2 \ldots r,\ F(x_r) = 1 - \exp(-l kx_r),\ x = (x_1, x_2 \ldots x_r) .$$

Then function L is

$$L = g(x \mid l) = \frac{n!}{(n-r)!}[\prod_{i=1}^{r} f(x_i \mid l)][1 - F(x_r)]^{n-r}$$

$$= \frac{n!}{(n-r)!}[\prod_{i=1}^{r} l k \exp(-l kx_i)][\exp(-l kx_r)]^{n-r}$$

$$= \frac{n!}{(n-r)!}(l k)^r \exp\{-l k[\sum_{i=1}^{r} x_i + (n-r)x_r]\} . \tag{2}$$

From Bayesian theorem, the posterior distribution of l can be written as

$$h(l\mid x)=\frac{g(x\mid l)p(l\mid b)}{\int_0^{+\infty}g(x\mid l)p(l\mid b)dl}=\frac{(lk)^r\exp\{-l[k\sum_{i=1}^{r}x_i+kx_r(n-r)+b]\}}{\int_0^{+\infty}(lk)^r\exp\{-l[k\sum_{i=1}^{r}x_i+kx_r(n-r)+b]\}dl}$$

$$=l^r[k\sum_{i=1}^{r}x_i+kx_r(n-r)+b]^{r+1}\exp\{-l[k\sum_{i=1}^{r}x_i+kx_r(n-r)+b]\}[\Gamma(r+1)]^{-1}$$

That is $l\mid x\sim\Gamma(r+1,k\sum_{i=1}^{r}x_i+kx_r(n-r)+b)$. From [6], we can get

$$2(k\sum_{i=1}^{r}x_i+kx_r(n-r)+b)l\sim\Gamma(r+1,2^{-1})=c^2(2r+2)$$

Where $x^2(m)$ denotes a central chi-square distribution with m degree of freedom.

Let $c_a^2(\mathrm{m})$ be a α quantile of probability distribution $c^2(\mathrm{m})$, where m = 2r + 2. $c_a^2(\mathrm{m})$ satisfies the equality $P\{2(k\sum_{i=1}^{r}x_i+kx_r(n-r)+b)l\le c_a^2(m)\}=1-a$. Then the 1 – a upper confidence limit of λ can be written as :

$$l_U=c_a^2(m)[2(k\sum_{i=1}^{r}x_i+kx_r(n-r)+b)]^{-1}. \qquad (3)$$

As b is an unknown constant, l_U can not be used. In order to estimate b, we need to use the the maximum likelihood estimation approach. As the probability distribution of every unit X is a exponential distribution exp (l), the margin density function X is

$$f_X(x)=\int_0^{\infty}f(x\mid l)p(l\mid b)dl=\int_0^{\infty}lk\exp(-lkx)b\exp(-bl)dl=\frac{kb}{(kx+b)^2}.$$

Let density function $h(x)$ and distribution function $F(x_r)$ be $f_x(x)$ and $F_x(x_r)$ respectively in equation (1), where

$$1-F_X(x_r)=\int_{x_r}^{+\infty}f_X(x)dx=\int_{x_r}^{+\infty}\frac{kb}{(kx+b)^2}dx=\frac{b}{kx_r+b}.$$

From [2], the associated density function of ($X_{(1)}$, $X_{(2)}$, $KX_{(r)}$) is

$$L=\frac{n!}{(n-r)!}[\prod_{i=1}^{r}f_X(x_i)][1-F_X(x_r)]^{n-r}=\frac{n!}{(n-r)!}[\prod_{i=1}^{r}\frac{kb}{(kx_i+b)^2}](\frac{b}{kx_r+b})^{n-r}$$

$$=\frac{(kb)^r n!}{(n-r)!}[\prod_{i=1}^{r}\frac{1}{(kx_i+b)^2}](\frac{b}{kx_r+b})^{n-r}$$

$$\lg L=\lg\frac{n!}{(n-r)!}+r(\lg k+\lg b)-2\sum_{i=1}^{r}\lg(kx_i+b)+(n-r)[\lg b-\lg(kx_r+b)]$$

$$\frac{d\lg L}{db}=\frac{r}{b}-2\sum_{i=1}^{r}\frac{1}{kx_i+b}+(n-r)(\frac{1}{b}-\frac{1}{kx_r+b}).$$

