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Editorial

Ethics and Technology Should Run Parallel

In the last half of the last century we have seen the massive outburst of the development in science & technology. The whole development got a tremendous acceleration in its progress towards a more beautiful future, as the world of information processing & artificially expert processing has supported the spirit in the above progress.

Inspite of the blooming phase of the development in our civilisation we have seen, the ethical world is still in dark. Lot of species in Zoological as well as Botanical world have been lost for ever from their mother world in absence of proper protection. A man in the street till now collects a little livelihood by pulling rickshaw in a bright morning of twenteeth century. Thousands examples can be cited in favour of the high ethical darkening of our civilisation. Now it is the high time to clear the darkness with the light from science & technology. Otherwise we may have to pay more, if we be late to catch the train.

Souransu Mukhopadhyay

Editor-in-Chief,
Vidyasagar University Journal of Physical Sciences

OPTICAL PARALLEL COMPUTATION : SOME IMPORTANT STUDIES IN LAST FEW DECADES

Partha Pratima Das and Sourangshu Mukhopadhyay
Department of Physics & Technophysics,,
Vidyasagar University,
Midnapur - 721102, West Bengal, India.

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Abstract:

All optical parallel computation is a novel challenge of the scientists & technologists since last few decades. Various approaches, methods and proposals by the scientists from the middle seventies of the last century made this field stronger. They believe, the days are not far, when people of the globe will see an all-optical super-fast computer with its tremendus applicational advantages. Here in this communication we express the establishment of milestones from various corners of the world in this field. The trend of research is also be discussed.

ANALOG & DIGITAL OPTICAL COMPUTING : INTRODUCTION

From last few decades the works/proposals referred by the scientists & technologists in the field of optical/opto-electronic parallel computation have shown a clear trend towards the goal of achieving a complete all-optical computer. Here we place a short report about these achievements. Various schemes of optical processors (logic/arithmetic/algebraic), optical storage unit, optical interconnection network, optical artificial intelligence based expert mechanism & optical neural network proposed from different corners of the world are important milestones established in the way towards its goal.

Here in this review work we collects some global & novel approaches in the field of "optical parallel computation".

OPTICAL LOGIC PROCESSOR :

Optics has already proved its strong importance in parallel computation due to its inherent parallelism. Considering this parallel behaviour there are several proposals of optical/opto-electronic logic processors around the world in last two to three decades. In this content we referred some novel work about logical processors. To implement the optical/opto-electronic logic gates, some of the proposals utilized Boolean logic,some used tristate and multi-valued logic approaches. Implementation of logic units are the primary requirement for any computer. In optical parallel computer all the above approaches & proposals exploited partially or in full the inherent parallelism of optics. The speed of operation in these schemes crosses the super-fast limit easily. Sometimes real time operation can also be required. The switching circuits here sometimes are achieved by high speed opto-electronic materials, nonlinear materials, Bi - stable materials, real time hologram etc. The references are 1-15.

OPTICAL ARITHMETIC PROCESSOR :

Arithmetic and logical unit in a computing system is one fundamental section. In digital computer this arithmetic section is generally achieved by the integration of many logic processors. Arithmetic operations in Binary system are conducted by this section.

In optical computer different concepts were proposed to implement this arithmetic unit. All those concepts tried to exploit the inherent parallelism of optics. Not only binary mechanisms are followed there. Tristate, Quarternary, Multi-valued digital mathematics are also used sometimes to get high speed arithmetic operation. Optical symbolic substitution, Residue arithmetic and many other concepts are included in many proposals by the scientists in this field. All those approaches are framed and proposed in such a way that those concepts become suitable to be accommodated in optics. These proposals are therefore very novel in respect of performing all-optical super-fast processing. The references are 16-30.

OPTICAL ALGEBRAIC PROCESSOR :

Optical algebraic processor is also an essential part in the proposed optical computer. The purpose of this algebraic processor is to perform different algebraic operations and solving the linear, differential, integral equations. Optical logic gates are also the basic building block in an optical algebraic processor, just like as it was for the arithmetic processor. In last three decades lots of proposals were seen to develop all - optical algebraic processors. These proposals followed not only the principle of conventional algebraic processors. Proposals were seen to get these processors with different angle of views to consider very high speed operation and/or real time operations. Therefore conventional approaches were not seen in many cases. The references are 31-35.

OPTICAL STORAGE MEDIUM :

Memory section is a very very important unit in any processing. In digital optical processor both 2D and 3D storage are required. Optical storage units/systems are very successful candidate to do that Optical flip-flop mechanism, holographic storage, real time holographic storage & retrieval etc. are used very nicely to store and to restore 3D and 2D optical data. Multiple storage & retrieval and conditional storage are successfully done by many proposed methods till now. We here refer few of these works. However optical storage unit can take us in a world of very high speed storage, very high speed reconstruction, very high resolution of data, multiple storage, and very high capacity of storage (i.e. number of data per unit volume can be done very very high using of some optical material). The references are 36-43.

OPTICAL INTERCONNECTS NETWORK :

To achieve a super-fast optical parallel computer one of the main feature is interconnects network. To connects the various parts of the computer the basic need is optical interconnection. These interconnection is channel less in a major portion of the computer. In channel based section fibre optic wave guide are generally used. Here optical interconnection system is capable to carry a massive data per second which is very very high than conventional cable in electronic computer.

Various aspects of optical interconnects network are proposed by the scientists & technologists and some of these proposals are given. The references are 44-50.

OPTICAL ARTIFICIAL INTELLIGENCE, EXPERT-MECHANISM & NEURAL NETWORK :

Scientists of the globe are not only thinking on the implementation of the discussed issues in optical computer. They are parallelly thinking about the expert mechanism. Optical artificial intelligent system can give light about such expert mechanism. These expert operations are followed sometimes by neural network mechanisms. Artificial intelligent system, though, is in the thought stage of the scientist, still some novel methods are proposed by them to get few expert operation with optics. However scientists & technologists believe that optical expert unit in an optical computer will be possible to be developed in future. Researches are going on this area satisfactorily. The references are 51-58.

CONCLUSION

Optical parallel computer, once upon a time, was fantacy, but now scientists and researcher believe that it will be fact in near future, where we will get very very high speed even real time operational speed from a very little volume of optical computer. Lots and lots of computational advantages will come from this computer. The global horizon of computation will be extend far and far if this computer touches the hands of human civilization. Though the paths of researchers are not kepts with roses all the times. There are many portion of paths where corns are also seen. So avioding the problems, scientists are going to the goal of optical computer slowly.

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GROWTH OF IONIZATION IN SUBNORMAL REGION IN DISCHARGE IN AIR

D.C. Jana and S.S. Pradhan
Department of Physics Technophysics,
. Vidyasagar University
Midnapur - 721102, West Bengal, India.

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Abstract:
The variation of post-breakdown ionization current with voltage in uniform field in air for different pressures has been investigated. The multiplication ratio (i/i_0) where i is the instantanious and i_0 is the initial current has been noted for four different pressures. A study of the growth of power as the inverse function of the applied field or (X / P) where X is the applied field and P is the pressure, is presented.

Kew words : Post-breakdown, ionization, multiplication ratio, growth of power.

INTRODUCTION

At a low pressure (< 1 torr) in a gas discharge just after breakdown, the current-voltage characteristics has a negative slope in the subnormal region of the discharge. The discharge current in this region fluctuates [1,2,3,4] due to motion of space charges. Subnormal discharges are still not completely understood and there are a numbr of areas in which closer investigations have been suggested [5]. It is important in the study of the ionization process which gives rise to the process of growth, of ionization.

EXPERIMENTAL PROCEDURE

The apparatus and electric circuit used is shown in figure 1. The cylindrical pyrex glass discharge tube of length 22 cm inner diameter 3.7 cm with two plane parallel Ni electrodes of diameter 3 cm, gap 16.7 cm was throughly cleaned and properly aged for the purpose. Pure and dry air was used which was passed through phosphorous pentaxide to remove traces of water vapour. To start with the pressure has been kept fixed by a micro-leak needle valve at 0.836, 0.756, 0.672, and 0.509 torr and the output voltage of the power supply adjusted so as to obtain average tube current in the circuit. The voltage across the electrodes as well as the discharge current are noted by the V T V M and nanometre and micrometre for different pressures.

RESULTS AND DISCUSSIONS

The variation of power and ion pairs per cm per torr (α / P) with ionization field (X / P) for different values of pressure is shown in figure 2 and 3. It is observed that the power, and the ion pairs per cm per torr increase gradually with the decrease of the ionization field. The ionization coefficient (number of ions produced by electron collision per unit potential drop) ion pairs per volt also as a inverse function of X / P for air in different pressures is shown in figure 4. Since the electron causes ionization by colliding with the gas molecules, the total number of electrons available will increase. The newly liberated electrons will similarly gain kinetic energy from the field and are able to ionize further. The number of electrons will increase continuously from cathode to anode; for instance, after the first ionizing collision, one electron is liberated which moves with the initial electron. These two electrons collide and ionize, liberating another two electrons. The total of four electrons again ionize the atoms, yielding a new total of eight electrons and so on the number of electrons goes on increasing. Thus due to electron multiplication for decreasing electric field per torr or lowering the ionization per volt per torr increases the power and the ion pairs per cm per torr. The increase of power with the lowering of applied voltage can thus be attributed to the rapid growth of ionization.

The avarage power decreases with the decrease of pressure which indicates that in the region of subnormal discharge multiple ionization decreases with lowering of pressure.

ACKNOWLEDGEMENT

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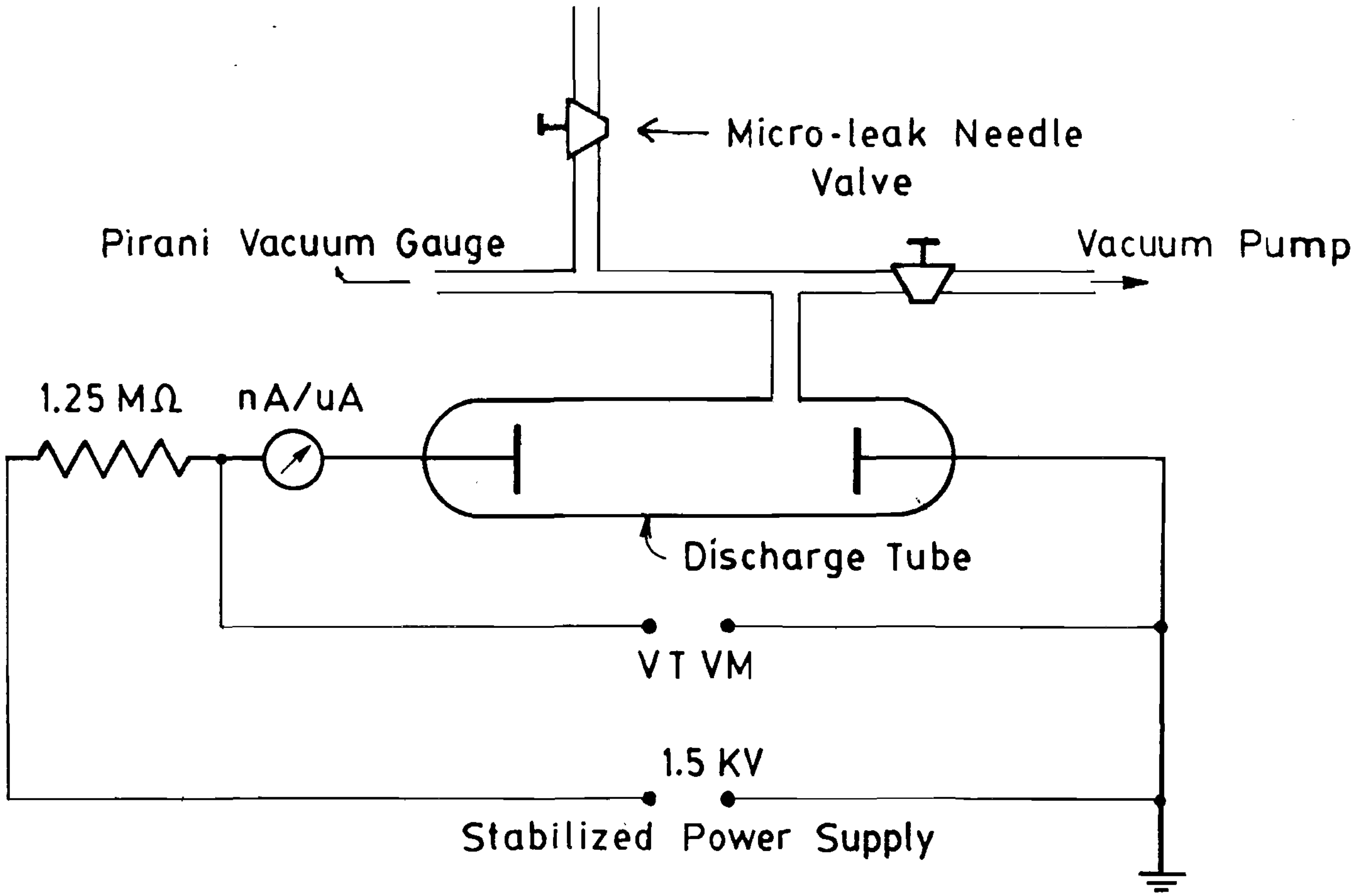

Figure 1 Apparatus and electric circuit.

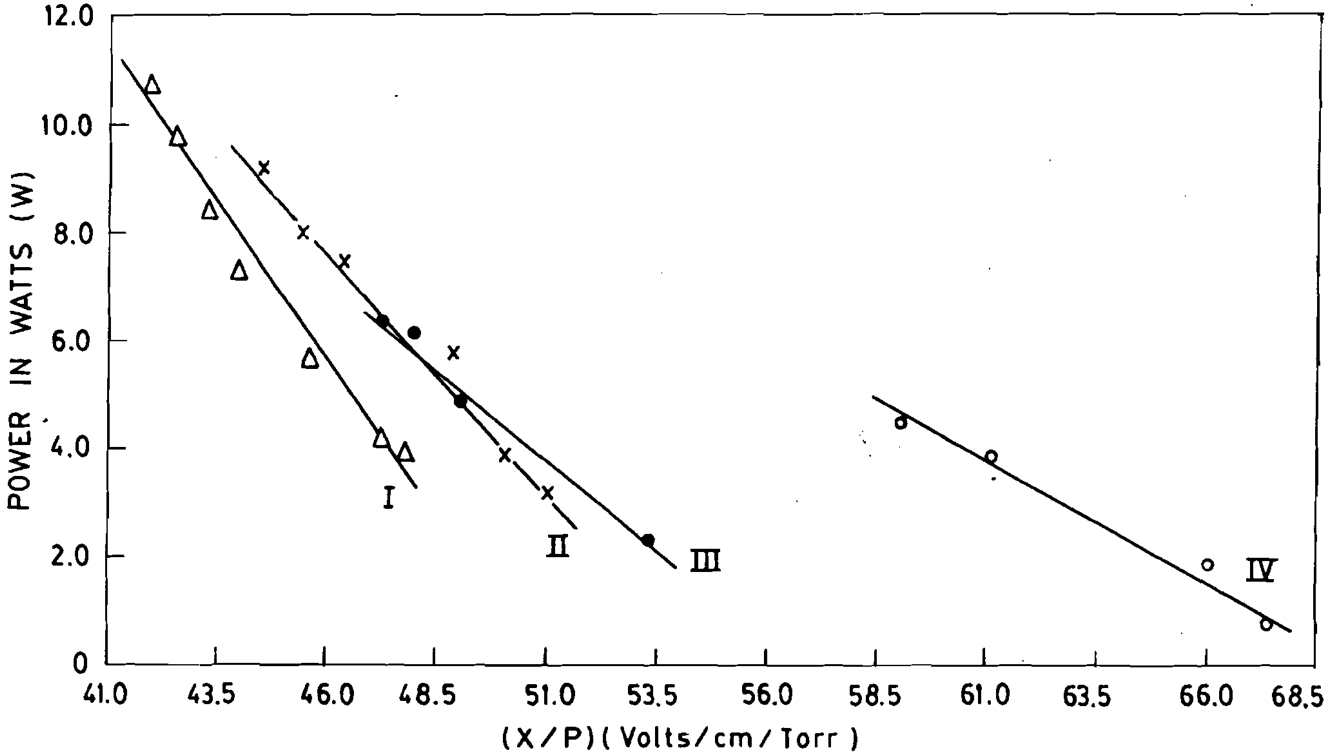

Figure 2 Variation of Power (W) with ionization field (X / P) Volt per cm per torr (I) - 0.836 torr, (II) - - - - - 0.756 torr, (III) - - 0.672 torr, (IV) - - 0.509 torr.

