2023

M.Sc.

4th Semester Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER : MTM-401

(FUNCTIONAL ANALYSIS)

Full Marks : 40

Time : 2 hours

The figures in the right-hand margin indicate marks.

The symbols used have their usual meanings.

Answer **all** questions.

1. Answer *any* **four** questions from the following : 2×4=8

(a) State with justification, whether the following statement is true of false :

Let Y be a proper dense subspace of a Banach space X. Then Y is not a Banach space with respect to the induced norm.

(b) Let $A, B \in BL(H)$ be self-adjoint where H is a Hilbert space and $\alpha, \beta \in \mathbb{R}$. Show that $T = \alpha A + \beta B$ is self-adjoint.

(c) Let H be a Hilbert space and y be a fixed element of H. Define $f : H \to \mathbb{C}$ by $f(x) = \langle x, y \rangle$ for all $x \in H$. Find $\| f \|$.

(d) Let X be a normed space and Y be a closed subset of X. If $x_n \xrightarrow{w} x$ in X, then show that $x_n + Y \xrightarrow{w} x + Y$ in X/Y.

(e) Show that an inner product function is continuous.

(f) Give an example of an operator on a Hilbert space which is normal but not self-adjoint.

2. Answer *any* **four** questions from the following : $4 \times 4 = 16$

(a) When is an operator said to be self-adjoint on a Hilbert space? Let H be a Hilbert space and let $T : H \to H$ be a bounded linear operator. Prove that T is self-adjoint if and only if $\langle Tx, x \rangle$ is real for all $x \in H$. $1+3=4$

(b) Let C be a closed convex subset of a Hilbert space H. Prove that C contains an unique element of smallest norm. 4

(c) Let S be a subset of a normed space X. Show that $f(S)$ is bounded for all $f \in X^*$ if and only if $\sup\{\| x \| : x \in S\} < \infty$. 4

(d) Let $X = \mathbb{C}^3$. For $x = (x(1), x(2), x(3)) \in X$, let $\| x \| = [(|x(1)|^2 + |x(2)|^2)^{\frac{3}{2}} + |x(3)|^3]^{\frac{1}{3}}$. Show that $\| \bullet \|$ is a norm on X. 4

(e) Examine if $C[0,1]$, the space of all real valued continuous functions over the closed interval $[0,1]$ is a Banach space with respect to the norm defined by $\| f \| = \int_0^1 | f(t) | \, dt$, where $f \in C[0,1]$ and the integral is taken in Riemann sense. 4

(f) Let X and Y be inner product spaces. Then show that a linear map $F : X \to Y$ satisfies $\langle F(x), F(y) \rangle = \langle x, y \rangle$ for all $x, y \in X$ if and only if it satisfies $\| F(x) \| = \| x \|$ for all $x \in X$ where the norms on X and Y are induced by the respective inner products. 4

Answer *any* **two** questions from the following : 8×2=16

3. (a) Let X and Y be Banach spaces and $A \in BL(X, Y)$. Show that there is a constant $c > 0$ such that $\| Ax \| \geq c \| x \|$ for all $x \in X$ if and only if $\text{Ker}(A) = \{0\}$ and $\text{Ran}(A)$ is closed in X. 3

(b) Let X, Y be Banach spaces and Z be a normed space. Consider $G \in B(X, Z)$ and $H \in B(Y, Z)$. Suppose that for every $x \in X$, there is a unique $y \in Y$ such that $G(x) = H(y)$ and define $F(x) = y$. Show that $F \in B(X, Y)$. 5

4. *(a)* Let M be a closed subspace of a Hilbert space H and $x \in H$. Then show that there exist unique $y \in M$ and $z \in M^{\perp}$ such that $x = y + z$. 3

(b) Show that a normed space X is a Banach space if and only if every absolutely summable series of elements of X is summable in X. 5

5. *(a)* Let F and G be subspaces of a Hilbert space H. Show that $(F+G)^{\perp} = F^{\perp} \cap G^{\perp}$. 3

(b) Let $T \in BL(H)$ where H is a Hilbert space. Then show that

$$\|T\|^2 = \|T^*\|^2 = \|T^*T\| = \|TT^*\|$$ 5

6. *(a)* Let X be an inner product space and $A, B \subseteq X$. Then show that

(i) $A \subseteq B \Rightarrow B^{\perp} \subseteq A^{\perp}$

(ii) $A \subseteq A^{\perp\perp}$

(iii) $A^{\perp} = A^{\perp\perp\perp}$ 6

(b) If $\|x + \lambda y\| = \|x - \lambda y\|$ is true for all scalar λ, then show that $x \perp y$. 2

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