We consider function $g_1(b)=\frac{r}{b}+(n-r)(\frac{1}{b}-\frac{1}{kx_r+b})$ and $g_2(b)=2\sum_{i=1}^{r}\frac{1}{kx_i+b}$. As the maximum likelihood estimation (MLE) of b is needed, we just draw a conclusion that equation $g_1(b)=g_2(b)$ have only one root. The reasons are as follow :

For any $\beta>0$, we have

$$g_1(\beta)>0, g_1(\beta)\to 0, (\text{as } \beta\to\infty), \text{and } g_1(\beta)\to\infty, (\text{as } \beta\to 0),$$

$$g_1'(b)=-\{(rb^{-2}+kx_r(n-r)(kx_r+2b)[b(kx_r+b)]^{-2}\}<0$$

$$g_1''(b)=2rb^{-3}+2kx_r(n-r)[b(kx_r+b)]^{-3}(k^2x_r^{\,2}+3b^2+3kx_rb)>0.$$

One arrives that $g_1(b)$ is strict monotone increasing concave function in $(0,+\infty)$. Similarly

For any $\beta>0$, $g_2(\beta)>0$, $g_2(\beta)\to 0, (\beta\to\infty), \text{and } g_2(\beta)\to 2\sum_{i=1}^{r}\frac{1}{kx_r}, (\beta\to 0)$

$$g_2'(b)=-2\sum_{i=1}^{r}(kx_i+b)^{-2}<0 \text{ £ } g_2''(b)=4\sum_{i=1}^{r}(kx_i+b)^{-3}>0.$$

We also get that $g_2(b)$ is strict monotone increasing concave function in $(0,+\infty)$. Moreover

$$\lim_{b\to\infty}\frac{g_1(b)}{g_2(b)}=\lim_{b\to\infty}[\frac{r}{b}+(n-r)(\frac{1}{b}-\frac{1}{kx_r+b})][2\sum_{i=1}^{r}\frac{1}{kx_i+b}]^{-1}=\frac{1}{2}<1$$

From above conclusion, the equation $\frac{d\lg L}{db}=0$ has only one root and the expression is

$$b^{(l+1)}=r[2\sum_{i=1}^{r}\frac{1}{kx_i+b^{(l)}}-(n-r)\frac{kx_r}{b^{(l)}(kx_r+b^{(l)})}]^{-1}. \quad (4)$$

Where $b^{(l)}$ is l th iteration value (l = 1,2,Λ). $b^{(0)}$ is an initial value. If the iteration solution is denoted by $\hat{b}$ and replacing b in (3) by $\hat{b}$, we can obtain the estimator of Bayesian upper confidence limit $\hat{l}_U$

$$\hat{l}_U = c_a^2(m)[2(k\sum_{i=1}^{r} x_i + kx_r(n-r) + \hat{b})]^{-1} \tag{5}$$

From (5), we can get the Bayesian approximate confidence limits of reliability function and average life. From [2], the reliability function *R(t)* and average life *MTTF* for a cold standby system are respectively given as

$$R(t) = \sum_{i=0}^{n} (l kt)^i \exp(-l kt)/i!, \quad MTTF = \frac{n}{l k}$$

For any real number $l > 0$, $R(t)$ and *MTTF* are respectively strict monotone decreasing functions with respect to l (see [6]). Thus when the degree of confidence 1 – a is given, lower confidence limits of *R(t)* and *MTTF* are respectively written as

$$R_L(t) = \sum_{i=0}^{n-1} \frac{(l_U kt)^i}{i!} \exp(-l_U kt) \tag{6}$$

$$MTTF_L = \frac{n}{l_U k} \tag{7}$$

When λ_U in (6) and (7) are replaced by $\hat{l}_U$ in (5), the estimators of $1-a$ Bayesian lower confidence limit for *R(t)* and *MTTF* are given as follows

$$\hat{R}_L(t) = \sum_{i=0}^{n-1} \frac{(\hat{l}_U kt)^i}{i!} \exp(-\hat{l}_U kt) \tag{8}$$

$$M\hat{T}TF_L = \frac{n}{\hat{l}_U k} \tag{9}$$

3. A Numerical Example

When the value of b is given, from the prior distribution function of l, that is, $F(l \mid b) = 1 - \exp(-b l)$, $b, l > 0$, a group of values of l are generated by using Monte-Carlo method. For every l value, from (2), we obtain the corresponding samples as $(x_{i1}, x_{i2,\cdots,} x_{ir})$ by using Monte-Carlo method again, where $i = 1, 2, ..., N$. From