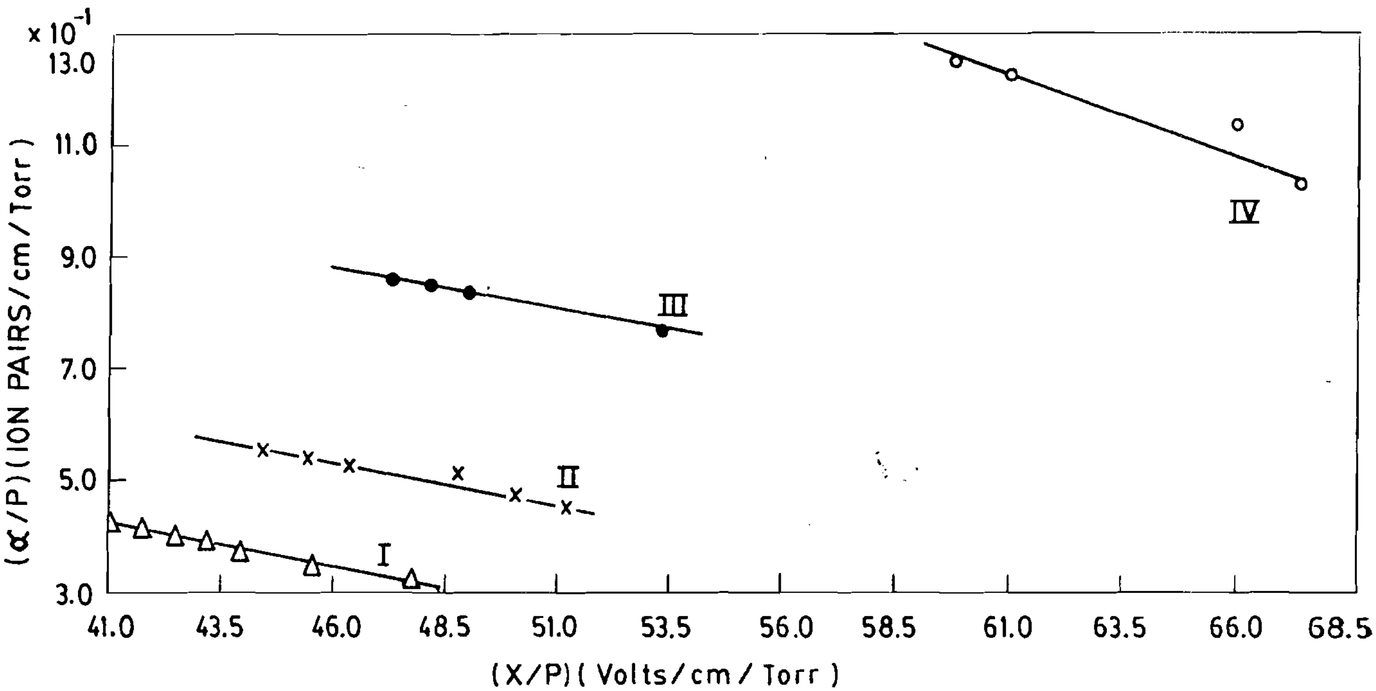

Figure 3 Variation of ion pairs per cm per torr (α / P) with ionization field (X / P) Volt per cm per torr (I) - - - - 0.836 torr, (II) - - 0.754 torr, (III) - - 0.672 torr, (IV) - - 0.509 torr.

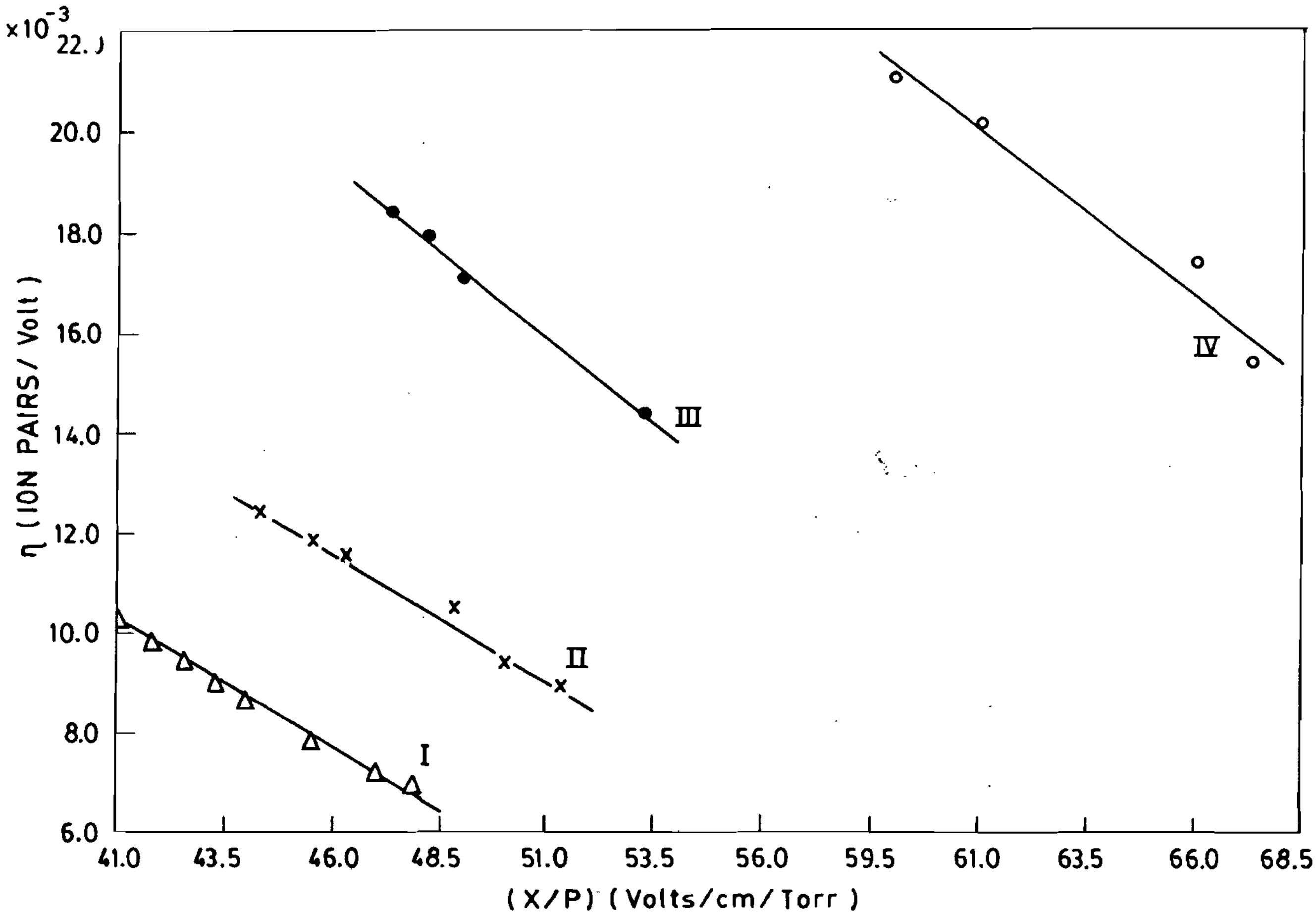

Figure 4 Variation of ionization coefficient (η) with ionization field (X / P) Volt per cm per torr for (I) - - 0.836 torr, (II)- - 0.756 torr. (III) - - 0.672 torr, (IV) - - 0.509 torr.

OPTICAL CHARACTERIZATION OF GaP AND Ge CRYSTALS

N. Karak, N.K.Ghosh, G. Maity, P. Misra and S. Saha
Department of Physics & Technophysics,,
Vidyasagar University,
Midnapur W. B..

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Abstract:
Optical absroption study has been done on GaPand Ge crystals in bisible and infrared region respectively. From the optical absorption spectrum band gap has been determined for each sample. It has been shown that phonon is involved in both of these materials during optical transition.

INTRODUCTION

The progress of solid state device technology since the invention of transistor has depended not only on the development of device concepts but also on the improvement of materials. Not only these crystals must be available in large single crystal but also the purity must be controlled within extremely close limits. Ge and GaP single crystal are two important semiconductors (1-6) as far as device is concerned. It is very important to characterize different properties of these crystals. In the present case optical characterization has been done on both these materials.

EXPERIMENTAL DETAILS

The optical absorption measurement has been done on both GaP and Ge crystals using a spectrophotometer. Before measurement both these crystals are cleaned with T.C.E. The spectrophotometer consist of a source and, light falls through a slit into a mirror and then falls on the grating. The grating is slowly rotated so that the light produces different wavelength depending on the angle. This falls onto a chopper to convert it into pulse. After crossing the chopper the light splits into two parts so that one part goes through the sample and the other part goes through the reference. Finally these two lights are received by photomultiplier tube. The absorption of light by a sample is given by the relation

Hence, $$\alpha = \frac{1}{x}\log\frac{I_0}{I} \qquad (35)$$

$$I = I_o e^{-\alpha x}$$

Hence, $$\alpha = \frac{1}{x}\log\frac{I_o}{I} \qquad (35)$$

The absorption coefficient is calculated for each wavelength of incident light. For GaP crystal the variation of wavelength is within the visible region while in case of Germanium it is in the near infrared region.

THEORY

If the bottom of the conduction band E_C occupies the position whose value of k differs from that for the top of the valence band E_V as in the case of sillicon, apart from direct (band to band) transition, indirect transition also takes place. An electron in the state $E_1(o)$ goes over to the state $E_2(o)$ absorbing a photon and then finally goes to $E_2(k_0)$ by absorbing or emitting a phonon. The electron transition takes place via an intermediate state in which the long wave phonon is transformed into a short wave phonon. In other words, the transition of electron from the valence band to the conduction band takes place at the expense of the photon energy, the change of electron momentum being compensated by the lattice. To arrive at the frequency dependence of α one should take into account the energy and quasi momentum conservation laws :

$$E_2(k_2) = E_1(k_1) + \hbar\omega \pm \hbar\omega_{phonon} \qquad (36)$$

$$K_2 = k_1 \pm K_{phonon} \qquad (37)$$

where ω_{phonon}, K_{phonon} are the frequency and the wave vector of the absorbed (plus) and emitter (minus sign) phonon. If we presume now that the perturbation includes some characteristics of the phonon, we shall have to determine the electron transition probability both by the matrix element of the perturbation due to electromagnetic field and by the matrix element of the perturbation due to the lattice.

In indirect transition the absorption coefficient for allowed transition will follow the following relation :

$$\alpha = \frac{c}{\hbar\omega}(h\omega - E_{gi} \pm \hbar\omega_{phonon})^2 \qquad (38)$$

$$(\alpha h\nu)^{1/2} = c(h\nu - E_{gi} \pm \hbar\omega_{phonon}) \qquad (39)$$

where positive sign corresponds to absorption and negative sign corresponds to emission. of Phonon. Hence the plot of $(\alpha h\nu)^{1/2}$ vs. hv will give a curve. The extrapolation of linear portion to energy axis will give indirect band gap of the material.

EXPERIMENTAL DATA FOR Ge CRYSTAL Thickness of the sample = 0.1 mm

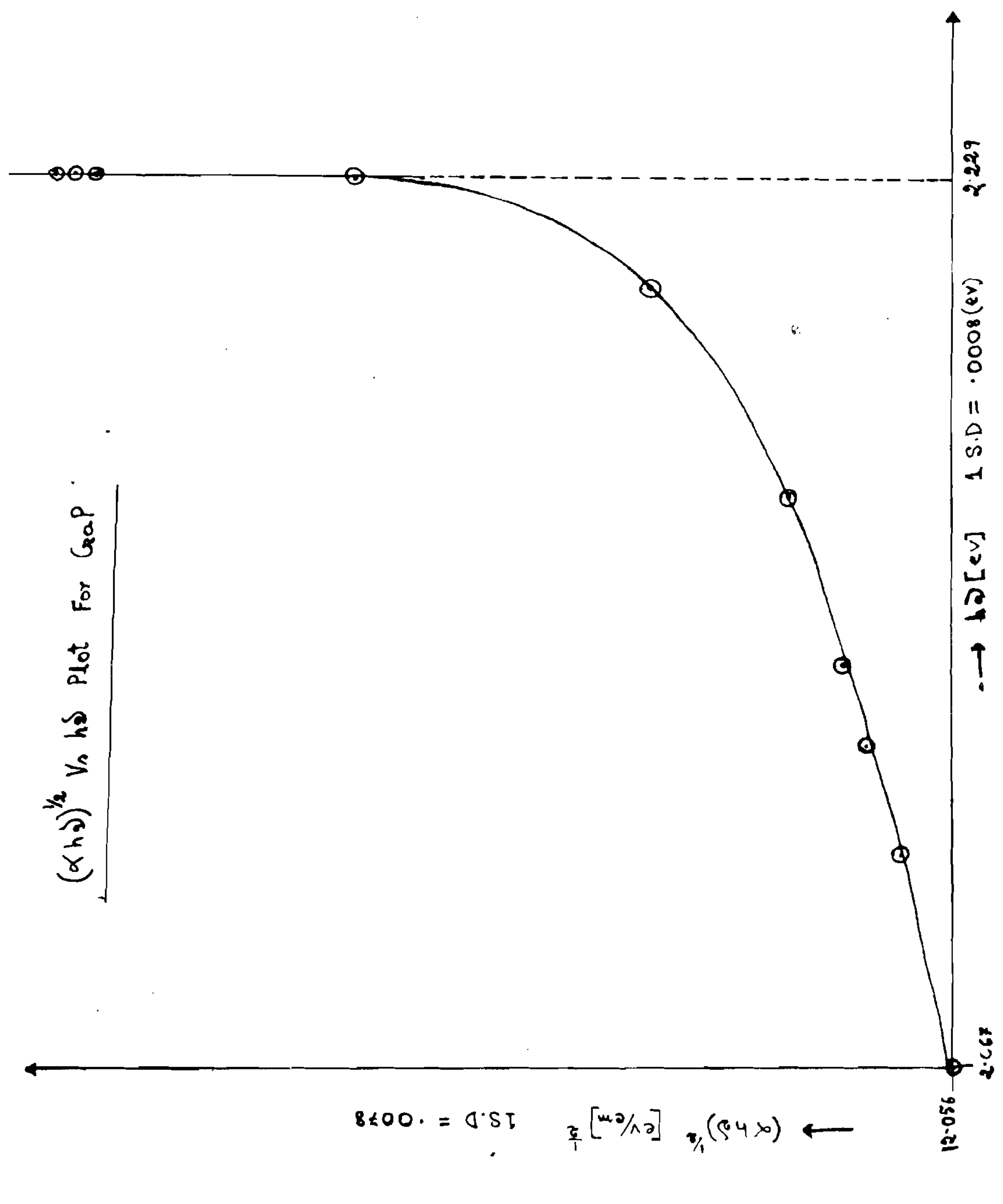
No. of Obs.	Wave length in nm.	$\log \frac{I_o}{I}$	absorption Coeff. (α) in cm.$^{-1}$	($h\nu$) in ev	$(\alpha h\nu)^{1/2}$	$(\alpha h\nu)^2$
1.	1850	0.187	1.87	0.670	1.11	1.56
2.	1800	0.229	2.29	0.689	1.25	2.48
3.	1750	0.721	7.21	0.708	2.25	26.05
4.	1740	0.570	5.70	0.712	2.01	16.47
5.	1710	0.932	9.32	0.725	2.59	45.65
6.	1680	1.522	15.22	0.738	3.35	126.16
7.	1650	2.053	20.53	0.751	3.92	273.71
8.	1620	2.355	23.55	0.765	4.24	324.56
9.	1590	2.586	25.86	0.780	4.49	406.86
10.	1580	3.063	30.63	0.784	4.90	576.66
11.	1560	3.750	37.50	0.795	5.46	888.78
12.	1550	3.750	37.50	0.805	5.47	900.00
13.	1540	3.750	37.50	0.805	5.49	911.28

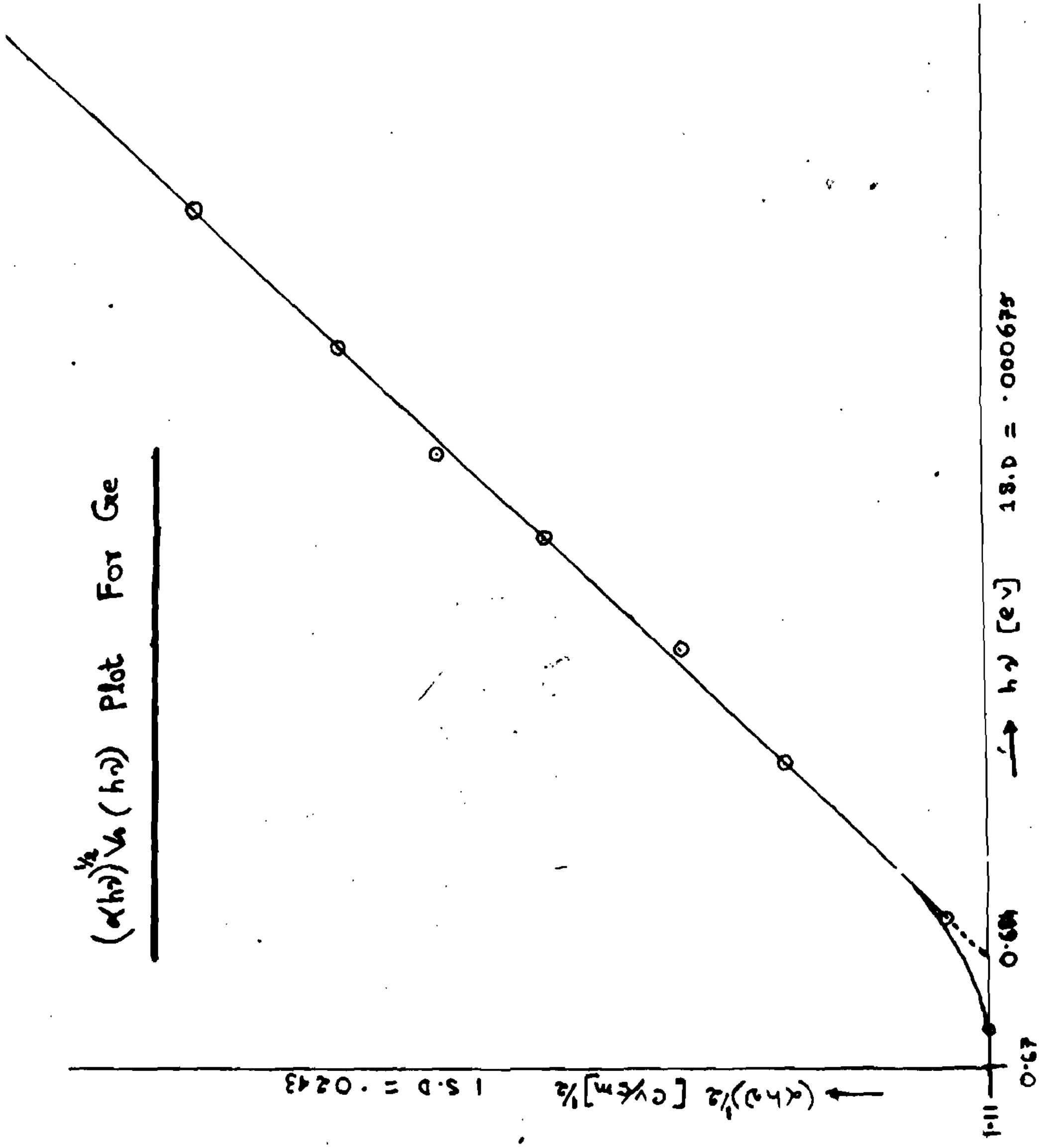
CALCULATION OF BAND GAP ENERGY (E_G) [FROM THE EXTRAPOLATION OF THE LINEAR PORTION OF $(\alpha h\vartheta)^{1/2}$ VS. $(h\vartheta)$ PLOT] Eg = 0.684ev.

