PG/1st Sem/Math/22

2022

1st Semester Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

Paper : MTM - 103

(ODE and Special Functions)

Full Marks : 40 Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

The symbols have their usual meaning.

1. Answer any ***four*** questions : 2×4=8

   (a) Let $P_n(z)$ be the Legendre's polynomial of degree n. If $1+z^5=\sum_{n=0}^{5} C_n P_n(z)$, then find the value of C_5.

   (b) What are Bessel's functions of order n? State for what values of n the solutions are independent of Bessel's equation of order n.

   (c) Define the utility of Wronskian in connection with ODE.

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(d) Find all the singularities of the following differential equation and then classify them :

$$(z-1)\frac{d^2w}{dz^2}+(\cot\pi z)\frac{dw}{dz}+\left(\operatorname{cosec}^2\pi z\right)w=0.$$

(e) Define eigenvalues and eigenfunctions associated with Sturm-Liouville problem.

(f) Show that $\int_{-1}^{1} P_n(z)\,dz=\begin{cases}0, & n\neq 0\\ 2, & n=0\end{cases}$.

2. Answer any ***four*** questions : 4×4=16

(a) Show that $J_0(kz)$ where k is a constant, satisfies the differential equation $xy''(x)+y'(x)+k^2xy=0$.

(b) Construct Green's function for the differential equation $xy''(x)+y'(x)=0$, with the following conditions : $y(x)$ is bounded as $x\to 0$, $y(1)=ay'(1)$, $a\neq 0$.

(c) Prove that $\int_{-1}^{1} P_m(z)P_n(z)\,dz=\frac{2}{2n+1}\delta_{mn}$, where δ_{mn} and $P_n(z)$ are the Kroneker delta and Legendre's polynomial, respectively.

(d) Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of $x^2y''(x)-2xy'(x)-4y=0$ for all x in $[0, 10]$. Consider the Wronskian

$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$. If $W(1) = 1$ then find the value of $W(3) - W(2)$.

(e) Show that $J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin(z)$ and $J_{-\frac{3}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[-\frac{\cos(z)}{z} - \sin(z)\right]$.

(f) If the vector functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ defined as follows:

$$\varphi_1 = \begin{bmatrix} \varphi_{11} \\ \varphi_{21} \\ \cdot \\ \varphi_{n1} \end{bmatrix}, \varphi_2 = \begin{bmatrix} \varphi_{12} \\ \varphi_{22} \\ \cdot \\ \varphi_{n2} \end{bmatrix}, \ldots, \varphi_n = \begin{bmatrix} \varphi_{1n} \\ \varphi_{2n} \\ \cdot \\ \varphi_{nn} \end{bmatrix}$$ be n

solutions of the homogeneous linear differential equation $\frac{dX}{dt} = A(t)X(t)$ in the interval $a \le t \le b$, then these n solutions are linearly dependent in $a \le t \le b$ iff Wronskian $W[\varphi_1, \varphi_2, \ldots, \varphi_n] = 0 \; \forall \, t$, on $a \le t \le b$.

3. Answer any *two* questions : 8×2=16

(a) (i) Find the general solution of the non

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homogeneous system $\frac{dX}{dt} = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} X + \begin{pmatrix} e^{5t} \\ 4 \end{pmatrix}$

where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(ii) Show that $nP_n(z) = zP_n'(z) - P_{n-1}'(z)$, where $P_n(z)$ denotes the Legendre Polynomial of degree n. 5+3

(b) (i) If α and β are the roots of the equation $J_n(z) = 0$, then show that

$$\int_0^1 J_n(\alpha z) J_n(\beta z)\, dz = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \frac{1}{2}\left[J_n'(z)\right]^2 & \text{if } \alpha = \beta. \end{cases}$$

(ii) Deduce the integral formula for confluent hypergeometric function. 5+3

(c) (i) Find the characteristics values and characteristic functions of the Sturm-Liouville problem $\left(x^3 y'\right)' + \lambda xy = 0$; $y(1) = 0$, $y(e) = 0$.

(ii) Let the Legendre equation $\left(1 - z^2\right) w''(z) - 2zw'(z) + n(n+1)w(z) = 0$. The nth degree polynomial solution $w_n(z)$ such that $w_n(1) = 3$. If $\int_{-1}^{1}\left[w_n^2(z)+w_{n-1}^2(z)\right]dz = \frac{144}{15}$, then find the value of n. 5+3

(d) (i) Find the general solution of the equation $2z(1-z)w''(z)+w'(z)+4w(z)=0$ by the method of solution in series about $z=0$, and show that the equation has a solution which is polynomial in z.

(ii) Prove that $\frac{d}{dz}\left[z^{-n}J_n(z)\right] = -z^{-n}J_{n+1}(z)$, where $J_n(z)$ is the Bessel's function. 5+3