$(x_{i1}, x_{i2,\cdots,} x_{ir})$ and (4), $\hat{b}_i$ is also obtained. To decrease error and improve precision, we should take $\hat{b}$ as the average estimation of $\hat{b}_i$ ($i = 1, 2, \ldots, N$). For the given degree of confidence $1 - a$, the quantile $c_a^2(\mathrm{f})$ is obtained through the data table of Chi-square distribution. Note $\hat{b}$, $c_a^2(\mathrm{f})$ and (5), we can get $1 - a$ Bayes upper confidence limit $\hat{l}_U$ of l. From (5), (8) and (9), we also get $\hat{R}_L(t)$ and $M\hat{T}TF_L$. We take the average value of parameter estimation as the estimation value of the parameters in Tab. 1.

Tab1 : The simulation result of confidence limits

The value of the parameter	$\hat{b}$	$1-a$	$\hat{l}_U$	$\hat{R}_L(t)$	$M\hat{T}TF_L$
b =3,n=3,k=2, r=1	2.7949	0.90	0.2938	0.8819	3.3902
		0.95	0.3616	0.8376	2.7801
b =3,n=4,k=2, r=2	3.3871	0.90	0.4772	0.7532	2.0953
		0.95	0.5651	0.6878	1.7802
b =2,n=2,k=1, r=1	1.8726	0.90	0.4435	0.9253	4.5110
		0.95	0.5421	0.8968	3.6971

4. The Discussion on the Accuracy of Approximate Confidence Limits

A1). Take b^*, t^* as the value of and t respectively. From the prior distribution function of l, that is, $F(l \mid b) = 1 - \exp(-b\,l)$, $b, l > 0$, a group of values of l are generated by using Monte-Carlo method. Taking one of them as l^*, and replacing l by l^* in the expression of $R(t)$ and *MTTF*, one can find the value of $R(t^*)$ and $MTTF^*$.

A2). For one of l, note (2) and using Monte-Carlo method again, we can generate the corresponding samples $(x_{i1}\ x_{i2} \ldots x_{ir})$. From these samples and (4), the value of $\hat{b}$ can be obtained.

A3). From (5), (8) and (9), we can get the values of $\hat{l}_U, \hat{R}_L(t)\ M\hat{T}TF_L$.

A4). Repeating A2)-A3) N times, N groups of $\hat{l}_U, \hat{R}_L(t)$ and $M\hat{T}TF_L$ can be obtained respectively.

A5). We compare $\hat{l}_U, \hat{R}_L(t)$ and $M\hat{T}TF_L$ with $l^*, R(t^*)$ and $MTTF^*$ respectively, and we consider the covering percentage for l^* (that is $\geq l^*$), $R(t^*)$ (that is $\leq R(t^*)$), (that is $\leq MTTF^*$). Denoting these covering percentages by P_{l^*}, P_{R^*}, P_{M^*} respectively, these covering percentages with the degree of confidence $1-a$ are compared in Tab.2.

Tab. 2 The simulation comparison of approximate confidence limits ($t^*=1$)

The true value of parameters	l^*	$R(t^*)$	$MTTF^*$	$1-a$	P_{l^*}	P_{R^*}	P_{M^*}
b =3,n=3,k=2, r=1	0.3171	0.9732	9.4702	0.90	0.906	0.912	0.912
				0.95	0.964	0.971	0.981
b =3, n=4, k=2, r=2	0.3405	0.8601	2.9723	0.90	0.916	0.918	0.935
				0.95	0.969	0.962	0.971
b =2, n=2, k=1, r=1	0.2606	0.9698	7.4590	0.90	0.919	0.933	0.925
				0.95	0.976	0.968	0.959

In this experiment, we take N as 200 and t^* as 1. From the simulation results, we can see that the approximate confidence limits for $l, R(t)$ and $MTTF$ are efficient.

5. Conclusion

By using empirical Bayes approach, we obtain the approximate confidence limits of the reliability indexes for a cold standby series system under type : II censoring life test. The Monte-Carlo simulation was used to examine the result of reliability estimation. The simulation results show the accuracy of approximate confidence limits of $l, R(t)$ and MTTF are good.

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