EXPERIMENTAL DATA FOR GaP CRYSTAL - Thickness of the sample = 0.5 m.m.

No. of Obs.	Wave length in nm.	$\log \frac{I_o}{I}$	absorption Coeff. (α) in cm.$^{-1}$	$(h\vartheta)$ in ev	$(\alpha h\vartheta)^{1/2}$	$(\alpha h\vartheta)^2$
1.	600	3.516	70.32	2.067	12.056	21127.04
2.	595	3.465	69.30	2.084	12.017	20857.48
3.	590	3.516	70.32	2.102	12.157	21848.57
4.	585	3.516	70.32	2.120	12.209	2222.36
5.	581	3.516	70.32	2.134	12.250	22518.86
6.	573	3.495	69.90	2.164	12.298	22880.67
7.	571	3.474	69.48	2.171	12.281	22753.03
8.	567	3.694	73.98	2.187	12.719	26177.38
9.	563	3.914	78.28	2.202	13.129	29712.29
10.	561	4.185	83.70	2.210	13.600	34216.29
11.	561	4.188	83.76	2.210	13.605	34265.56
12.	561	4.200	84.00	2.210	13.624	34462.20

CALCULATION OF BAND GAP ENERGY (E_G) [FROM THE EXTRAPOLATION OF THE LINEAR PORTION OF $(\alpha h\nu)^{1/2}$ VS. $(h\nu)$ PLOT] Eg = 2.229 ev.


(αhν)^1/2 Vs hν Plot For GaP
→ hν [ev]
1 S.D = ·0008 (ev)
2·229
← (αhν)^1/2 [ev/cm]^1/2
1S.D = ·0078
12·056


(αhν)^½ Vs (hν) Plot For Ge
hν [ev]
1S.D = ·000675
(αhν)^½ [ev/cm]^½
1 S.D = ·0243
0·64
0·67
1·11

RESULTS & CONCLUSION

Absorption coefficient is calculated for each wavelength from the absorbance data. Now simultaneously $(\alpha h\nu)^{1/2}$ and $(\alpha h\nu)^2$ is calculated from the experimental data for each wavelength. But $(\alpha h\nu)^2$ and Vs $h\nu$ plot do not satisfy the experimentally observed points. Hence it can be concluded that both Gap and Ge crystals are not direct band gap materials. But it is clearly observed that in both cases $(\alpha h\nu)^{1/2}$ Vs $h\nu$ plot will match all the points in the curve. Hence it can be concluded that both GaP and Ge crystal are indirect band gap materials. The intercept of the linear portion on the energy axis will give the corresponding indirect band gaps in GaP in and Ge crystals. In case of GaP the energy in found to be 2.229 ev which is smaller then the actual band gap by .011 ev. Hence phonon is absorbed in the case and the phonon energy is .011 ev. While in case of Ge the energy is found to be .684 ev which is greater than the actual band gap by .024 ev. Therefore phonon is emitted in this case and the phonon energy is 024 ev. Hence from the optical study the band gaps of these two materials are accurately calculated and also the involvement of phonon during the transition is also confirmed.

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A FUZZY INVENTORY MODEL OF DAMAGABLE ITEMS WITH VARIABLE REPLENISHMENT AND SHORTAGE UNDER SPACE CONSTRAINT

M. Mandal and M. Maiti*
+Department of Mathematics, Midnapore College,
Midnapore - 721 101, India.

* Department of Applied Mathematics with Occanology and Computer Programming,
Vidyasagar University,
Midnapore - 721 102, India

Abstract:
A profit maximization inventory model for damagable items with inventory dependent replenishment and shortages is developed in fuzzy environment. Here, a constraint is imposed on the available space for inventory storage and selling price is dependent on purchasing price. The fuzzy environment is created making the inventory costs, purchasing price, demand, rate of damagability, storage space and rate of replenishment imprecise and vague in non-stochastic sense. The imprecisenss of these paremeters are expressed by linear membership functions. The fuzzy model is solved by fuzzy non linear programming (FNLP) method and illustrated with numerical examples. A sensitivity analysis due to the profit goal and warehouse space has been also presented.

Kew words : Inventory, Variable Production, Damagable Items, Fuzzy Programming.

INTRODUCTION

In classical inventory models, normally rate of replenishment/production is assumed to be a constant. In practice, in many situations, the production/replenishment rates are either demand or stock dependent. If the stock in the godown is less or reaches bottom-level, managerial action is taken to increase the production/replenishment. If the stock accumulated in the warehouse is more, then the process of production/replenishment is made slower to avoid unnecessary idle inventory. Again, there is also a relationship between replenishment and demand though it is different from the above and production/replenishment goes up and down directly with demand. Inventory policy with finite (constant) and infinite replenishemnt is available in the classical books like Hadley and Whitin [1], Naddor [2], Arrow, Kerlin and Scarf [3], etc. A very few O.R. scientists have considered the inventory problems with variable replenishment/production. Goswami and Chaudhuri [4] solved an inventory problem of deteriorating items with production rate as $K(t)=\beta D(t)$. Recently, Bhunia and Maiti [5] considered two models-one with K (t)= $\alpha+\beta$ I (t) and other with $K(t)=\alpha+\beta D(t)$. Later, Balkhi and Benkherouf [6] and Mandal and Maiti [7] took the production to be of the forms $K(t)=\alpha+\beta D(t)-\gamma I(t)$, and $K(t)=\alpha+\beta D(I(t))$ respectively where I(t) is the inventory, D (t) the demand and K(t) the rate of replenishment at time, t.

Now-a-days, a product is promoted in the society through advertisements in the modern mass/electronic media and/or by the attractive display of units in the show-rooms at the marketting places. Glamorous display in large numbers by modern light and electronic arrangements attracts the people and brings more customers for the item. According to Levin et al [8]? " At times, the presnece of inventory has a motivatinal effect on the people around it. It is a common beliet that large piles of goods displayed in a supermarket will lead the customer to buy more". Marketting research also supports this relationship and due to this fact, operations research scientists have recently concentrated on analysing the inventory problems taking the effect of exihibited inventory on demand into account. Some of them (Baker and Urban [9], Datta and Pal [10], Urban [11], Pal, Goswami and Chaudhuri [12], etc.) took the demand to be the power function of on-hand inventory and others (Gupta and Vart [13], Mandal and Phaujdar [14], etc.) considered linear stock dependent demand.

In reality, many physical goods deteriorate due to dryness, spoilage, vaporization ect. during long storage period. The deterioration always depends upon the preserving facilities and environmental conditions of the warehouse. So, due to deterioration, a certain fraction of items fails to satisfy the customers. A number of research papers has already been published for deteriorating items with stock-dependent demand (cf. Dave and Patel [15], Sachan [16], Goswami and Chowdhury [4], Mandal and Phaujdar [17] etc.) But, criteria of breakability for the items made of glass, ceramics, Chinaclay, etc. is different from the deterioration criteria of food grains, fruits, etc.. During inventory, glass, ceramics and chain-clay items are normally placed one upon the other and kept as a heaped stock and as a result, an item gets damaged mainly due to the accumulated stress of the stock-placed above it. Only Mandal and Maiti [7] have recently considered an inventory model of these items. Till now, none has tried to incorporate the stock-depndent replenishment /production rate in the inventory models of these damagable items.

In real life, the inventory paremeters-demand, holding cost, set-up cost, purchasing price, storage area, production rate and the rate of damagability are normally imprecise, vague and flexible in nature. Their values, instead of being fixed, may vary with-in some ranges. In our complex world, there exits so many factors for which the purchasing prices changes from day to day, but it varies with-in a reasonable range. Similarly, demand, holding cost, set-up cost, purchasing price, production rate, the rate of damagability are imprecisely and vaguely defined with-in some ranges. These paremeters can be realistically represented in fuzzy environment by fuzzy linear or non-linear membership functions and then the problem is solved by fuzzy non-linear programming (FNLP) technique.

Zadeh [18] first introduced the concept of fuzzy set theory, For further discussion about fuzzy set theory, the reader may refer to Kaufmann and Gupta [19,20], Zimmermann [21] and Dubois and Prade [22]. The first fuzzification of a linear programming problem was due to Zimmermann [23], he applied fuzzy set theory concept with suitable choices of membership functions and derived a fuzzy linear programing problem. He showed that the solution obtained by FNLP method is always an efficient solution and this methodology also gives an optimal comprise solution.

To the best of our knowledge, very few inventory models have been developed in fuzzy environment. Park [24] examined the EOQ formula with fuzzy inventory costs represented by Trapezoidal fuzzy member [Tr FN]. Recently Roy and Maiti [25,26] considered two EOQ models under limited storage capacity; Mandal, Roy and Maiti [27], Roy and Maiti [28] examined two fuzzy inventory models of deteriorating items. Roy and Maiti [29] also considered an inventory model with fuzzy demand and fuzzy cost. But all the above inventory models are with-out variable production and shortages.

In this paper, under limited storage area, a profit making inventory model of damagable items with inventory dependent production and shortages is formulated in fuzzy environment. Here selling price is linearly dependent on purchasing price . The profit goal, rate of production, purchasing price, set-up cost, holding cost, shortage cost, total storage area and rate of damagability are fuzzy in nature. These fuzzification of the model, it is solved by FNLP method. The model is illustrated numerically. A sensitivity analysis on fuzzy goals is presnted with different tolerances on profit and storage area.

MODEL FORMULATION

1. Assumptions and Notations :

An inventory model of damagable-items with variable rate of replenishment, purchasing price dependent selling price, constant demand, shortages and the constraint on storage space is developed under the following assumptions.

n=number of items,
W=available floor-space or shelf-space,
PF (Q, Q_l)=total average profit of the system,
(where Q and Q_l is the vector of n decision variables Q_i andQ_{li}, i=1,2,.....n. respectively),

For the i-th item, (i=1,2,3,4,......,n), let
Q_i=total inventory.
Q_{li}=highest stock level,
p_i=purchase cost per unit item,
s_i=selling price per unit item=$m_i p_i$, m_i is mark-up,
c_{1i}=inventory holding cost per unit item per unit time,
c_{2i}=inventory shortage cost per unit quantity per unit time,
c_{3i}=set-up cost per period,
a_i=rate of damagablity per unit time at time t,
w_i=space area required for per unit quantity,
T_i=time period per cycle,
$K_i(q_i)$=replenishment rate at time t, given by

$$K_i(q_i) = \alpha_i + \beta_i q_i(t), for q_i(t) \geq 0$$
$$= \alpha_i, for q_i(t) < 0$$

where $q_i(t)$ is the inventory level at time t, $\alpha_i and \beta_i$ are constant; $a_i > 0, 0 \leq \beta_i < 1$.
Hence, replenishment rate is on-hand inventory dependent when q_i (t) ≥ 0 and it becomes constant when $q_i(t) < 0$,
D_i= consumption rate which is constant,
shortages are allowed and fully back-logged,
lead time is zero,
the time horizon of the inventory system is infinite.
breakable units B(q) is a known function of current stock level and $B(q_i)=a_i q_i^{\gamma}, 0 < a_i < 1, 0 \leq \gamma \leq 1$

2. Mathematical Formulation

If $q_i(t)$ is the inventory level at time t of the i-th item, then

$$\begin{aligned} \frac{dq_i}{dt} &= K_i(q_i) - D_i - B(q_i), 0 \leq t \leq t_{1i} \\ &= -D_i - B_i(q_i), t_{i1} \leq t \leq t_{1i} + t_{2i} \\ &= -D_i, t_{1i} + t_{2i} \leq t \leq t_{1i} + t_{2i} + t_{3i} \\ &= K_i(q_i) - D_i, t_{1i} + t_{2i} + t_{3i} \leq t \leq t_{1i} + t_{2i} + t_{3i} + t_{4i} \end{aligned} \quad (1)$$

So the length of the cycle for i-th item is

$$T_i = t_{1i} + t_{2i} + t_{3i} + t_{4i} \quad (2)$$

where

$$t_{1i} = \int_0^{Q_{1i}} \frac{dq_i}{K_i - D_i - B_i(q_i)}$$
$$t_{1i} = \int_0^{Q_{1i}} \frac{dq_i}{D_i + B_i(q_i)}$$
$$t_{1i} = \int_0^{-(Q_i - Q_{1i})} \frac{dq_i}{D_i(q_i)}$$
$$t_{1i} = \int_{-(Q_i - Q_{1i})}^{0} \frac{dq_i}{K_i - D_i}$$

The holding cost for each cycle of i-th item is given by $C_{1i}G_i(Q_{1i})$, where

$$G_i(Q_{1i}) = \int_0^{Q_{1i}} \frac{q_i dq_i}{K_i - D_i - B_i(qi)} + \int_0^{Q_{1i}} \frac{q_i dq_i}{D_i + B_i(qi)} \tag{3}$$

The total noumber of damage units are

$$\theta_i(Q_{1i}) = \int_0^{Q_{1i}} \frac{B_i(q_i)dq_i}{K_i - D_i - B_i(q_i)} + \int_0^{Q_{1i}} \frac{B_i(q_i)dq_i}{D_i + B_i(q_i)} \tag{4}$$

The total shortage cost are $C_{3i}\phi_i(Q_i, Q_{1i})$, where

$$\phi_i(Q_i, Q_{1i}) = \int_0^{-(Q_i - Q_{1i})} \frac{q_i dq_i}{D_i} + \int_{-(Q_i - Q_{1i})}^{0} \frac{q_i dq_i}{K_i + D_i} \tag{5}$$

Net revenue from the firt cycle is

$$\psi(Q_i, Q_{1i}) = (s_i - p_i)(\psi(Q_i, Q_{1i}) - \theta_i(Q_{1i})) - p_i\theta_i(Q_{1i}) \tag{6}$$

The total average profit is given by

$$PF(Q_i, Q_{1i}) = \sum_{i=1}^{n} [N_i(Q_i - Q_{1i}) - C_{3i} - C_{1i}G_i(Q_{1i}) - C_{2i}\phi_i(Q_i, Q_{1i})] / T_i \tag{7}$$

Model 1. When $\gamma = 1, B_i(q_i) = a_i q_i i,e$ for linear damage function, from equations (2),(3),(4)(6),(7), we have

$$T_i = t_{1i} + t_{2i} + t_{3i} + t_{4i}$$

$$t_{1i} = \frac{1}{a_i + \beta_i} \log \frac{\alpha_i - D_i - (a_i + \beta_i)Q_{1i}}{\alpha_i - D_i}$$

$$t_{2i} = \frac{1}{a_i} \log \frac{D_i + a_i Q_{1i}}{\alpha_i - D_i}$$

$$t_{3i} = \frac{Q_i - Q_{1i}}{D_i}$$

$$t_{4i} = \frac{Q_i - Q_{1i}}{\alpha_i + D_i}$$

$$G_i(Q_{1i}) = \frac{Q_{1i}}{a_i+\beta_i} - \frac{\alpha_i - D_i}{(a_i+\beta_i)}\log\frac{\alpha_i - D_i - (a_i+\beta_i)Q_{1i}}{\alpha_i - D_i} + \frac{Q_{1i}}{a_i} - \frac{D_i}{a_i^2}\log\frac{D_i + a_i Q_{1i}}{D_i}$$

$$\theta_i(Q_{1i}) = a_i G_i(Q_{1i})$$

$$\phi_{1i}(Q_i, Q_{1i}) = \frac{(Q_i - Q_{1i})}{2D_i}\frac{(\alpha_i - D_i)\beta_i t_{1i}}{a_i - \beta_i}$$

$$\psi_i(Q_i, Q_{1i}) = \alpha_i t_{1i} + \alpha_i t_{4i} + \frac{\beta_i Q_{1i}}{a_i+\beta_i} - \frac{(\alpha_i - D_i)\beta_i t_{1i}}{\alpha_i + \beta_i}$$

Model - 2. When $\gamma = 1/2, B_i(q_i) = a_i\sqrt{q_i}$ i,e, for non-linear damage function, we get from equations (2),(3),(4),(5),(6),(7),

$$T_i = t_{1i} + t_{2i} + t_{3i} + t_{4i}$$

$$t_{1i} = \frac{1}{\beta_i}\log\frac{(\alpha_i - D_i - \beta_i)Q_{1i} - a_i\sqrt{Q_{1i}}}{\alpha_i - D_i} + \frac{a_i}{\beta_i X_1}\log\frac{(X_1+Y_1)(X_1-a_i)}{(X_1-Y_1)(X_1+a_i)}$$

$$t_{2i} = \frac{2}{a_i}\left(\sqrt{Q_{1i}} - \frac{D_i}{a_i}\log\frac{D_i + a_i\sqrt{Q_{1i}}}{D_i}\right)$$

$$t_{3i} = \frac{Q_i - Q_{1i}}{D_i}$$

$$t_{4i} = \frac{Q_i - Q_{1i}}{\alpha_i - D_i}$$

$$G_i(Q_{1i}) = I_1 + I_2$$

$$I_1 = \frac{1}{\beta_i}((\alpha_i - D_i)t_{1i} - Q_{1i}I_3)$$

$$I_2 = \frac{2}{a_i}\left(\frac{Q_{1i}^{3/2}}{3} - \frac{D_i Q_{1i}}{2a_i} + \frac{D_i^2\sqrt{q_{1i}}}{a_i^2} + \frac{D_i^3}{a_i^3}\log\frac{D_i + a_i\sqrt{Q_{1i}}}{D_i}\right)$$

$$I_3 = \frac{2a_i}{\beta_i}\left(\frac{\alpha_i - D_i}{X_1}\log\frac{(X_1 + Y_1(X_1 - a_i)}{(X_1 - Y_1)(X_1 + a_i)} - \frac{a_i^2 t_{1i}}{\beta_i}\right)$$

$$\theta_i(Q_{1i}) = I_3 + I_4$$

$$I_4 = Q_{1i} - D_i t_{2i}$$

$$\phi_i(Q_i, Q_{1i}) = \frac{(Q_i - Q_{1i})}{2D_i} - \frac{(Q_i - Q_{1i})^2}{2(\alpha_i - D_i)}$$

$$\phi_i(Q_i, Q_{1i}) = \alpha_i t_{1i} + \alpha_i t_{4i} + \beta_i I_1$$

$$X_1 = \sqrt{4\beta_i(\alpha_i - D_i) + a_i^2}$$

$$Y_1 = 2\beta_i\sqrt{Q_{1i}} - a_i$$

Crisp Problem :
Hence our problem is to maximize the total average profit under the limitation of total storage area.

$$MaxPF(Q,Q_1) = \sum_{i=1}^{n} PF(Q_i, Q_{1i}) \tag{8}$$

subject to

$$\sum_{i=1}^{n} w_i Q_i \le W, Q_{1i}, Q_i > 0, Q_i > Q_{1i}, i = 1,2,3,\ldots,n.$$

Fuzzy Problem :

When the inventory parameters such as purchasing price, set-up cost, holding cost, storage area, the investment cost, shortage cost, rate of replenishment, rte of damage, consumption rate and objective goal are fuzzy, the said crisp problem (8) is transformed to a fuzzy problem and is represented as

$$\tilde{M}axPF(Q,Q_1) = \sum_{i=1}^{n}(m_i\tilde{p}_i Q_i - \tilde{a}_i G_i(Q_i) - \tilde{c}_{1i} G_i(Q_i) - \tilde{c}_{2i}\phi_i(Q_i, Q_{1i}) - \tilde{c}_{3i} - \tilde{p}_i Q_i / T_i \tag{9}$$

subject to

$$\sum_{i=1}^{n} w_i Q_i \le \tilde{W}, Q_{1i}, Q_i > 0, Q_i > Q_{1i}, i = 1,2,\ldots,n.$$

(wave bar '~' represents fuzziness of the paremeter)

MATHEMATICAL ANALYSIS

Fuzzy non-linear programming technique
FNLP algorithm has been used here to solve fuzzy inventory problem (9) with single objective. In fuzzy set theory, the fuzzy objective, constraints, costs rate of deterioration and rate of back-logging are defined by their membership functions which may be linear or non-linear. Following to Zimmermann (21), the linear membership functions are

$$PF\mu(Q,Q_1) = 0 \quad \text{for } PF(Q,Q_1) \le B_0 - P_{PF} \qquad (10)$$

$$= 1 - \left(\frac{B_O - PF(Q,Q_1)}{P_{PF}}\right) \quad \text{for } B_0 - P_{PF} < PF(Q,Q_1) \le B_o$$

$$= 1 \quad \text{for } PF(Q,Q_1) > B_0$$

$$w\mu(Q) = 1 \quad \text{for } \sum_{i=1}^{n} w_i Q_i \le W$$

$$= 1 - \left(\frac{\sum_{i=1}^{n} w_i Q_i - W}{pw}\right) \quad \text{for } W \le \sum_{i=1}^{n} w_i Q_i \le W + P_W \qquad (11)$$

$$= 0 \quad \text{for } \sum_{i=1}^{n} w_i Q_i > W + P_W$$

$$D_i\mu(u) = 1 \quad \text{for } u \le D_i$$

$$= 1 - \left(\frac{u - D_i}{P_{Di}}\right) \quad \text{for } D_i \le u \le D_i + P_{Di} \qquad (12)$$

$$= 0 \quad \text{for } u > D_i + P_{Di}$$

$$\mu_{pi}(u) = 1 \qquad \text{for } u \le p_i$$

$$= 1 - \left(\frac{u - p_i}{P_{pi}}\right) \qquad \text{for } p_i \le u \le p_i + p_{pi} \qquad (13)$$

$$= 0 \qquad \text{for } u > p_i + p_{pi}$$

$$\mu_{cli}(u) = 1 \qquad \text{for u >c}_{li}$$

$$= 1 - \left(\frac{c_{li} - u}{p_{cli}}\right) \qquad \text{for } c_{li} - p_{cli} < u < c_{li} \qquad (14)$$

$$1 = \qquad \text{for } u \le c_{li} - p_{cli}$$

l=1,2,3

$$\mu_{\beta i}(u) = 1 \qquad \text{for } \mu > \beta_i$$

$$= 1 - \left(\frac{\beta_i - u}{p_{\beta i}}\right) \qquad \text{for } \beta_i - p_{\beta i} \le u < \beta_i \qquad (15)$$

$$= 0 \qquad \text{for } u \le \beta_i - P_{\beta i}$$

Here, objective goal, total storage area, purchasing price, set-up cost, holding cost, shortage cost, rate of damage and rate of replenishment are respectively $B_o, W, p_i C_{3i} c_{li}, c_{2i}, a_i, \beta_i$ having corresponding tolerances $P_{PF}, P_W, P_{pi}, P_{c3i}, P_{cli}, P_{2i}, P_{ai} and p_{\beta i}$ which are positive real numbers. Using the above membership functions, the fuzzy problem (9) is transformed to an equivalent crisp problem,

Max α
subject to

$$(1+\frac{B_o - PF(Q_i,Q_{li})}{P_{PF}}) \geq \alpha \qquad (16)$$

$$\sum_{i=1}^{n} w_i Q_i \leq W + (1-a)P_W,$$

$$Q_i, Q_{li} > 0, Q_i > Q_{li}, i = 1,2,\ldots\ldots,n, \alpha \in [0,1]$$

where

$$PF(Q,Q_1) = \sum_{i=1}^{n} (m_i \mu_{pi}^{-1}(\alpha)Q_i - \mu_{ai}^{-1}(\alpha)\theta_i(Q_i) - \mu_{cli}^{-1}(\alpha)\phi_i(Q_i) - \mu_{c2i}^{-1}(\alpha)\phi_i(Q_i) - \mu_{c3i}^{-1}(\alpha)) / T_i$$

Here, P_{PF} is the minimum and P_W is the maximum acceptable-violation of the aspiration level PF and W respectively.

NUMERICAL EXAMPLES

Here, we solve both crisp and fuzzy problems given respectively by (8) and (16) for some numerical values of the inventory parameters. The non-linear programming problems (8) and (16) are solved by a computer algorithm based on gradient search technique (generalised reduced gradient method) for the following numerical data.

For all models, let us assume, n=2, a_1=150, a_2= 160, c_{11}=Rs. 60, c_{12}=Rs. 50, c_{21}=Rs. 4.00, c_{22}=Rs. 3.70, c_{31}=Rs. 50.00, c_{32}=Rs. 45.00, m_1=1.20, m_2=1.20, D_1=40, D_2=50,p_1=Rs 8.0, p_2=Rs 9.0, β_1=.50, β_2=.05, a_1=0.1, a_2=0.12, w_1=0.5 sq. ft., w_2=0.6 sq. ft.

(a) Crisp Problem :
Solving the models (1) and (2) with the above parametric values, we get the following optimal values

PF* = Rs 46.20, $Q_1^* = 75.93, Q_{11}^* = 47.76, Q_2^* = 38.91, Q_{21}^* = 15.30$ for model-1,

and PF* = Rs 52.32, Q_1^* = 87.17, Q_{11}^* = 65.97, Q_2^* = 62.28, Q_{21}^* = 41.57 for model - 2

(b). Fuzzy Problem :

To illustrate the above fuzzy problem, we assume that all the crisp paremeter's values remain same. With P_{PF}=Rs35.00, B_0=Rs 70.00, D_1=4, D_2=6, P_w=20, p_{p1}=Rs3, p_{p2}=Rs 2, P_{a1}=.04, P_{a2}=Rs.02, P_{c11}=.2,P_{c12}=.3, P_{c31}=Rs8.0, P_{c32}=Rs9, $P\beta 1=.03$, $P_{\beta 2}=0.2$, we obtain the following optimal values solving the models-1 and -2 by FNLP method.

Model-1

PF*=Rs64.70, $p_1^* = Rs8.45$,

$p_2^* = Rs9.3, s_1^* = Rs10.15, s_2^* = Rs11.16, c_{11}^* = Rs0.57, c_{12}^* = Rs0.46,$

$c_{21}^* = Rs3.92, c_{22}^* = Rs3.64, c_{31}^* = Rs48.79, c_{32}^* = Rs42.60,$

$W^* = 33.03, a_1^* = 0.09, a_2^* = 0.11, D_1^* = 44.62, D_2^* = 52.91, a^* = 0.85, \beta_1^* = .052, \beta_2^* = .045.$

$Q_1^* = 80.04, Q_{11}^* = 49.55, Q_2^* = 39.18, Q_{21}^* = 13.76.$

Model - 2 :

$PF^* = Rs66.27, P_1^* = Rs8.62, P_2^* = Rs9.32, S_1^* = Rs10.21, s_2^* = Rs11.20, C_{11}^* =$

$Rs0.56, C_{12}^* = Rs0.456, C_{21}^* = Rs3.87, C_{22}^* = Rs3.52, C_{31}^* = Rs48.59, C_{32}^* =$

$Rs42.26, W^* = 35.03, a_1^* = 0.087, a_2^* = 0.107, D_1^* = 45.23, D_2^* = 54.11, \alpha^* =$

$0.82, \beta_1^* = .053, \beta_2^* = .044. Q_{11}^* = 72.61, Q_2^* = 69.32, Q_2^* = 49.16$

SENSITIVITY ANALYSIS

Now, we perform some sensitivity analyses upon the profit goal and warehouse space in the fuzzy model due to the changes in the tolerance limits of P_{PF} andP_W and the results are given in Table-1 and Table-2 respectively.

Table-1

Effect due to incremental changes of P_{PF} on PF and W

P_{PF}	PF	W	a*	Model-1
25	65.88	33.29	0.835	,,
30	65.27	33.15	0.84	,,
35	64.70	33.03	0.846	,,
40	64.18	32.91	0.85	,,
45	63.40	34.70	0.834	,,
25	67.10	35.29	0.08	Model-2
30	66.67	35.15	0.815	,,
35	66.27	35.03	0,82	,,
40	65.82	34.91	0.827	,,
45	65.40	34.70	0.834	,,

Table-2
Effect due to incremental changes of P_W on PF and W

P_w	PF	W	a*	Model-1
15	64.63	32.30	0.847	,,
20	64.70	33.03	0.849	,,
25	64.76	33.74	0.85	,,
30	64.81	34.45	0.852	,,
35	64.85	35.15	0.853	,,
15	66.19	34.30	0.817	Model-2
20	66.27	35.30	0,82	,,
25	66.37	35.84	0.825	,,
30	66.45	36.60	0.831	,,
35	66.60	37.32	0.842	,,

CONCLUSION

A real life inventory problem with fuzzy inventory parameters have been solved by FNLP method. This derivation and procedure are quite general and can be used for the inventory models with price discount, etc.Different membership functions can be tried to fit the variations of inventory parameters realistically and then can be used for analysis. For these types of problems, fuzzy numbers also can be used and optimum results may be obtained applying iteration method and arithmetic operations on fuzzy numbers. These models can also be formulated in fuzzy-stochastic environment and solved by using this method with some modification.

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SENSITIVITY ANALYSIS IN LINEAR FRACTIONAL PROGRAMMING PROBLEM

Md. T. Hossain and Tapan K. Pal
Department of Applied Mathematics with Oceanology
and Computer Programming,
Vidyasagar University, Midnapore - 721 102, India
July 21, 1999

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Abstract:
Sensitivity analysis have been studied on the new method developed in [1] for solving linear fractional programming problem. The range of the coefficients in the objective function as well as the components of the requirement vector are obtained to maintain the optimality conditions when the components are changed one at a time. Final optimal table is used efficiently to study this sensitivity analysis.
The advantage of this method is that in each step the number of variables remains always same as the number of original variables. The less number of variables leads the solution of the problem in less number of calculations.

Kew words : Linear Programming, Fractional Programming, optimal solution Sensitivity Analysis.

INTRODUCTION

The linear fractional programming problems (LFPP) are frequently encountered in many real life problems. It is used in finance as to minimize debt to equity ratio, maximize return on investment, minimize actual cost to standard cost, maximize actual capital to required capital. In health care, it is used to minimize cost to bed, nurse to doctor and doctor to patient ratios. In university planning, we may have student teacher ratios, tenured to non tenured faculty ratios and so forth. The fractional programming method is useful in solving the problem in Economics whenever the different economic activities utilize the fixed resources in proportion to the level of their values. In addition to these fractional objectives occur in many other areas also.

In most of the above mentioned problems, one becomes interested not only to find the optimal solutions, but also to know how the solution changes when the different parameters of the problem change. Moreover, in practice the values of these parameters are seldom known precisely as most of them are functions of some uncontrollable parameters. Hence the solution of a practical problem is not complete with the mere determination of the optimal solution of a fixed problem but it is necessary to study how the current optimal solution changes with the possible change of the parameters. Thus the of study of sensitivity analysis is important for fractional programming problem.

In all the available methods [2,3,4,5,6] to find solution of fractional linear programming problems the number of variables in each table are greater than the number of original variables (as it includes all slack, surplus and artificial variables.) But in [1] a method has been developed to solve LFPP which contains the number of variables in all the steps as the same number of original variables. Because of less number of variables, the solution is achieved in less number of computations in comparison to the available methods. For a maximization problem in each iteration of this method one variables is removed and in its place one new variable is introduced in such a manner that the value of the objective function becomes more at the new origin than its value at the old origin if the old origin be in the feasible region. On the other hand if the old origin be not in the feasible region, the outgoing and incoming variables are chosen in such a way that the new origins gradually approach to the feasible region until a new origin becomes a feasible point. Once a new origin becomes a feasible point the value of the objective function gradually increases at the new origins till the condition of optimality is satisfied. Thus finally the optimal solution is obtained at the last origin.

In this paper a technique has been developed on this method for the post optimality analysis when the coefficients in the objective function and the right hand side of the constraints are changed one at a time.

MATHEMATICAL ANALYSIS

Mathematically, the linear fractional programming problem (LEPP) can be formulated as follows :

Maximize $z = \dfrac{(\Sigma_{j=1}^{n} c_j x_j + p)}{(\Sigma_{j=1}^{n} d_j x_j + q)}$

subject to $\Sigma_{j=1}^{n} a_{ij} x_j \leq = \geq b_i (\geq 0), i = 1, 2 \ldots, m$

and $x_j \geq 0, j = 1, 2, \ldots, n$

We call this problem as LFPP1.

It is customary to assume that the linear fractional programming problem is well posed in the sense that denominator is positive everywhere in the feasible region.

In brief the method described in [1] to solve LFPP1 may be stated as the following Algorithm.

Algorithm LFPP

Step 1 : If the problem is of minimization it is converted to maximization.

Step 2 : If $c_j q - d_j p \leq 0$ for all j and if all the constraints are associated with $\leq$ type inequality then the corresponding origin is the optimal solution.

If some of the constraints are associated with $\geq$ type inequality and / or some

$(c_j q - d_j p) > 0$

go to step 3.

Step 3 : For each j calculate $\min_i\{b_i / a_{ij} > 0\}$. Let this minimum occurs for $i = p_j$, i.e., the minimum value is $b_{p_i} / a_{p_i j} = \alpha_j$ (say).

Step 4 : Find $\max_j\{(c_j\alpha_j + p) / (d_j\alpha_j + q)\}$. Let this maximum occurs for $j = k$. Take $r = p_k$.

Step 5 : Three cases arise.

(i) If the r-th constraint is associated with '$\geq$' type relation then take a_rk as key element and go to step 6.

(ii) If the r-th constraint is associated with '$\leq$' type relation but there exists some other constraints associated with '$\geq$' relation then choose any p_j - th constraint associated with '$\geq$' type relation as key constraint with $a_{p_j j}$ as key element i.e. take this j as k and go to step 6.

(iii) If r-th as well as all other constraints are associated with '$\leq$' type relation then take a_rk as key element and go to step 6.

Step 6 : Take $a_{rk}(>0)$ as the key element. Obtain the following LFPP2 by replacing x_k from LFPP1 with the help of the key equation $\Sigma_{j \neq k} a_{rj}x_j + a_{rk}x_k \pm x_s = b_r$. ...... (1)

LFPP2 :

$$\text{Maximize } z = \frac{[\Sigma_{j\neq k} x_j(c_j - c_k a_{rj} / a_{rk}) \mp x_s c_k / a_{rk} + c_k b_r / a_{rk} + p]}{[\Sigma_{j\neq k} x_j(d_j - d_k a_{rj} / a_{rk}) \mp x_s d_k / a_{rk} + d_k b_r / a_{rk} + q]}$$

subject to

$$\Sigma_{j\neq k}(a_{ij} - a_{ik}a_{rj} / a_{rk})x_j \mp a_{ik}x_s / a_{rk} \leq = \geq b_i - a_{ik}b_r / a_{rk}, i = 1,2\ldots,r-1,r+1,\ldots,m$$

$$[a_{rk}x_k] \quad \Sigma_{j\neq k} a_{rj}x_j \pm x_s \leq b_r$$

and $x_j \geq 0, j = 1,2,\ldots,k-1,k+1,\ldots,n$

$x_s \geq 0.$

Go to step 2.

Note : $[a_{rk}x_k]$ in the rth constraint indicates the fact that if $a_{rk}x_k$ is added to the LHS of the constraint then it will reduce to the key equation (1).

Using Algorithm LFPP let the final LFPP of LFPP1 be

$$\text{Maximize } z = \frac{(\Sigma_j c'_j x_j + p')}{(\Sigma_j d'_j x_j + q')}$$

subject to $\Sigma_j a'_{ij}x_j \leq b'_i(\geq 0), i = 1,2,\ldots,m$

and $x_j \geq 0,$

where j runs over all suffixes of the variables present.

Call this LFPP as LFPP3.

Now we consider sensitivity on LFPP1. The purpose of this investigation is to reduce the computational time required to obtain the changes in the optimal solution due to the changes in the parameters involved in the LFPP. Linear variations of two types of parameters viz., (i) variations in the objective function coefficient and (ii) variations in the resource availability i.e., right hand side values will be investigated.

Variations in the objective function coefficients :

In LFPP1 let the coefficient c_t be changed to $c_t + \theta$ and call it as LFPP4. We note that θ is then associated with the variable x_t.

For sensitivity analysis of this LFPP4 we first solve the LFPP4 taking $\theta = 0$ i.e. we solve LFPP1. Two cases will arise

(i) the optimal LFPP contains x_t as a variable

(ii) the optimal LFPP does not contain x_t as a variable.

Case (i) : Here x_t is a variable present in the final table. Therefore x_t remains present in all the tables from first to last. Hence to obtain the objective function of the LFPP4 corresponding to the final LFPP of LFPP1 we have just to add θx_t in the numerator of the final objective function of LFPP1. All the constrints of the final LFPP of LFPP1 are the corresponding constraints of LFPP4. To maintain optimality, the range of θ is obtained from the optimality condition $(c_t' + \theta)q' - d_t'p' \leq 0$. Example 4.1.1 illustrates the situation.

Using same technique we can solve the problem in which d_t in the objective function is replaced by $d_t + \theta$. The same technique also applies if both c_t and d_t are replaced simultaneously by $c_t + a\theta$ and $d_t + b\theta$ for any fixed positive constants a and b.

Case (ii) : Here x_t is not present in the final LFPP. So at some intermediate step,, x_t was replaced by some slack/surplus variable with the help of some key eqution. This key equation changes with the necessary transformations done for getting the final solution and this changed final equation can be obtained from the final LFPP just by adding the term present in the third bracket to the corresponding constraint. From this equation we find x_t in terms of the variables present in the final LFPP. To get the LFPP of LFPP4 corresponding to the final LFPP of LFPP1 we are to only add the term $x_t\theta$ in the numerator of the objective function and replace x_t in terms of the final variables with the help of the above mentioned equation. We note that all constraints of the final variables with the help of the above mentioned equation. We note that all constraints of the final LFPP of LFPP1 and corresponding LFPP for LFPP4 are same. The range of θ is then obtained easily using optimality conditions of the form $c_j q - d_j p \leq 0$ for all j. This situation is explained in Example 4.1.2. The same process is applicable if d_t in the objective function is replaced by $d_t + \theta$.

Variations in the components of the requirement vector :

In LFPP1 let the coefficient b_t be changed to $b_t + \theta$ and call it as LFPP5.

We first solve the LFPP5 taking $\theta = 0$ and note that here θ is associated with the t-th constraint. Three cases will arise

(i) t-th constraint has never become key constraint at any iteration

(ii) at some iteration t-th constraint becomes a key constraint and the corresponding slack/surplus variable is not present in the optimal LFPP

(iii) at some iteration t-th constraint becomes a key constraint and the corresponding slack/surplus variable is present in the optimal LFPP.

Case (i) : Here the t-th constraint has never become key constraint at any iteration. So LFPP's occurring in all iterations of LFPP1 and the corresponding LFPP's of LFPP5 differ only in the RHS of the t-th constraint. As the RHS of the t-th constraint of the final LFPP of LFPP1 is b_t', the RHS of the t-th constraint of LFPP5 corresponding to the final LFPP of LFPP1 will be $b_t' + \theta$. All the other terms of the two LFPP will remain same. Thus the range of θ is obtained from the non-negativity condition $b_t' + \theta \geq 0$.

Case (ii) : Here at some iteration t-th constraint becomes a key constraint and the corresponding slack/surplus variable x_s is not present in the final optimal LFPP. However, x_s may remain present in ahead of some constraint in the third bracket. We note that addition of this term reduces the inequality constraint to equality. Thus in the final LFPP of LFPP1 just by replacing x_s (a) by $x_s - \theta$ if x_s is a slack variable, and (b) by $x_s + \theta$ if x_s is a surplus variable in the term present in the third bracket and then ignoring x_s term we get the corresponding LFPP of LFPP5.

Obviously, the objective function and other constraints remain unchanged here. Imposing non-negativity restriction on the RHS, the range of θ is obtained. Example 4.1.3 illustrates the situation.

Case (iii) : Here at some iteration t-th constraint becomes as a key constraint and the corresponding slack/surplus variable x_s is present in the final optimal LFPP. Then just by replacing x_s (a) by x_s - θ if x_s is a slack variable and (b) by x_s + θ if x_s is a surplus variable. in the final LFPP of LFPP1 we get the corresponding LFPP of LFPP5.

Optimality conditions of the form $c_j q - d_j p \leq 0$ for all j are imposed. These inequalities together with inequalities obtained by non-negativity restriction on the RHS give the range of θ. Example 4.1.4 illustrates the situation.

ILLUSTRATIVE EXAMPLES

To illustrate the method we consider one main example and study sensitivity analysis on it.

Example 4.1

Let us consider the maximization problem

Maximize $z = (-3x_1 - x_2)/(x_1 + 2x_2 + 5)$

subject to constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0.$$

Using Algorithm LFPP we solve this LFPP. Here $c_1 q - d_1 p$ = (-3)5 - 1.0 < 0 and $c_2 q - d_2 p$ = (-1)5 - 2.0 < 0.

But the origin $x_1 = x_2 = 0$ is not a feasible point. Therefore we first consider transformations to convert all inequalities into ≤ types so that the new origin becomes a feasible point. The first table is

b	a_1	a_2	b_i / a_{i1}	b_i/a_{i2}
1	1	1	1	1
2	2	3 *	1 $= \alpha_1$	2/3 $= \alpha_2$
	$\frac{(c_j \alpha_j + p)}{(d_j \alpha_j + q)}$		-1/2	((-2/19))

Table : 4.1 (i)

Here $p_1 = 2$ and $p_2 = 2$ and the maximum of $(c_j \alpha_j + p)/d_j \alpha_j + q)$ occurs for $j = 2$. Therefore $k = 2$. Since $p_k = p_2 = 2$ and $p_k = r$ we have $r = 2$. Thus the key element is $a_{rk} = a_{22} = 3$. Therefore we introduce the surplus variable x_3 to the second constraint to obtain the equation $2x_1 + 3x_2^* - x_3 = 2$ and with the help of it eliminate x_2 from the LFPP to get the second LFPP as

Maximize $z = (-7x_1/3 - x_3/3 - 2/3)/(-x_1/3 + 2x_3/3 + 19/3)$

subject to constraints

$$x_1/3 + x_3/3 \geq 1/3$$
$$[3x_2] \qquad 2x_1 - x_3 \leq 2$$
$$x_1, x_3 \geq 0.$$

We note that the second constraint has been converted to an inequation containing $\leq$ type inequality. Here $c_1q - d_1p$ = (-7/3) (19 / 3) - (-1 / 3) (-2 / 3) < 0 and $c_2q - d_2p$ = (-1 / 3) (19 / 3) -(2 / 3) (-2 / 3) 0. But as the first constraint is associated with $\geq$ type inequality, optimality condition is not satisfied. The second table is

b	a_1	a_3	b_i / a_{i1}	b_i / a_{i3}
1	1	1 *	1	$1 = \alpha_3$
2	2	-1	$1 = \alpha_1$	- -
	$\dfrac{(c_j\alpha_j + p)}{(d_j\alpha_j + q)}$		- 1/2	((-1/7))

Table : 4.1 (ii)

Here $p_1 = 2$ and $p_3 = 1$ and the maximum value of $(c_j\alpha_j + p)/(d_j\alpha_j + q)$ occurs for $j = 3$. Therefore $k = 3$. Since $p_k = p_3 = 1$ and $p_k = r$ we have $r = 1$. Thus the key element is $a_{rk} = a_{13} = 1$. Therefore we introduce the surplus variable x_4 to the first constraint to get the equation $x_1 + x_3^* - x_4 = 1$ and use it to eliminate x_3 from the second LFPP. We thus have the third equivalent LFPP as

Maximize $z = (-2x_1 - x_4 - 1)/(-x_1 + 2x_4 + 7)$

subject to constraints

$$[x_3 / 3] \qquad x_1 / 3 - x_4 \leq 1/3$$

$$[3x_2] \qquad 3x_1 - 3x_4 \leq 3$$

$$x_1, x_4 \geq 0.$$

Here $c_1q - d_1p$ = (-2) (7) - (-1) (-1) < 0 and $c_2q - d_2p$ = (-1 / 3) (7) - (2 / 3) (-1) < 0 and all the constraints have been converted to $\leq$ type inequation. Hence the origin $x_1 = x_4 = 0$ is the optimal solution. From $x_1 + x_3 - x_4 = 1$ we get $x_3 = 1$ and from $2x_1 + 3x_2 - x_3 = 2$ we get $x_2 = 1$. Therefore, the optimal solution is $x_1 = 0$, $x_2 = 1$ and maximum value of z is -1 / 7.

In Examples 4.1.1 and 4.1.2 we study sensitivity analysis for the coefficient in the objective function.

Example 4.1.1

Maximize $z = [-3x_1 - x_2]/[(1 + \theta)x_1 + 2x_2 + 5]$

subject to constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0.$$

We note that here θ is associated with the variable x_1. The optimal LFPP of this example with $\theta = 0$ is the optimal LFPP of Example 4.1. As this optimal LFPP contains x_1, the objective function of Example 4.1.1 corresponding to the optimal LFPP of Example 4.1 is obtained just by adding $x_1\theta$ in the denominator of the optimal LFPP of Example 4.1. Hence this objective function is given by $z = [-2x_1 - x_4 - 1]/[(\theta - 1)x_1 + 2x_4 + 7]$. To get the range of θ we impose the optimality condition $-2.7 - (-1).(\theta - 1) \leq 0$ giving $\theta \leq 15$. Hence if $\theta \leq 15$ then the optimal solution remains optimal.

Example 4.1.2

Maximize $z = [-3x_1 + (-1 + \theta)x_2]/[x_1 + 2x_2 + 5]$
subject to constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0.$$

We note here that θ is associated with x_2 and x_2 is not present in the final LFPP obtained taking $\theta = 0$. The objective function of Example 4.1.2 corresponding to the final LFPP of Example 4.1 is obtained just by adding the term $x_2\theta$ in the numerator of the objective function of the final LFPP of Example 4.1. So it is $z = [-2x_1 - x_4 - 1 + x_2\theta]/[-x_1 + 2x_4 + 7]$. Now x_2 is to be expressed in terms of the variables x_1 and x_4 with the help of the second constraint of the final LFPP of Example 4.1 which can be expressed as an equation as $3x_2 + 3x_1 - 3x_4 = 3$. This gives the objective fuction as $z = [(-\theta - 2)x_1 - (1 - \theta)x_4 + (-1 + \theta]/[-x_1 + 2x_4 + 7]$. The range of θ is obtained by the optimality conditions
$(\theta - 2).7 - (-1).(-1 + \theta) \leq 0$ and $-(1 - \theta).7 - 2.(-1 + \theta) \leq 0$ which gives $-5/2 \leq \theta \leq 1$.

To study sensitivity analysis for the components of the requirement vector *i.e.* RHS vector, we consider Examples 4.1.3 and 4.1.4.

Example 4.1.3

Maximize $z = (-3x_1 - x_2)/(x_1 + 2x_2 + 5)$
subject to constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq (2+\theta)$$
$$x_1, x_2 \geq 0.$$

In this example we note that θ is associated with second constraint and this constraint is a key constraint in a step of the solution of the LFPP with $\theta = 0$. Also we note that the surplus variable x_3 is associated with this key constraint and x_3 is not present in the final LFPP. Thus the LPFF of Example 4.1.3 corresponding to the final LFPP of Example 4.1 is obtained by replacing x_3 by $x_3 + \theta$ in the final LFPP of Example 4.1. We see that x_3 is present in the first constraint in third bracket and is $[(1/3)x_3]$. Hence the corresponding constraint of the given problem is
$(1/3)x_1 - x_4 \leq 1/3 - (1/3)\theta$. The non-negativity restriction on the RHS *i.e.* on 1 / 3 -(1 / 3) θ gives the range of θ as $\theta \leq 1$.

Example 4.1.4

Maximize $z = (-3x_1 - x_2)/(x_1 + 2x_2 + 5)$
subject to constraints

$$x_1 + x_2 \geq (1+\theta)$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0.$$

In this example we note that θ is associated with the first constraint and this constraint is a key constraint in a step of the solution of the LFPP with $\theta = 0$. Also we note that the surplus variable x_4 is associated with this key contraint and x_4 is present in the final LFPP. Thus the LPFF of Example 4.1.4 corresponding to the final LFPP of Example 4.1 is obtained just by replacing x_4 by $x_4 + \theta$ in the final LFPP of Example 4.1 and is given by Maximize
$z = (-2x_1 - x_4 - 1 - \theta)/(-x_1 + 2x_4 + 7 + 2\theta)$ subject to constraints

$$x_1/3 - x_4 \leq 1/3 + \theta$$
$$3x_1 - 3x_4 \leq 3 + 3\theta$$
$$x_1, x_4 \geq 0.$$

The non-negative restriction on RHS and the optimality conditions (-2). (7 + 2θ) - (-1). (-1-θ) $\leq$ 0 and (-1). (7 + 2θ) -2 (- 1 - θ) $\leq$ 0 gives the range of θ to maintain optimality as $\theta \geq -1/3$.

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AN OPTIMAL ALGORITHM FOR COMPUTING ALL-PAIR SHORTEST PATH ON WEIGHTED CACTUS GRAPHS

Kalyani Maity, Madhumangal Pal* and Tapan K. Pal
Department of Applied Mathematics with Oceanology
and Computer Programming,
Vidyasagar University, Midnapore - 721 102, India
July 31, 2000

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Abstract:
A cactus graph is a connected graph in which every block is either an edge or a cycle. Some interesting properties of general as well as cactus graph have been studied in this paper. Also, to find all-pair shortest paths on a weighted cactus graph an optimal algorithm has been presented here which takes $O(n^2)$ time where n is the number of vertices.

Kew words : Design and analysis of algorithms, all-pair shortest path, cactus graph.

INTRODUCTION

The class of cactus graph is an important subclass of general planar graphs. Let G = (V, E) be a finite, connected, undirected simple graph of n vertices and m edges, where V is the set of vertices and E is the set of edges. A vertex u is called a *cutvertex* if removal of ϑ and all edges incident on ϑ disconnect the graph. A connected graph without a cutvertex is called a *non-separable* graph. A *block* of a graph is a maximal non-separable subgraph. A *cycle* is a connected graph (or subgraph) in which every vertex is of degree two. A block which is a cycle is called a *cyclic block*. A *cactus graph* is a connected graph in which every block is either an edge or a cycle. A *weighted* graph G is a graph in which every edge is associates with a weight. Without loss of generality we assume that all weights are positive. A *weighted cactus graph* is a weighted, connected graph in which every block containing two vertices is an edge and three or

* E-mail : madhumangal@mailcity.com

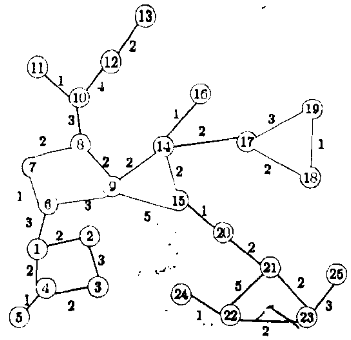

Figure 1 : A weighted cactus graph

more vertices is a cycle. A *path* of a graph G is an alternating sequence of distinct vertices and edges which begins and ends with vertices in G. The *length* of a path is the sum of the weights of the edges in the path. A path from vertex u to ϑ is a *shortest path* if there is no other path from u to ϑ with lower length. The *distance* d (u, ϑ) between vertices *u* and ϑ is the length of shortest path between u and ϑ in G.

As an example, Figure 1 represents a weighted cactus graph. The numbers within circles represent the vertices and the numbers adjacent to edges represent corresponding weights.

A large number of optimization problems are mathematically equivalent to finding shortest path in a graph. Consequently, shortest path algorithms have been designed more thoroughly than any other algorithm in graph theory. There are different types of shortest path problems among which we frequently encounter the following :

1. Shortest paths between two specified vertices.
2. Shortest paths between all pair of vertices.
3. Shortest paths from a specified vertex to all other vertices.
4. Shortest paths between two specified vertices that pass through certain specified vertices.

Computationally, problems of type 1, type 2 and type 3 are somewhat similar. In this paper we consider problem of type 2.

For any general graph with *n* vertices, solution to the all-pair shortest path problem takes $O(n^3)$ time [1]. A lot of works have been done in improving this running time using randomization and probabilistic methods for general as well as special kinds of graphs. Ahuja et. al. [2] have given a faster sequential algorithm using Radix heap and Fibonacci heap for single source shortest path problem in $O(m+n\sqrt{\log C})$ time for a network with n vertices and m edges and non-negative integers arc costs bounded by C. In

[21], Seidal has given an $O(M(n)\log n)$ time sequential algorithm for all-pair shortest path problem for an undirected and unweighted arbitrary graph with n vertices, where M(n) is the time (best value of $M(n)$ is $O(n^{2.376})$) necessary to multiply two n x n matrices of small integers.

Alon et. al. [3] have reported a sub-cubic algorithm for computing APSP on directed graph with edge length which require $O(Mn^{\gamma})$ time, where $\gamma = (3+\omega)/2$, $\omega < 3$ and M is the largest edge-length. Galil and Margalit [12] have improved the dependence on M and have also given an $O(M^{(\omega+1)/2}n^{\omega}\log n)$ algorithm for undirected graph. Ravi et. al. [19] have given a sequential algorithm to solve all-pair shortest path (APSP) on interval graph in $O(n^2)$ time. Pal and Bhattacharjee in [18] have given an $O(n^2)$ time algorithm for finding the distance between all pair of vertices on interval graphs. In this paper, a sequential algorithm is presented to compute all-pair shortest distance on weighted cactus graphs, which takes $O(n^2)$ time.

DETERMINATION OF BLOCKS AND CUTVERTICES

As described in [20] the bi-connected components as well as cutvertices of a graph G can be determined by applying DFS technique. Using this techniques we obtain all bi-connected components and cutvertices of the cactus graph G = (V, E). Let the bi-connected components be $B_1, B_2, B_3, \ldots, B_N$ where N is total number of blocks, and the cutvertices be $c_1, c_2, \ldots, c_R$ where R is the total number of cutvertices.

For the cactus graph of Figure 1 the blocks thus obtained, are $B_1 = \{1, 2, 3, 4\}$; $B_2 = \{4, 5\}$; $B_3 = \{1, 6\}$; $B_4 = \{6, 7, 8, 9\}$; $B_5 = \{8, 10\}$; $B_6 = \{10, 11\}$ $B_7 = \{10, 12\}$; $B_8 = \{12, 13\}$; $B_9 = \{9, 14, 15\}$; $B_{10} = \{14, 16\}$; $B_{11} = \{14, 17\}$; $B_{12} = \{17, 18, 19\}$; $B_{13} = \{15, 20\}$; $B_{14} = \{20, 21\}$; $B_{15} = \{21, 22, 23\}$; $B_{16} = \{22, 24\}$; $B_{17} = \{23, 25\}$ and cutvertices are $c_1 = 4$; $c_2 = 1$; $c_3 = 6$; $c_4 = 8$; $c_5 = 10$; $c_6 = 12$; $c_7 = 9$; $c_8 = 14$; $c_9 = 17$; $c_{10} = 15$; $c_{11} = 20$; $c_{12} = 21$; $c_{13} = 22$; $c_{14} = 23$.

The following property is important for cactus graph.

Lemma 1 *The maximum number of edges in a simple cactus graph with n vertices is* $\lfloor 3(n-1)/2 \rfloor$ [22].

A necessary and sufficient condition for a vertex ϑ to be a cutvertex is stated below.

Theorem 1 *A vertex u in a connected graph G is a cutvertex if and only if there exist two vertices x and y in G such that every path between x and y passes through* ϑ *[7].*

From the definition of block it follows that each block is non-separable, i.e., if we regard this block as a subgraph then it has no cutvertex. So we have the following lemma.

Lemma 2 *In Theorem 1, x and y always belongs to different blocks.*

The following lemma gives the intersection property of two blocks.

Lemma 3 *For any two blocks B_i and B_j of a simple connected graph either $|B_i \cap B_j| = 0$ or $|B_i \cap B_j| = 1$.*

Proof : For any two block B_i and B_j either $B_i \cap B_j = \phi$ or $B_i \cap B_j \neq \phi$. If $B_i \cap B_j = \phi$ then $|B_i \cap B_j| = 0$. If $B_i \cap B_j \neq \phi$ then $|B_i \cap B_j| \geq 1$.

If possible let $|B_i \cap B_j| > 1$. Then there exist at least two vertices u and v common to both B_i and B_j. We first prove that the subgraph $B_i \cup B_j$ is non-separable. If we remove u then any two elements of $B_i \cup B_j$ will be connected through ϑ. So, by Theorem 1, u cannot be a cutvertex of $B_i \cup B_j$. Similarly, ϑ is not a cutvertex. If w be any element of $B_i \cup B_j$ other than u and ϑ and is removed then also any elements of $B_i \cup B_j$ will be connected through u or ϑ. So w is not a cutvertex. This means that $B_i \cup B_j$ has no cutvertex *i.e.*, $B_i \cup B_j$ is non-separable. As B_i is a block, it is maximal non-separable. But $B_i \subset B_i \cup B_j$, so B_i is not maximal non-separable. This is a contradiction. Hence $|B_i \cap B_j| \not> 1, i.e, |B_i \cap B_j| = 1$.

The necessary and sufficient condition in terms of blocks for a vertex to be a cutvertex is given in the following theorem.

Theorem 2 *A vertex ϑ is a cutvertex if and only if it is contained in at least two blocks.*

Proof : *Necessary :* From the definition of cutvertex, the removal of the cutvertex ϑ and edges incident on it from a connected graph increases the number of components. Each of these components along with the cutvertex ϑ and the corresponding edges are either blocks or separable subgraphs.

If it is a separable subgraph then there must be a block containing 'u' in it. This is because each vertex of a subgraph is always contained in at least one block.

Thus, corresponding to each component obtained after removal of the cutvertex ϑ we get a block containing ϑ. Since number of components is at least two we get the necessary part of the Lemma.

Sufficient : Without loss of generality we assume that the vertex $\vartheta \in G$ belongs to B_i and B_j. Let $x \in B_i$ and $y \in B_j$. By Lemma 3, B_i and B_j can not have any other common vertex

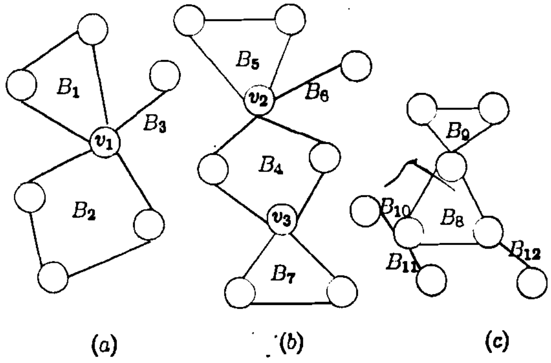

Figure 2 :

other than ϑ. Then every path from x to y passes through ϑ. Therefore, by Theorem 1, ϑ is a cutvertex.

Def. 1 Adjacent block : *Two blocks are said to be adjacent if they have a common cutvertex.*

Def. 2 Non-adjacent block : *If two blocks have no vertices in common then they are called non-adjacent.*

In any grapn every block B_j contains at least one cutvertex and at most $|B_j|$ cutvertices. In Figure 2 (a), the blocks B_1, B_2 and B_3 contain only one cutvertex ϑ_1 and in Figure 2(b), the block B_4 cotains two cutvertices ϑ_2 and ϑ_3. In Figure 2(c), all the vertices of block B_8 are cutvertices.

Lemma 4 *If u be any non-cutvertex of a graph then u belongs to only one block.*

Proof : We prove this by contradiction.

If possible let u belongs to more than one blocks. Now from Theorem 2, if a vertex is contained in at least two blocks, then it is a cutvertex. So u is a cutvertex, which contradicts the assumption that u is a non-cutvertex.

Hence u can not belongs to more than one block *i.e.*, u belongs to only one block.

CONSTRUCTION OF THE TREE T_{BC}

In this section we construct a tree using the blocks obtained from Section 2. Before constructing the tree, we define an intermediate graph G' as follows :

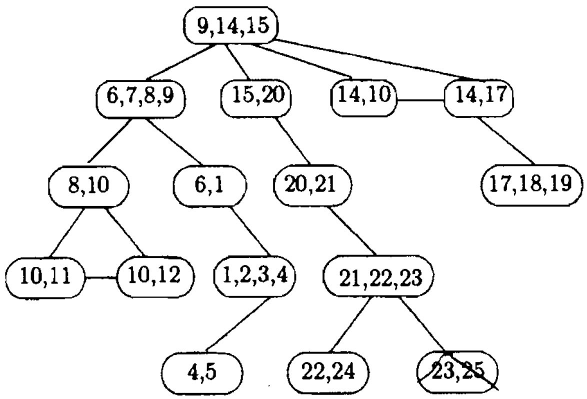

Figure 3 : The graph G' of the cactus graph of Figure 1.

$G' = (V', E')$ where $V' = \{B_1, B_2, \ldots, B_N\}$ and $E' = \{(B_i, B_j) : i \neq j; i, j = 1, 2, \ldots, N$ and B_i and B_j are adjacent blocks$\}$.

The graph G' for the graph G of Figure 1 is shown in Figure 3.
Now the tree T_{BC} is constructed from G' as follows :

We remove some suitable edges from G' in such a way that the resultant graph becomes a tree. The procedure for reduction is given below :

Let us take any arbitrary block of G' as root of the tree T_{BC} and mark it. All the adjacent blocks of this root are taken as children and placed in level one and are marked. If there are edges between the blocks of this level, then those edges are removed. Each block of level one is considered one by one to find their children. The blocks which are adjacent but unmarked to the blocks of level one are taken as children and are placed at level two. These children at level two are marked and if there by any edge between them then they are removed. This process is continued until all the blocks are marked.

Thus the tree $T_{BC} = (V', E'')$ where $V' = \{B_1, B_2, \ldots, B_N\}, i = 1, 2, \ldots, N$ and $E'' \subset E'$ is obtained.

Since we have two different graphs under consideration G and G' and a tree T_{BC}, we shall refer to a vertex of G and G' as a vertex and a vertex of T_{BC} as node.

We note that each node of this tree is a block of the graph G = (V, E).

The parent of the node B_i in the tree T_{BC} will be denoted by $Parent\ (B_i)$.

The tree T_{BC} constructed from G' of Figure 3 is given in Figure 4.

By Lemma 3, every node B_i (except the root) of the tree T_{BC} has only one common element with its parent $Parent\ (B_i)$. We define this common element as **entry point** of B_i and denote it by e_i. By Theorem 2, e_i is a cutvertex.

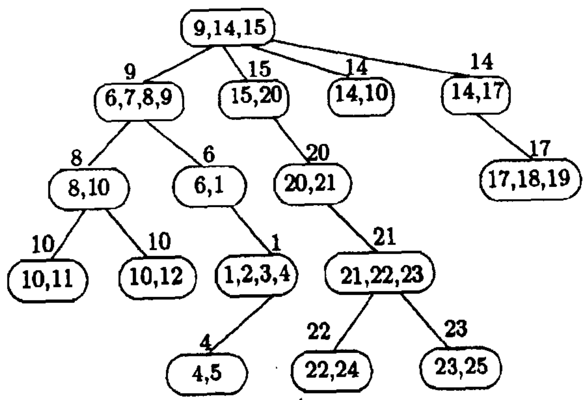

Figure 4 · The tree T_{BC} of the graph of Figure 1, the numbers outside the ovals represent the entry point

From the construction of the tree T_{BC} we have Lemma 5 as :

Lemma 5 *Each block corresponding to an internal node of T_{BC} contains at least two cutvertices.*

Lemma 6 *Each node of T_{BC} other than the root has unique entry point.*

Proof : The entry point e_i of the block B_i is the edge between B_i and $Parent\ (B_i)$.. Since B_i has only one parent *viz*, $Parent(B_i)$, it follows that e_i is unique.

Lemma 7 *The level difference of the nodes corresponding to the blocks with a common cutvertex can not be more than one.*

Proof : Let the blocks B_i and B_j have a common cutvertex. Since B_i and B_j have a common element, they are adjacent. Therefore, in the tree T_{BC} the blocks are either in the same level or one of them is a child of the other. Therefore, there level difference is one or zero, *i.e.*, not more than one. •

Lemma 8 *For the nodes of different levels the entry point of the corresponding blocks are different.*

Proof : Without loss of generality, let the nodes B_i and B_j be in different levels and the level of B_i is less than the level of B_j. Now the entry points of B_i and B_j are e_i and e_j respectively.
Then

$$e_i \in B_i \text{ and } e_i \in Parent(B_i)$$
$$\& \; e_j \in B_j \text{ and } e_j \in Parent(B_j).$$

Since B_i and B_j are in different levels, their parents are also in different levels. As level of $B_i <$ level of B_j, level difference of B_i and $Parent\ (B_j)$ is more than one.

If $e_i = e_j$ then e_i is a member of $B_i, B_j, Parent\ (B_i)$. But level difference between B_i and $Parent\ (B_j)$ is always more than one. This contradicts Lemma 7. Hence $e_i \neq e_j$. . Therefore, e_i and e_j does not represent the same cutvertex.

When a cutvertex is contained in more than two blocks, we take one of them as parent and other adjacent blocks as children and put them at same level. Now the blocks corresponding to these children and parent have same cutvertex in common. This cutvertex is the entry point of these children which imply that the entry point may be same for different nodes in the same level.

As an example in Figure 4 the blocks $B_6 = \{10,11\}$ and $B_7 = \{10,12\}$ are at the same level and they have same entry point '10'. Again the block $B_{16} = \{22,24\}$ and $B_{17} = \{23,25\}$ are also at same level, but they have different entry points '22', '23'.

Theorem 3 *If u be a cutvertex and belongs to the blocks corresponding to the nodes at level k and $k+1$, then there does not exist any edge in G between the vertices u and ϑ of G where ϑ is any vertex of the blocks whose corresponding nodes are at level less than $k-1$ and greater than $k+2$.*

Proof : Here u belongs to some blocks at level k and $k+1$. Let ϑ be any vertex adjacent to the vertex u. Then clearly there is only one block containing both u and ϑ. The level of this block may be k or $k+1$. Thus two cases arise.

Case 1 : The level of the block containing u and u is k.

As ϑ may or may not be cutvertex, we have two cases.

Case 1 (a) : ϑ is a cutvertex.

Since both u and ϑ are cutvertices and the level of the block in which they belongs is k it follows by Lemma 8 that ϑ belongs to one or more blocks (not containing u) of level $(k+1)$ or $(k-1)$ or k but not at level less than $k-1$ or at level more than $k+1$.

Case 1(b) : v is a non-cutvertex.

If v is a non-cutvertex then by Theorem 2, ϑ belongs to only one block. So this block must be the same block containing u at level k.

Case 2 : The level of the block containing u and ϑ is $k+1$.

Let this block be B'. Then as u belongs to blocks at level k and $k+1$ and is a cutvertex, it is clear that B' and *Parent* (B') is connected by u. Two cases arise as may or may not be a cutvertex.

Case 2(a) : ϑ is a cutvertex.

Since B' has only one parent and ϑ is a cutvertex, ϑ can not belong to a block at level k.

Since u is a cutvertex of the blocks at level k and $k+1$, and ϑ can not belong to any block at level k,, it follows that u belongs to the blocks at levels $k+1$ and $k+2$ only but not at levels greater than $k+2$, by Lemma 8.

Case 2(b) : u is not a cutvertex.

The same argument given in Case 1(b) holds good here.

Corollary 1 *If u be a cutvertex and belongs to the block corresponding to the nodes at level k and $(k+1)$ then there may be an edge between the vertices u and ϑ where ϑ is a vertex of the blocks whose corresponding nodes are at levels $(k-1), k, (k+1), (k+2)$.*

Theorem 4 *All paths between any two vertices u and ϑ of the graph G passes through the same set of blocks and same set of cutvertices.*

Proof : Let the vertices u and ϑ belongs to the blocks B and B' respectively. If possible let the blocks B and B' are connected by more than one sequences of blocks and cutvertices. Then there exists a finite sequences of blocks and cutvertices forming a cycle. This contradicts the property of cutvertices and blocks since in that case the cutvertices contained in the cycle will not remain cutvertices as their removal will not form any block.

From the construction of tree T_{BC} the following lemma is obvious.

Lemma 9 *The sequences of blocks and cutvertices connecting two blocks B_i and B_j in the graph G has one to one correspondence with the path from B_i to B_j (or from B_j to B_i) in the tree T_{BC} formed with B_i (or B_j) as root.*

COMPUTATION OF DISTANCE FROM ENTRY POINT TO ANY OTHER VERTICES WITHIN A BLOCK

There are two types of blocks in a cactus graph, cyclic and non-cyclic. Blocks with more than two vertices are cyclic and blocks with two vertices are non-cyclic.

In this section we find the shortest distance between the entry point and other vertices of a block.

Obviously, distance from entry point e_j to other vertex ϑ in a non-cyclic block is $d(e_j, \vartheta)$ which is the weight of the edge (e_j, ϑ).

To find the distance from entry point e_j to some vertex ϑ of a cyclic block we proceed as follows.

As the block is cycle, there are only two possible paths from the entry point e_j to any vertex ϑ. The minimum of the sum of the weights associated with these two paths gives the distance between e_j and ϑ, and the corresponding path is the shortest path between e_j and ϑ.

SHORTEST PATH FROM A SPECIFIED VERTEX TO ALL OTHER VERTICES IN G

Let the specified vertex be x. First we have to find a block which contains x. We construct a tree T_{BC} taking this block as root by the method described in Section 3.

After constructing the tree we first compute the distance between x and all other vertices in this root. Then we compute the distance from x to vertices (other than x) of the blocks corresponding to the nodes in level one as follows. Let B_i be a node at level one. Then e_i is its entry point. We compute the distances of every vertex ϑ of B_i from e_i and adding $d(x, e_i)$ with these distances we obtain the distance and so shortest path of the vertices of the block B_i from x. Similarly, we compute the distance from x to other vertices of the blocks at level one.

For the nodes, *i.e.*, blocks of the remaining levels, the distance from x can be computed by the same process.

In general, let us consider a block B_j at level i and assume that the distance between x and all vertices of the blocks at level '$i-1$' have been calculated. The entry point of B_j is e_j. Then $d(x,e_j)$ is known, as $e_j \in$ a block at level '$i-1$'. We now compute $d(e_j, \vartheta)$ for all $\vartheta \in B_j$.Then $d(x,\vartheta) = d(x,e_j) + d(e_j,\vartheta)$ for all $\vartheta \in B_j$. This determine the distance between x and any vertex of any block at level i.

This process is continued upto the blocks at last level.

Algorithm SSSP presented below compute the shortest distance from a specified vertex to all other vertices :

Algorithm SSSP.

Input : The cactus graph G and the vertex x from which the distance is to be calculated.

Output : The distance $d(x,u), \forall u \in V$.

Step 1 : Compute the blocks and cutvertices of G .

Step 2 : Find a block at which the vertex x belongs to, let this block be B_p.

Step 3 : Construct a tree T_{BC} taking B_p as root.

Step 4 : Compute the entry point e_j of all $B_j, j = 1,2,\ldots,N.(Here|T_{BC}| = N)$

Step 5 : Compute $d(x,u)$ for all $u \in B_p$.

Step 6 : Select any block B_j at level $k, k = 1,2,\ldots,h$ where h is the height of the tree T_{BC}. Compute $d(x,u) = d(x,e_j) + d(e_j,u)$ for all vertex $u \in B_j, j = 1,2,\ldots,N$.

end SSSP

The. time complexity of Algorithm SSSP is calculated in the following theorem.

Theorem 5 The shortest distances from a specified vertex to all other vertices of a weighted cactus graph can be computed in $O(n)$ time.

Proof : The blocks and cutvertices of any graph $G = (V,E)$ can be computed in $(|V| + |E|)$ time [20]. But for cactus graph $|E| = O(n)$ (Lemma 1), and hence Step 1 of Algorithm SSSP can be carried out in $O(n)$ time. The output of the algorithm of [20] is an explicit ordering of the blocks and cutvertices. Let the total number of blocks and cutvertices of G be N and R respectively. It is easy to observe that $N = O(n)$ and $R = O(n)$. From the ordering of blocks and cutvertices, one may construct the tree T_{BC} in $O(n)$ time.

The entry points are nothing but the cutvertices, so the entry points of all blocks can be computed in $O(n)$ time. Step 6 computes the distance from x to all other vertices $u, u \in V$ using the relation $d(x,u) = d(x,e_i) + d(e_i,u)$ where e_i is the entry point of B_i. Therefore, this step takes only $O(n)$ time. Hence the over all time complexity is of $O(n)$.

Using Algorithm SSSP one may compute the shortest distance from a vertex $i, i \in V$, to all other vertices in $O(n)$ time. The all pair shortest distance can be computed by repeated use of Algorithm SSSP and we can conclude the following result.

Theorem 6 *The all pair shortest distance of a weighted cactus graph can be computed in $O(n^2)$ time, where n represents the total number of vertices of the graph.*

The correctness of the algorithm follows from the lemmas and theorems discussed earlier.

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PROPAGATION OF WAVES IN SOLID-SOLID VISCO-ELASTIC SEMI-SPACES WHEN A COMPRESSIONAL WAVE SOURCE BEING PRESENT IN THE UPPER SOLID SUBSTRATUM

Sisir Nath and P. R. Sengupta
Indian Institute of Mechanics of Continua,
201, Manicktala Main Road, Suite No. 42, Calcutta - 700054, West Bengal, India

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Abstract:
The present paper is concerned with the investigation of propagation of waves in a solid-solid visco-elastic semi-spaces in presence of a compressional wave source in the upper solid substratum. The result obtained are in good agreement with corresponding classical problem when the effect of viscosity is neglected.

Kew words : Compressional wave source / visco-elastic semi-space / strain rate and stress rate of first order / Stoneley waves.

INTRODUCTION

The usefulness of the study of the wave propagation in the above media is well recognized in the field of earth quake and seismology. Sommerfeld (1) Jeffreys (2), Ewing et al (3) Muskat (4) have studied the wave propagation in presence of a source. In all the above investigations the effect of viscosity involving strain rate and stress rate of first order have been neglected. The study of waves and vibrations under the influence of visosity by Flugge (5) Hunter (6) and Bland (7) are note worthy. Sengupta and his research workers have also published a good number of papers (8-12) in this field. Following the above theories, the authors have investigated the titled axis-symmetric problem in cylindrical co-ordinate system assuming the surface of separation is a plane surface.

BASIC EQUATIONS AND RELATIONS

The two dimensional equation of motion in absence of body forces can be written in cartesian co-ordinates (x, y, z) as (13)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2}; \qquad \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{2.1}$$

Considering the axial symmetry, the equation of motion in cylindrical system of co-ordinates can be written as (13)

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=\rho\frac{\partial^2 q}{\partial t^2} \qquad (2.2)$$

$$\frac{\partial\sigma_{rz}}{\partial r}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}=\rho\frac{\partial^2 w}{\partial t^2}$$

where q and w are the displacements in the r and z direction. The stress-strain relation in an isotropic visco-elastic solid medium (5, 14) of first order involving strain rate and stress rate as

$$(\eta_0+\eta_1\frac{\partial}{\partial t})\sigma_{ij}=(\lambda_0+\lambda_1\frac{\partial}{\partial t})\Delta\delta_{ij}+2(\mu_0+\mu_1\frac{\partial}{\partial t})e_{ij} \qquad (2.3)$$

where η_0,λ_0,μ_0 are the elastic constants with η_1,λ_1,μ_1 accounting for viscosity, e_{ij} is the strain tensor, σ_{ij} is the stress tensor, δ_{ij} is the Kronecker symbol and Δ is the dilatation. Now expressing q and w interms of the potentials φ and ψ as (3)

$$q=\frac{\partial\varphi}{\partial r}+\frac{\partial^2\psi}{\partial r\partial z} \qquad w=\frac{\partial\varphi}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) \qquad (2.4)$$

The following expressions can be written from (2.3) in cylindrical co-ordinates

$$\left.\begin{aligned}
&(\eta_0+\eta_1\frac{\partial}{\partial t})\sigma_{zz}=(\lambda_0+\lambda_1\frac{\partial}{\partial t})\nabla^2\varphi+2(\mu_0+\mu_1\frac{\partial}{\partial t})\frac{\partial}{\partial z}[\frac{\partial\varphi}{\partial z}+\nabla^2\psi-\frac{\partial^2\psi}{\partial z^2}]\\
&(\eta_0+\eta_1\frac{\partial}{\partial t})\sigma_{rz}=(\mu_0+\mu_1\frac{\partial}{\partial t})\frac{\partial}{\partial r}(2\frac{\partial\varphi}{\partial z}+\nabla^2\psi-2\frac{\partial^2\psi}{\partial z^2})\\
&(\eta_0+\eta_1\frac{\partial}{\partial t})\sigma_{rr}=(\lambda_0+\lambda_1\frac{\partial}{\partial t})\nabla^2\varphi+2(\mu_0+\mu_1\frac{\partial}{\partial t})\frac{\partial}{\partial r}(\frac{\partial\varphi}{\partial r}+\frac{\partial^2\psi}{\partial r\partial z})\\
&(\eta_0+\eta_1\frac{\partial}{\partial t})\sigma_{\theta\theta}=(\lambda_0+\lambda_1\frac{\partial}{\partial t})\nabla^2\varphi+2(\frac{\mu_0+\mu_1\frac{\partial}{\partial t}}{r})(\frac{\partial\varphi}{\partial r}+\frac{\partial^2\psi}{\partial r\partial z})
\end{aligned}\right\} \qquad (2.5)$$

where

$$\nabla^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$$

Introducing (2.5) into (2.2) we obtain

$$\frac{\partial^2\varphi}{\partial t^2}=(v_{0T}^2+v_{1T}^2\frac{\partial}{\partial t})\nabla^2\varphi/L, \qquad \frac{\partial^2\psi}{\partial t^2}=(v_{0s}^2+v_{1s}^2\frac{\partial}{\partial t})\nabla^2\psi/L$$

where

$$v_{0T}^2=(\lambda_0+2\mu_0)/\rho, \qquad v_{1T}^2=(\lambda_1+2\mu_1)/\rho$$

$$v_{0s}^2=\mu_0/\rho. \qquad v_{1s}^2=\mu_1/\rho$$

$$L\equiv\eta_0+\eta_1\,\partial/\partial t$$

BOUNDARY CONDITIONS

Let us consider two homogeneous semi-infinite visco-elastic solid medin M and M' welded in contact at Z = 0 in the cylindrical system of co-ordinates. The axis oz being vertically upwards and M is above M'. $p,\eta_0,\eta_1,\lambda_0,\lambda_1,\mu_0,\mu_1$ denote the properties for M and those with dashes denote the properties of the medium M' we suppose that source lies at (0, 0 h) in the upper solid semi-space. The boundary conditions for the problem are at Z = 0 (3)

$$q=q', \qquad w=w', \qquad \sigma_{zz}=(\sigma_{zz})', \quad \sigma_{zr}=(\sigma_{zr})' \tag{3.1}$$

SOLUTION OF THE PROBLEM

As the potential φ in the upper solid can be written as (3)

$$\varphi=\varphi_0+\varphi_1 \tag{4.1}$$

where φ_0 is the primary disturbance created by the source and the presence of the second part is due to the boundary, we obtain the following expressions as studied by Ewing at al (3)

$$\varphi=\int_0^\infty\frac{k}{\upsilon_1^0}J_0(kr)e^{-\upsilon_1^0|z-h|}dk+\int_0^\infty Q(k)J_0(kr)e^{-\upsilon_1^0(z-h)}dk \tag{4.2}$$

$$\psi=\int_0^\infty S(k)J_0(kr)e^{-\upsilon_2^0(z-h)}dk \qquad \text{at } z>0 \tag{4.3}$$

$$\varphi'=\int_0^\infty Q'(k)J_0(kr)e^{\upsilon_1^{0'}(z-h)}dk \tag{4.4}$$

$$\psi'=\int_0^\infty S'(k)J_0(kr)e^{\upsilon_2^{0'}(z-h)}dk \qquad \text{at } z<0 \tag{4.5}$$

A time factor exp (iwt) is understood
where

$$\upsilon_1^{02} = k^2 - k_\alpha^{02}, \upsilon_2^{02} = k^2 - k_\beta^{02}, \upsilon_1^{0/2} = k^2 - k_\alpha^{0/2}, \upsilon_2^{0/2} = k^2 - k_\beta^{0/2} \quad (4.6)$$

$$k_\alpha^{02} = \frac{w^2}{D_T^{*2}}, \quad k_\beta^{02} = \frac{w^2}{D_S^{*2}}, \quad k_\alpha^{0/2} = \frac{w^2}{D_T^{*'2}}, \quad k_\beta^{0/2} = \frac{w^2}{D_S^{*'2}}$$

$$D_T^{*2} = \frac{(v_{0T}^2 + iwv_{1T}^2)}{(\eta_0 + iw\eta_1)} \qquad D_s^{*2} = \frac{(v_{0s}^2 + iwv_{1s}^2)}{(\eta_0 + iw\eta_1)} \quad (4.7)$$

substituting the values of φ, ψ, φ' and ψ' from (4.2) to (4.5) in the boundary conditions (3.1) we obtain the following relations

$$\hat{Q} - \hat{Q}' - \upsilon_2^0 \hat{S} - \upsilon_2^{0'} \hat{S}' = \frac{-k}{\upsilon_1^0} e^{-\upsilon_1^0 h} \quad (4.8)$$

$$\upsilon_1^0 \hat{Q} + \upsilon_1^{0'} \hat{Q}' - k^2 \hat{S} + k^2 \hat{S}' = k.e^{-\upsilon_1^0 h} \quad (4.9)$$

$$-a_1^0 \hat{Q} + a_1'^0 \hat{Q}' - \frac{2\mu_k^* k^2}{\eta_k^*} \upsilon_2^0 \hat{S} + \frac{2\mu_k^{*'} k^2 \upsilon_2^{0'} \hat{S}'}{\eta_k^{*'}} = \frac{k}{\upsilon_1^0} \left(\frac{2\mu_k^* k^2}{\eta_k^*} - pw^2 \right) e^{-\upsilon_1^0 h} \quad (4.10)$$

$$\frac{2\mu_k^*}{\eta_k^*} \upsilon_1^0 \hat{Q} + \frac{2\mu_k^{*'} \upsilon_1^{0'}}{\eta_k^{*'}} \hat{Q}' - a_1^0 \hat{S} + a_1^{0'} \hat{S}' = \frac{2k\mu_k^*}{\eta_k^*} e^{-\upsilon_1^0 h} \quad (4.11)$$

where $a_1 = \frac{2\mu_k^* k^2}{\eta_k^*} - \rho w^2, \qquad a_1^{0'} = \frac{2\mu_k^{*'} k^2}{\eta_k^{*'}} - \rho' w^2$

$$\hat{Q} = Qe^{\upsilon_1^0 h}, \qquad \hat{Q}' = Q' e^{-\upsilon_1^{0'} h}, \qquad \hat{S} = Se^{\upsilon_2^0 h}, \qquad \hat{S}' = S' e^{-\upsilon_2^{0'} h}, \quad (4.12)$$

From (4.8) to (4.11) we obtain the following expression

$$Q = \triangleleft_1 / \Delta e^{-2\upsilon_1^0 h}, \qquad S = \triangleleft_2 / \Delta e^{-(\upsilon_2^0 + \upsilon_1^0) h}, \qquad Q' = \triangleleft_1' / \Delta e^{-(\upsilon_1^0 - \upsilon_1^{0'}) h}$$

$$S' = \triangleleft_2' / \Delta e^{-(\upsilon_1^0 - \upsilon_2^{0'})}. \qquad \mu_k^* = \mu_0 + iw\mu_1, \qquad \eta_k^* = \eta_0 + iw\eta_1 \quad (4.13)$$

where $\triangleleft$ is the determinant of the system given by (4.8) to (4.11).

The factors $\triangleleft_1/\triangleleft$, $\triangleleft_2/\triangleleft$, $\triangleleft'_1/\triangleleft$, $\triangleleft'_2/\triangleleft$ have the form of reflection and transmission coefficients for plane simple harmonic waves. So from (4.2) to (4.5) and (4.13) the solution of the problem of wave propagation in two solids is represented by

$$\varphi = \int_0^\infty \frac{k}{\nu_1^0} J_0(kr) e^{-\upsilon_1^0 |z-h|} dk + \int_0^\infty \frac{\triangleleft_1}{\triangleleft} J_0(kr) e^{-\upsilon_1^0 (z+h)} dk \qquad (4.14)$$

$$\psi = \int_0^\infty \triangleleft_2/\triangleleft \, J_0(kr) e^{-\upsilon_2^0 z - \upsilon_1^0 h} dk \qquad (4.15)$$

$$\phi' = \int_0^\infty \frac{\triangleleft'_1}{\triangleleft} J_0(kr) e^{\upsilon_1'^0 z - \upsilon_1^0 h} dk \qquad (4.16)$$

$$\psi' = \int_0^\infty \frac{\triangleleft'_2}{\triangleleft} J_0(kr) e^{\upsilon_2^{0'} z - \upsilon_1^0 h} dk \qquad (4.17)$$

The first term in (4.14) represent the direct compressional waves, all the other terms in (4.14) to (4.17) represent waves generated in both media by it. Different types of waves are determined by a set of branch line integrals corresponding to $k = k_\alpha^0, k_\beta^0, k_\alpha^{0'}, k_\beta^{0'}$ and residues corresponding to the roots k of the equation

$$\triangleleft(k) = 0 \qquad (4.18)$$

where

$$\triangleleft(k) = \begin{vmatrix} 1 & -1 & -\upsilon_2^0 & \upsilon_2^{0'} \\ \upsilon_1^0 & \upsilon_1^{0'} & -k^2 & k^2 \\ -a_1^0 & a_1^{0'} & \dfrac{2\mu_k^* \upsilon_1^0}{\eta_k^*} & \dfrac{2\mu_k^{*'} \upsilon_1^{0'}}{\eta_k^{*'}} \\ \dfrac{2\mu_k^* \upsilon_1^0}{\eta_k^*} & \dfrac{2\mu_k^{*'} \upsilon_1^{0'}}{\eta_k^{*'}} & -a_1^0 & a_1^{0'} \end{vmatrix} \qquad (4.10)$$

DISCUSSION

It is obvious that the expression for the potentials in (4.14) to (4.17) are affected by the viscous field. Here each of the wave types associated with the branch points of equations (4.14) to (4.17) may be considered to travel along a path composed of three parts (1). Source to interface (2) along the interface (3) Interface to receiver. The coefficient of h in the exponential indicates wheather the first part of the path is traversed by compressional or shear waves, the coefficient of z gives the same information about the third part, while the value of k at the branch points indicates the mode of tracel along the interface.

Now if $\eta_0 = 1$, $\eta_1 = \lambda_1 = \mu_1 = 0$ we obtain the classical result as studied by Ewing et al (3). Under the above assumption the equation (4.18) reduces to

$$4(\mu' - \mu)^2 [k^2 \{k^2 - \frac{w^2(\rho' - \rho)}{2(\mu' - \mu)}\}^2 - \upsilon_1 \upsilon_2 (k^2 - \frac{\rho' w^2}{2(\mu' - \mu)})^2 \qquad (5.1)$$

$$- \upsilon_1' \upsilon_2' (k^2 + \frac{\rho w^2}{2(\mu' - \mu)})^2 - (\upsilon_1 \upsilon_2' + \upsilon_1' \upsilon_2) \frac{w^4 \rho \rho'}{4(\mu' - \mu)^2} + \upsilon_1 \upsilon_2 \upsilon_1' \upsilon_2' k^2 = 0$$

where

$$\upsilon_1 = \sqrt{k^2 - k_\alpha{}^2}, \qquad \upsilon_2 = \sqrt{k^2 - k_\beta{}^2}, \qquad \upsilon_1' = \sqrt{k^2 - k_\alpha'^2}$$

$$\upsilon_2' = \sqrt{k^2 - k_\beta'^2}, \qquad k_\alpha = w / \alpha, \qquad k_\beta = w / \beta, \qquad k_\alpha' = \frac{w}{\alpha}$$

$$k_\beta' = w / \beta', \qquad \alpha = V_{OT}, \qquad \beta = V_{OS}, \qquad \alpha' = V_{OT}', \qquad \beta' = V_{OS}'$$

The characteristic equation (5.1) in the form given later investigated first by Stoneley (15). Thus equation (4.18) represents the Stoneley waves in visco-elastic medium in a different form. Now the equation (5.1) can be written in the following form

$$c^4 \{(\rho - \rho')^2 - (\rho c_1' + \rho' c_1)(\rho D_1' + \rho' D_1)\} + 2kc^2 \{\rho c_1' D_1' - \rho' c_1 D_1 - \rho + \rho'\}$$

$$+ k^2 (c_1 D_1 - 1)(c_1' D_1' - 1) = 0 \qquad (5.2)$$

which is the frequency equation obtained by Stoneley and known as Stoneley waves (15). In the above equation

$$c_1^2 = (1 - c^2 / \alpha^2), \qquad c_1'^2 = 1 - \frac{c^2}{\alpha'^2}, \qquad D_1^2 = 1 - \frac{c^2}{\beta^2}, \qquad D_1'^2 = \frac{1 - c^2}{\beta_2'^2}$$

$$K = 2(\mu - \mu'), \qquad c = w / k, \qquad \upsilon_1 = kA_1, \qquad \upsilon_1' = kB_1$$

$$\upsilon_2 = kA_1', \qquad \upsilon_2' = kB_1'$$

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THE EFFECT OF TEMPERATURE DEPENDENT FREQUENCY FACTOR ON THE DETERMINATION OF ACTIVATION ENERGY OF A DIFFERENTIAL THERMAL ANALYSIS PEAK

Manabesh Bhattacharya[a], W. G. Devi[b], and S. Dorendrajit Singh[c]

[a]Department of Physics,
Krishnath College,
Berhampore - 742101, Murshidabad, West Bengal, India

[b]Department of Chemistry,
T.S. Paul Manipur Womens' College
Monsangei, Imphal 795008, Manipur, India

[c]Department of Physics,
School of Science
Manipur University, Chanchipur, Imphal 795003, Manipur, India

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Abstract:
A detailed analysis of the determination of the activation energy of a differential thermal analysis peak is made by considering the temperature dependence of frequency factor. We also suggest a technique for the determination of the temperature exponent.

Kew words : Differential thermal analysis, order of kinetics, activation energy, temperature exponent.

INTRODUCTION

Now-a-days there is an enormous groth of interest in thermal analysis i.e. techniques which measure the change in physical and / or chemical properties of materials as a function of temperature or time. Differential thermal analysis (DTA) is one such technique which is popularly used for the quantitative identification of organic and inorganic compounds [1-4]. Devi et al [5] presented a method for the determination of activation energy and order of kinetics of DTA peaks. But they did not take into account the possible temperature dependence [6-9] of the type T^{-a} $(-2 \leq a \leq 2)$ of the frequency factor. In the present paper we reinvestigate the problem by considering the temperature dependence of the frequency factor and we also propose a method of determination of temperature exponenet a.

THEORY

Following Devi et al [5] we can write

$$\Delta T = A\beta \exp(-E/RT)\exp[-\frac{1}{\phi}\int_{T_0}^{T} A\exp(-E/RT')dT'] \quad (n=1) \qquad (1)$$

and for $n \neq 1$

$$\Delta T = A\beta \exp(-E/RT)\left[1 \pm \frac{n-1}{\phi}\int_{T_0}^{T} A\exp(-E/RT')dT'\right]^{-n/(n-1)}$$

where R is the universal gas constant. ΔT is the temperature deviation from the horizontal base line. β is the proportionality constant. ϕ is the linear heating rate n is the order of kineties. A is the pre-exponential factor commonly known as frequency factor. T is the temperature at time t and T_0 is the initial temperature. Now considering the temperature dependence of the type [6-9]

$$A = A_0 T^{a} \qquad (-2 \le a \le 2) \qquad (3)$$

of the frequency factor, where A_0 is a constant. Equations (1) and (2) can be recast as

$$\Delta T = \beta A_0 T^{a} \exp\left[-\frac{A_0}{\phi}\int_{T_0}^{T} T'^{a}\exp(-E/RT')dT'\right]\exp(-E/RT) \quad (n=1) \qquad (1)$$

and

$$\Delta T = \beta A_0 T^{a}\left[1 + (n-1)\frac{A_0}{\phi}\int_{T_0}^{T} T'^{a}\exp(-E/RT')dT'\right]^{-n/(n-1)} \times\exp(-E/RT) \quad (n \neq 1) \qquad (5)$$

The peak temperature T_m of the DTA curve can be obtained from the equation [5]

$$\left[\frac{d(\Delta T)}{dT}\right]_{T=T_m} = 0 \qquad (6)$$

From equations (4) - (6) the peak temperatures T_m of a DTA peak can be obtained from the following equations

$$\frac{E}{RT_m^2}-\frac{A_0T_m^a}{\phi}\exp(-\frac{E}{RT_m})+\frac{a}{T_m}=0\;(n=1) \tag{7}$$

and

$$\frac{E}{RT_m^2}+\frac{a}{T_m}=n\frac{A_0T_m^a}{\phi}\exp(-\frac{E}{RT_m})[1+\frac{A_0(n-1)}{\phi}\int_{T_0}^{T_m}T'^a\exp(-E/RT')dT']^{-1}(n\neq 1) \tag{8}$$

Without any loss of generality [3,4] T_0 can be replaced by zero and following Balarin [9] one gets

$$\int_0^T T'^a\exp(-E/RT')dT'=(E/R)^{a+1}\Gamma(-a-1,u) \tag{9}$$

with $u=\frac{E}{RT}$ and $\Gamma(a,u)$ is the complementary incomplete Gamma function [10]. Now

eliminating $\frac{A_0}{\phi}$ between equations (4), (5), (7) and (8) we can write

$$\frac{\Delta T}{(\Delta T)_m}=(\frac{u_m}{u})^a\exp[u_m-u+F(u,u_m)]\quad (n=1) \tag{10}$$

and

$$\frac{\Delta T}{(\Delta T)_m}=(\frac{u_m}{u})^a\exp(u_m-u)[1-\frac{n-1}{n}F(u,u_m)]^{-\frac{n}{n-1}}\quad (n\neq 1) \tag{11}$$

with

$$F(u,u_m)=(au_m+u_m^2)u_m^a\exp(u_m)[\Gamma(-a-1,u_m)-\Gamma(-a-1,u)]. \tag{12}$$

$(\Delta T)_m$ is the value of ΔT at peak temperature T_m. If T_1 and T_2 are the temperatures for which $\frac{\Delta T}{(\Delta T)_m}=0.5$, following Devi et al [5] we can write

$$u_m=\frac{C_1u_1}{(u_1-u_m)}+D_1 \tag{13}$$

$$u_m = \frac{C_2 u_2}{(u_m - u_2)} + D_2 \quad (14)$$

$$u_m = \frac{C_3 u_1 u_2}{u_m (u_1 - u_2)} + D_3 \quad (15)$$

where $u_1 = \frac{E}{RT_1}$, $u_2 = \frac{E}{RT_2}$ and $u_m = \frac{E}{RT_m}$. The co-efficients C_i and D_i (i = 1,2,3) occuring in the above equations depend both on a and n. For a particular value of n we get

$$C_i = c_{oi} + c_{1i} a \quad (16)$$

$$D_i = d_{oi} + d_{1i} a \quad (17)$$

Finally equations (13) - (15) can be expressed as

$$E_1 = \frac{(c_{01} + c_{11} a) RT_m^2}{(T_m - T_1)} + (d_{01} + d_{11} a) RT_m \quad (18)$$

$$E_2 = \frac{(c_{02} + c_{12} a) RT_m^2}{(T_2 - T_m)} + (d_{02} + d_{12} a) RT_m \quad (19)$$

$$E_3 = \frac{(c_{03} + c_{13} a) RT_m^2}{(T_2 - T_1)} + (d_{03} + d_{13} a) RT_m \quad (20)$$

where E_1, E_2 and E_3 are, respectively, the activation energies calculated by using the temperatures (T_1 and T_m), (T_2 and T_m), and (T_1, T_2 and T_m).

RESULTS AND DISCUSSIONS

We follow the numerical procedure outlined by Devi et al [5]. The coefficients c_{ji} and d_{ji} (i = 1-3, j = 0,1) occuring in equations (18) - (20) have been determined by using the standard technique of linear regression [11]. They are presented in table 1, for n=1, 1.5 and 2. The complementary incomplete Gamma function has been evaluated by using the continued fraction technique [12].

It has been pointed out by Devi et al [5] that order of kinetics (n) of a DTA peak can be determined by using the symmetry factor μ_g defined by

$$\mu_g = \frac{T_2 - T_m}{T_2 - T_1} \tag{21}$$

They have shown that for the case of temperature independent frequency factor (a = 0) one can write

$$\mu_g = 0.2953 + 0.1858n - 0.0244n^2. \tag{22}$$

We have found that for the case of temperature dependent frequency factor $(a \neq 0)\mu_g$ depends weakly on a. And equation (22) can still be used to estimate the order of kinetics n even for $a \neq 0$.

Now we apply the present method to determine E and n of some numerically computed DTA peaks. The results are displayed in table 2. It is seen that the values of E_p and n_p as computed by the present method are in fair agreement with the corresponding input values E_{in} and n_{in}. It is to be noted that E_p is the average of E_1, E_2 and E_3 and n_p is evaluated from equation (22).

Finally to test the applicability of the current technique we apply it to the experimental DTA peak [12, 13] of the dehydration reaction Ni (mpipz)$_2$ (NCS)$_2$. $2H_2O \rightarrow Ni(mpipz)_2(NCS)_2$, where mpipz stands for N-methylpiperazine recorded with two different heating rates namely $\phi = 5^0 C/\min$ and $\phi = 10^0 C/\min$. The peak recorded with the lower heating rate has been analysed by using the rigorous method of curve fitting [12]. In this case also there is a fair agreement between the values E_{cf} and n_{cf} as obtained by curve fitting method [12] and E_p and n_p (table 3).

DETERMINATION OF THE TEMPERATURE EXPONENT OF THE FREQUENCY FACTOR

Following Kirsh and Townsend [14] a method of determination of the temperature exponenet a of frequency factor is presented. If the same DTA peak is recorded with two different heating rates resulting in three different sets of temperatures $(T_m, T'_m), (T_1, T'_1)$ and (T_2, T'_2) from equations (18) - (20) we get

$$a = -\frac{c_{01}F_1 + d_{01}G}{c_{11}F_1 + d_{11}G} \tag{23}$$

$$a = -\frac{c_{02}F_2 + d_{02}G}{c_{12}F_2 + d_{12}G} \quad (24)$$

$$a = -\frac{c_{03}F_3 + d_{03}G}{c_{13}F_3 + d_{13}G} \quad (25)$$

with

$$G = T_m - T'_m \quad (26)$$

$$F_1 = \frac{T_m^2}{(T_m - T_1)} - \frac{T'^2_m}{(T'_m - T'_1)} \quad (27)$$

$$F_2 = \frac{T^2{}_m}{(-T_m + T_2)} - \frac{T'^2_m}{(-T'_m + T'_2)} \quad (28)$$

$$F_3 = \frac{T_m^2}{(T_2 - T_1)} - \frac{T'^2_m}{(T'_2 - T'_1)} \quad (29)$$

Now we apply the equations (23) - (25) to the experimental DTA peak considered in section 3, which has been recorded with two different heating rates. The value of a as calculated by employing equations (23), (24) and (25) are denoted respectively by a_1, a_2 and a_3 a_{av} denotes the average of a_1, a_2 and a_3. Their values are displayed in table 1. Keeping in mind that the experimental errors which are likely to creep in we can say that the experimental DTA peak considered here nearly corresponds to the case of the temperature independent frequency factor that is a = 0. Now knowing E_p, n_p and a the frequency factor A_p can be determined either from equation (7) or (8) and is also presented in table 4. A_{cf} and A_p denote, respectively, the values of frequency factor A as obtained by curve fitting method and present method. A close inspection of table 4 reveals that the curve fitted values of the experimental DTA peak under consideration are in fair agreement with those calculated by the present method.

Table 1 :

Coefficients c_{ji}, d_{ji} occuring in equations (16) and (17). A (B) stands for A x 10^B.

n	i	c_{0i}	c_{1i}	d_{0i}	d_{1i}
1.0	1	1.4603	-1.30 (-3)	-2.1640	-0.9167
1.0	2	0.9852	-1.10 (-4)	-0.1745	-0.9903
1.0	3	2.4432	-9.10 (-4)	-1.2808	-0.9614
1.5	1	1.6269	-1.62 (-3)	-2.6209	-0.9045
1.5	2	1.3861	-3.60 (-4)	-0.4374	-0.9777
1.5	3	3.0098	-1.34 (-3)	-1.5209	-0.9536
2.0	1	1.7594	-1.89 (-3)	-3.0196	-0.8945
2.0	2	1.7621	-6.90 (-4)	-0.7641	-0.9649
2.0	3	3.5177	-1.91 (-3)	-1.7935	-0.9439

Table 2 :

Activation energies (E_p) and orders of kinetics (n_p) of some numerically computed DTA peaks. For all the cases E_m = 146.447 Kcal / mole and ϕ =5^{0}C / min.

n_{in}	T_m (K)	a	μ_g	n_p	E_1 (KJ/mole)	E_2 (KJ/mole)	E_3 (KJ/mole)	E_p (KJ/mole)
1.0	522.3224	0	0.417	1.07	146.507	146.453	146.576	146.512
1.0	522.3224	2	0.416	1.07	146.681	146.480	146.654	146.732
1.0	522.3224	-2	0.418	1.08	146.333	146.426	146.523	146.127
1.5	521.9445	0	0.476	1.56	146.560	146.465	146.619	146.518
1.5	521.9445	2	0.474	1.55	146.746	146.519	146.689	146.651
1.5	521.9445	-2	0.477	1.57	146.376	146.407	146.552	146.445
2.0	521.5771	0	0.516	1.97	146.620	146.491	146.653	146.588
2.0	521.5771	2	0.515	1.95	146.814	146.571	146.739	146.708
2.0	521.5771	-2	0.518	1.98	149.504	143.263	146.247	146.338

Table 3 :

Determination of the kinetic parameters of the experimental DTA peak of $Ni(mpipz)_2(NCS)_2 2H_2O \rightarrow Ni(mpipz)_2(NCS)_2$.

E_{cf} (KJ/mole)	n_{cf}	a_{cf} (sec^{-1})	E_p (Kj/mole)	n_p
143.90	1.5	7.83 (16)	145.40	1.53

Table 4 :

Determination of the temperature exponent of the experimental DTA peak of $Ni(mpipz)_2(NCS)_2 2H_2O \rightarrow Ni(mpipz)_2(NCS)_2$.

a_1	a_2	a_3	a_a	A_p (Ssec^{-1})
2.94 (-2)	7.05 (-3)	6.12 (-2)	3.26 (-2)	1.21 (17)

CONCLUSION

In the present paper we have reinvestigated the problem of the determination of the kinetic parameters of a DTA peak in the light of the possible temperature dependence of frequency factor. When we apply the present method both to the numerically generated and experimental DTA peaks we obtain encouraging results. Finally following Townsend and Kelly [14] we also present a method of determination of temperature exponent a.

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